Press-Schechter Formalism:
applications, origin of the `fudge' factor 2, modifications
Press-Schechter (PS) Formalism: Preface...

PS assumes

1. the mass-density field is **Gaussian** (accurate at early times).
2. time evolution governed by linear theory: i.e. \( \delta \propto t^{2/3} \).
3. a collapsed object has collapsed when its linear overdensity has exceeded \( \delta_{\text{crit}} = 1.69 \).
Press-Schechter Formalism

Smooth density field on length scale $R$, i.e. mass scale $M$.

Smoothed density field obeys Gaussian statistics.

Probability that mass at $q$ inside collapsed object is $P(\delta > \delta_{\text{crit}}|M)$.
Press-Schechter Formalism

If $\delta > \delta_{\text{crit}}$ on scales $R$, then there exists a radius $R' > R(M')$ at which $\delta = \delta_{\text{crit}}$. 
If $\delta > \delta_{\text{crit}}$ on scales $R$, then there exists a radius $R' > R(M')$ at which $\delta = \delta_{\text{crit}}$. Atom at $q$ must be part in object of mass $M' > M$. 
Press-Schechter Formalism

Probability that atom at $\mathbf{q}$ contained in object with mass $> M$

$\equiv$ fraction of all mass in Universe in objects $> M$

homogeneity

$$P(>M) = P(\delta > \delta_{\text{crit}} | M)$$
Press-Schechter Formalism

PS therefore assumed* that:

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Conceptually, this is (arguably) the least logical step of the derivation
Press-Schechter Formalism

PS therefore assumed* that:

\[ P(> M) = 2P(\delta > \delta_{\text{crit}} | M) \]

Conceptually, this is (arguably) the least logical step of the derivation.
The factor of `2' is the infamous fudge factor. More on this later.
Press-Schechter Formalism

\[ P(>M) = P(\delta > \delta_{\text{crit}} | M) \]

In previous lecture we derived mass function of collapsed objects

\[ n(M) = \frac{\rho_m}{M} \frac{\partial P}{\partial M} \]

Which upon substitution of the expression for \( P(>M) \) led to

\[ M^2 n(M) = \rho_m \sqrt{\frac{2}{\pi}} \nu \exp(-\nu^2/2) \left| \frac{\partial \ln \sigma}{\partial \ln M} \right| \]

\[ \nu \equiv \frac{\delta_{\text{crit}}}{\sigma(M)} \]
Press-Schechter Formalism

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From lecture on Gaussian random fields we learned that:

\[ \sigma^2(M) = \sigma^2(R) = \int_0^{2\pi/R} \frac{d^3k}{(2\pi)^3} P(k) \]
Some Comments on $\sigma(\mathcal{M})$

$$\sigma^2(\mathcal{M}) = \sigma^2(R) = \int_0^{2\pi/R} \frac{d^3k}{(2\pi)^3} P(k)$$

1. `Smoothing' on scales $\mathcal{M}$ (i.e. $R$) washes out fluctuations on scales smaller than $R$, i.e. suppresses power on scales with $k > 2\pi/R$.

2. The powerspectrum $P(k)$ fully specifies $\sigma(\mathcal{M})$.

3. Later, we will see that $P(k)$ is tightly constrained by the CMB.
Some Comments on \( \sigma(M) \)

\[
\sigma^2(M) = \sigma^2(R) = \int_0^{2\pi/R} \frac{d^3k}{(2\pi)^3} P(k)
\]

For \( P(k) \) in standard cosmological model

From Barkana & Loeb, 2001
We discuss the functional form of $P(k)$ later.

Some insight into the power of PS can be seen by writing $P(k) = A k^n$ then...

$$\sigma^2(M) \propto \int_{R^{-1}} dk \ k^{n+2} \propto R^{(-n-3)} \propto M^{(-n-3)/3}.$$ 

with this, we can write $n(M)$ as

$$M^2 n(M) = \rho_m \sqrt{\frac{2}{\pi}} \nu \exp(-\nu^2/2) \left| \frac{\partial \ln \sigma}{\partial \ln M} \right|.$$
Press-Schechter Formalism

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$$M^2 n(M) = \rho_m \sqrt{\frac{2}{\pi}} \nu \exp(-\nu^2/2) \left( \frac{\partial \ln \sigma}{\partial \ln M} \right) \nu \equiv \delta_{\text{crit}}/\sigma(M) \quad \Rightarrow \quad (n+3)/6$$
Press-Schechter Formalism

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$$= \mathcal{N} M^{(n+3)/6} \exp(-\nu^2/2)$$
Press-Schechter Formalism

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$$\Rightarrow n(M) = \mathcal{N} M^{(n-9)/6} \exp(-\nu^2/2)$$
Press-Schechter Formalism

\[ n(M) = N M^{(n-9)/6} \exp(-\nu^2/2) \]

This mass function already contains a lot of useful information.

