Course review and problemsets* AST5220 v2026

May 19, 2026

1 Background Cosmology

1.1 General Relativity

- **Assumption 1:** The Universe is a 4D manifold (spacetime) equipped with a metric tensor $g_{\mu\nu}$ and the evolution of this tensor is determined by Einstein's General Relativity.

The metric tensor tells us how to compute distances and dictates how stuff move in spacetime. The key principle behind GR:

- **The equivalence principle:** The local effects of motion in a curved spacetime (gravitation) are indistinguishable from those of an accelerated observer in flat spacetime, without exception. Gravity is a manifistation of spacetime curvature.

In GR particles move according to the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{1}$$

The Christoffel symbols are

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right] \tag{2}$$

The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{3}$$

and the Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{4}$$

where $\mathcal{R} \equiv R^\mu_\mu$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor of matter/energy in spacetime.

1.2 The cosmological principle

- **Assumption 2:** The cosmological principle holds. The Universe is statistically homogenous and isotropic when viewed on sufficiently large scales ($\gtrsim 200 - 300$ Mpc).

The last assumption implies that the metric is on the FRLW form (here in spherical coordinates (t, r, θ, ϕ)):

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{5}$$

where k is the curvature constant (in a closed Universe we have $k = 1/R^2$ with R being the curvature radius) and $a(t)$ is the scale factor which is commonly normalized such that $a = 1$ at the present time. We have three possible geometries: flat

*Problems are taken from previous exams and marked with the corresponding year. For a solution manual to the different problemsets see `https://cmb.wintherscoming.no/problems.php#Exams`

($k = 0$), closed/spherical ($k < 0$) and open/hyperbolic ($k > 0$). In a flat Universe and Cartesian coordinates this simplifies to

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{6}$$

Another important quantity is the expansion rate

$$H(a) = \frac{da/dt}{a} \tag{7}$$

The value of the expansion rate today, the Hubble parameter $H_0 = \frac{100h\text{km/s}}{\text{Mpc}}$ with $h \sim 0.7$, is a key cosmological parameter.

1.3 Matter energy content of the Universe

- **Assumption 3:** Matter/energy components in our Universe can be modelled as perfect fluids.

For a perfect fluid the state in its rest frame is determined by just the energy density ρ and the (for background cosmology, isotropic) pressure p. The energy momentum tensor for a perfect fluid is

$$T^{\mu\nu} = U^\mu U^\nu (p + \rho) + p g^{\mu\nu} \tag{8}$$

with U^μ being the 4-velocity of the fluid. For an observer comoving with the expansion of the Universe $U^\mu = (1, 0, 0, 0)$. For background cosmology this is most easily remembered as $T^\mu_\nu = \text{diag}(-\rho, p, p, p)$. Local energy-momentum conservation gives the continuity equation

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} = 0 \implies \dot{\rho} + 3H(p + \rho) = 0 \tag{9}$$

Different fluids are distinguished by their equation of state $w = p/\rho$. For a constant EoS we have $\rho \propto 1/a^{3(1+w)}$. Matter ($w = 0$) has $\rho \propto 1/a^3$, radiation ($w = 1/3$) has $\rho \propto 1/a^4$ and dark energy (a cosmological constant $w = -1$) has $\rho \propto$ constant.

1.4 Friedmann equations

The Einsteins equations gives us the Friedmann equations

$$\left(\frac{da/dt}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{10}$$

$$\frac{d^2a/dt^2}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{11}$$

For multiple fluids we simply have $\rho = \sum \rho_i$ and $p = \sum p_i$.

1.5 Density parameters

The critical density is the density of all matter/energy components of our Universe s.t. the Universe would be flat ($k = 0$): $\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$. Density parameters are defined as $\Omega_i = \rho_i/\rho_{\text{crit}}$. We can also model (in the Friedmann equations) the effect of curvature as a density parameter $\Omega_k = -k/a^2H^2$. The density parameters satisfy

$$\sum_{\text{all components}} \Omega_i = 1 \tag{12}$$

or $\sum_{\text{normal components}} \Omega_i = 1 - \Omega_k$. We also have what we call physical density parameters $\Omega_i h^2$ (or equivalently $\Omega_i H_0^2$). The density of any species is $\rho \propto \Omega_i H_0^2$ so this combination directly tells us the density. Cosmological observables are often more easy to understand in terms of these parameter combinations instead of just the density parameters.

1.6 The ΛCDM model

The contents of the Universe and the free parameters (plus derived parameters):

- Baryons ($w = 0$): $\Omega_{b0} \approx 0.05$
- Cold dark matter ($w = 0$): $\Omega_{\text{CDM0}} \approx 0.25$

- Total matter: $\Omega_{M0} = \Omega_{b0} + \Omega_{\rm CDM0}$
- Cosmological constant ($w = -1$): $\Omega_{\Lambda 0} \approx 0.7$
- Radiation ($w = 1/3$): $\Omega_{R0} \propto T^4_{\rm CMB0}/H_0^2 \sim 5 \cdot 10^{-5}$ with $T_{\rm CMB0} = 2.7255$ K so $\Omega_{R0}H_0^2$ is essentially fixed.
- Masseless neutrinos ($w = 1/3$): $\Omega_{\nu 0} \propto \Omega_{R0} N_{\rm eff} \sim 4 \cdot 10^{-5}$ with $N_{\rm eff} = 3.046$ so $\Omega_{\nu 0}H_0^2$ is also essentially fixed.
- Total radiation: $\Omega_{\rm Rtot0} = \Omega_{R0} + \Omega_{\nu 0}$
- Curvature: Ω_{k0} (we consider focus on a flat model $\Omega_{k0} = 0$ in this course).
- Expansion rate today: H_0 (or equivalently $h \approx 0.7$)

Note that we have $\Omega_{\Lambda 0} = 1 - \Omega_{M0} - \Omega_{\rm Rtot0} - \Omega_k$ so this parameter is given in terms of the others. The Hubble function is

$$H^2(a) = H_0^2 \left(\frac{\Omega_{M0}}{a^3} + \frac{\Omega_{\rm Rtot0}}{a^4} + \frac{\Omega_{k0}}{a^2} + \Omega_{\Lambda 0} \right) \tag{13}$$

The value of the free parameters must be found from fitting to observations.

2 Background questions

- What is the relationship between conformal time η and cosmic time t? Give a physical interpretation of conformal time (2024, 1g).
- The Einstein equation reads $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. What does the left- and right-hand sides of this equation describe, respectively? (2024, 1h)
- Why is the temperature of the cosmological neutrinos smaller than the temperature of the photons today (2022, 1a)
- What is the equation of state for a perfect fluid? Mathematically what does it mean that the Universe is accelerating and what condition must the equation of state of a fluid responsible for an accelerated expansion of the Universe satisfy? (2022, 1b)
- $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$ is called the geodesic equation. Explain what the different terms mean, what the equation describes, and what we need it for when doing cosmology. (2018x, 1c)
- What is a tensor, and why are tensors useful in physics? (2018x, 1e)

3 Project related questions

What quantities are shown in figure 1 (left). What do the intersection(s) of the different curves in the plot represent in our Universe history. Derive an expression for the scalefactor a for the two intersections you see.

In figure 1 (right) we show the conformal Hubble function $\mathcal{H}(a) = aH = da/dt$ as function of $x = \log(a)$. The minimum of this curve tell us about an important event in our Universe history, which one is it? Derive an expression for the scalefactor a corresponding to this minimum (Hint: ignore the contributions to the Hubble function which are irrelevant at this time).

4 Problem: Deriving the Friedmann equation (2018, 2)

Consider a spatially flat Friedmann-Robertson-Walker spacetime with metric given by

$$ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2)$$

Calculate the Christoffel symbols Γ^0_{ij},Γ^i_{0j},Γ^i_{j0} and show that

$$\Gamma^0_{ij} = a^2 H \delta_{ij} \tag{14}$$

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \delta^i_j H \tag{15}$$

where $H = da/dt/a$ is the Hubble function. All other symbols are zero. Use this to show that

$$R_{00} = -3\frac{d^2a/dt^2}{a}$$

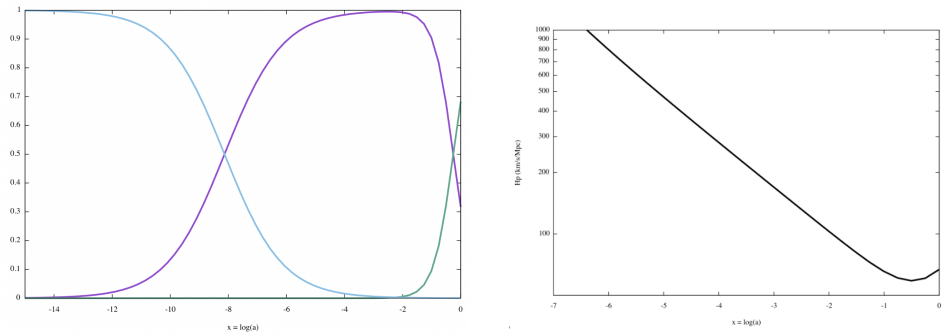


Figure 1: Left: A figure from milestone 1. Right: The conformal Hubble function.