Recall that \( \sigma(M) \) becomes smaller as we go to higher mass.

Therefore, \( \nu \equiv \delta_{\text{crit}}/\sigma(M) \) becomes larger as we go to higher mass.
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We can then see the following...
Press-Schechter Formalism

\[ n(M) = N M^{(n-9)/6} \exp(-\nu^2/2) \]

1. When \( \nu \ll 1 \), i.e. for low masses we have power-law slope.

On galaxy scales \( n \sim -3 \) (approximately), and \( n(M) \sim M^{-2} \).
The Press-Schechter mass function is given by:

\[ \frac{dn}{d \log M} \propto M n(M) \]

At different redshifts, denoted by \( z = 0, z = 5, z = 10, z = 20, z = 30 \), the mass function exhibits specific behaviors. The graph illustrates the distribution of matter in the universe at various cosmological epochs, with the redshift \( z \) indicating the time elapsed since the formation of structures.

Barkana & Loeb, 2001
Press-Schechter Formalism

\[ n(M) = N M^{(n-9)/6} \exp\left(-\nu^2/2\right) \]

2. Exponential cut-off occurs on mass-scale where i.e. when \( \nu = 1 \Rightarrow \sigma(M) = 1.69 = O(1) \)

In linear theory \( \delta \propto t^{2/3} \), and so \( \sigma(M) \propto t^{2/3} M^{(-n-3)/6} \)

A constant \( t^{2/3} M^{(-n-3)/6} \) corresponds to \( M \propto t^{4/(3+n)} \)

The cut-off mass-scale grows with cosmic time
The Press-Schechter mass function is given by

\[ \frac{dn}{d \log M} \propto Mn(M) \]

where \( n(M) \propto M^{-2} \) and the cut-off mass \( M_{\text{cut}} \propto t^{4/(3+n)} \) decreases towards higher redshifts (\( z = 30, 20, 10, 5 \)).

Barkana & Loeb, 2001
Press-Schechter: Comparison to Simulations

Springel et al. 2005, Nature

Wait!! Black solid line is not PS! But, blue-dotted is.
Press-Schechter: Comparison to Simulations

Springel et al. 2005, Nature

\[ M^2 n(M) \]

PS blue-dotted. Solid black line: modification of PS (later)
While agreement not perfect. PS captures main features of real halo mass function well.

PS blue-dotted. Solid black line: modification of PS (later)
Comparison to UV-Luminosity Function of Drop-Out Galaxies

The graph shows the comparison of the UV-luminosity function for drop-out galaxies at different redshifts (z≈4, z≈5, z≈6, z≈7, z≈8, z≈10). The brightness is represented on the x-axis (M_{1600,AB}) and the number of galaxies on the y-axis (log_{10} Number / mag / Mpc^3). The bright and faint regions are indicated on the graph.
Comparison to UV-Luminosity Function of Drop-Out Galaxies

**Bright**  
**Faint**

PS **qualitatively** describes observed z-evolution of starforming galaxies well.

- weak z-evolution at low mass/luminosity end
- steepening of the faint end slope of the mass/luminosity function towards high z
- most apparent evolution in characteristic (cut-off) mass/luminosity scale
Quantitatively there are outstanding issues:

• faint end slope of mass function steeper than faint end slope of luminosity function. This is also the case for luminosity function in other bands.
From Lecture 1: luminosity function local galaxies in `bj’ band.

Luminosity function: number density of galaxies as a function of abs. magnitude $M_X$.

Schechter function:

$$\phi(L) \equiv \frac{dn}{dL} = \phi_* \left( \frac{L}{L_*} \right)^\alpha \exp \left( - \frac{L}{L_*} \right)$$
Comparison to UV-Luminosity Function of Drop-Out Galaxies

Quantitatively there are outstanding issues:

- faint end slope of mass function steeper than faint end slope of luminosity function. This is also the case for luminosity function in other bands.
- massive end of the mass function cuts off `slower' than luminosity functions (more on this later)
Press-Schechter Theory: Strength & Weakness

**Strength.** PS theory provides intuition into the process of structure formation in the Universe:

- in the standard cosmological model [encoded in \( P(k) \)] low-mass objects form first, more massive objects later (‘**hierarchical**’ build-up of structure)
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First stars, galaxies & black holes must have formed in low-mass collapsed halos:

Additional physical restrictions:

- mass of collapsed object must exceed Jeans (or Filter) mass
- gas inside collapses object must have been able to cool.
Intermezzo: First Stars/Galaxies

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Recall: cosmological Jeans/Filter mass of the order $10^4$-$10^5$ solar masses

\[ \frac{dn}{d \log M} \propto M n(M) \]

Barkana & Loeb, 2001

\sim \text{comoving number density of star forming galaxies}
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not restrictive
Cooling inefficient when \( T < 1\times10^4 \text{K} \). Cooling only efficient when the virialization process heats gas to \( T_{\text{vir}} > 1\times10^4 \text{K} \). Collapsed structures in which \( T_{\text{vir}} \approx 1\times10^4 \text{K} \) are `atomically cooling' halos.

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**Recall:** cooling curve, which quantifies efficiency with which gas can cool.

\[
C(T) \equiv \frac{\Lambda(T)}{n_H^2}
\]

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Possible in collapsed structures inside `*atomically cooling*’ halos (i.e $T_{\text{vir}} > 1\,\text{e}^4 \, \text{K}$).

**Still plenty** of these halos, but $n(M,z)$ very sensitive to $M$ and $z$
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more restrictive than Jeans mass constraint

First stars, galaxies formed in `atomically cooling' halos with $T_{\text{vir}}=1\times10^4$ K -> high enough to trigger cooling via collisional excitation of HI.

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**THE FIRST GALAXIES: ASSEMBLY UNDER RADIATIVE FEEDBACK FROM THE FIRST STARS**

Andreas H. Pawlik, Miloš Milosavljević, and Volker Bromm

Submitted to ApJ

**ABSTRACT**

We investigate how radiative feedback from the first stars affects the assembly of the first dwarf galaxies. To this end we perform cosmological zoomed smoothed particle hydrodynamics simulations of a dwarf galaxy assembling inside a halo reaching a virial mass $\sim 10^8 M_\odot$ at $z = 10$. The simulations

Example of recent work on this...
Intermezzo: First Stars/Galaxies

First stars, galaxies & black holes must have formed in low-mass collapsed halos:
Additional physical restrictions:
• mass of collapsed object must exceed Jeans (or Filter) mass
• gas inside collapses object must have been able to cool.

First stars, galaxies formed in `atomically cooling’ halos with $T_{\text{vir}}=10^4 \text{ K}$ -> high enough to trigger cooling via collisional excitation of HI.

Caveat: cooling actually is possible in lower mass halos at high density via molecular hydrogen ($H_2$), enabling star formation in `minihalos'. However, `photoionization heating' (more later) by radiation from these first stars remove gas from minihalos, and suppress subsequent starformation.
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**Weaknesses.**
- Contains this fudge factor.
- Only treats gravity at the linear level (ignores all complexities of gravitational collapse!)
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**Weaknesses.**
- Contains this fudge factor.
- Only treats gravity at the linear level (ignores all complexities of gravitational collapse!)

**Good news.**
- Fudge factor can be understood and derived from ‘first’ principles.
Recall PS: \[ M^2 n(M) = \rho_m \sqrt{\frac{2}{\pi}} \nu \exp\left(-\nu^2/2\right) \left| \frac{\partial \ln \sigma}{\partial \ln M} \right| \]

Sheth & Tormen (1999) proposed the following modification:

\[ \nu \exp\left(-\nu^2/2\right) \rightarrow A\left[1 + (a\nu^2)^{-p}\right] \sqrt{a\nu^2} \exp\left(-a\nu^2/2\right) \]
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Modification is based on `ellipsoidal' collapse. Thus far, we assumed spherical collapses (spherical top-hat model). In Gaussian random fields, the average shape of fluctuations and the collapse criterion delta_crit depends on nu (more on this later).
Sheth-Tormen: Comparison to Simulations

Springel et al. 2005, Nature

$M^2 n(M)$

PS blue-dotted. Solid black line: Sheth Tormen modification