We also have

$$R_{ij} = \delta_{ij}[2H^2a^2 + a\frac{d^2a}{dt^2}]$$

Use this and the 00 component of the Einstein equation to derive the familiar (first) Friedmann equation. You can assume that the matter in the universe can be described as a perfect fluid with energy density ρ.

5 Thermodynamics and statistical physics

5.1 Distribution function

- **Key quantity:** The distribution function $f(t, \vec{x}, \vec{p})$ tells us how many particles (of a particular type - we have one distribution function for each species) we have in a small phase space volume.

$$\Delta N = \frac{g}{(2\pi)^3} f(t, \vec{x}, \vec{p})(\Delta\vec{x})^3(\Delta\vec{p})^3$$

  where g is the number of internal degrees of freedom (spin/polarization).

Macroscopic quantities (number-density, energy density, pressure) as weighted integrals (expectation values) over the distribution function. With $\langle Q \rangle \equiv \frac{g}{(2\pi)^3}\int fQ(\vec{p})d^3p$ we have

- **Number density:** $n = \frac{g}{(2\pi)^3}\int f d^3p = \langle 1 \rangle$
- **Energy density:** $\rho = \frac{g}{(2\pi)^3}\int Ef d^3p = \langle E \rangle$ where $E = \sqrt{p^2 + m^2}$.
- **Pressure:** $P = \frac{g}{(2\pi)^3}\int \frac{p^2}{3E} f d^3p = \langle p^2/3E \rangle$.
- **Velocity:** $n\vec{v} = \frac{g}{(2\pi)^3}\int \frac{\vec{p}}{m} f d^3p = \langle \vec{p}/m \rangle$.

In thermodynamical equilibrium the distribution function for a fluid of collection of particules is gives by

- **Fermions:** $f = \left(e^{\frac{E-\mu}{T}} + 1\right)^{-1}$ (Fermi-Dirac distribution)
- **Bosons:** $f = \left(e^{\frac{E-\mu}{T}} - 1\right)^{-1}$ (Bose-Einstein distribution)

where μ is the chemical potential. In the low-temperature limit both of these reduce to the Maxwell distribution $f = e^{-\frac{E-\mu}{T}}$. In the relativistic limit we have $n \propto T^3$ and $\rho \propto T^4$. In the non-relativistic limit we have

$$n^{\text{eq}} = \left(\frac{Tm}{2\pi}\right)^{3/2} e^{-\frac{m-\mu}{T}}$$

For a $1 + 2 \to 3 + 4$ process the chemical potentials satisfy $\mu_1 + \mu_2 = \mu_3 + \mu_4$. Photons have zero chemical potential.

5.2 Boltzmann equation

The evolution of the distribution function is given by

$$\frac{df}{d\lambda} = C$$

where λ is an affine parameter and C is the collision term. With coordinates x^μ and momentum $P^\mu = dx^\mu/d\lambda$ we can expand it as

$$\frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} + \frac{\partial f}{\partial P^\mu}\frac{dP^\mu}{d\lambda} = C$$

The momentum satisfy $g_{\mu\nu}P^\mu P^\nu = -m^2$ and the usual 3-momentum is defined via $p^2 = g_{ij}P^iP^j$, the direction $\hat{p}^i$ is a unit-vector proportional to P^i. The $\frac{dP^\mu}{d\lambda}$ term can be computed using the geodesic equation.

5.2.1 Master equation for a $1 + 2 \to 3 + 4$ process

For a $1 + 2 \to 3 + 4$ process the Boltzmann equation (when integrated over all momenta) in a FRLW universe takes the form

$$\frac{1}{n_1a^3}\frac{d(n_1a^3)}{dt} = -\langle \sigma v \rangle n_2 \left(1 - \frac{n_3n_4}{n_1n_2}\left(\frac{n_1n_2}{n_3n_4}\right)_{\text{eq}}\right)$$

When the interaction strength $n_2 \langle \sigma v \rangle \ll H$ the right hand side vanishes and the interaction siezes to be relevant. We say that the interaction decouples. If the interactions are efficient then we will be close to equilibrium and the RHS must vanish. This gives us the **Saha approximation**

$$\frac{n_1n_2}{n_3n_4} \approx \left(\frac{n_1n_2}{n_3n_4}\right)_{\text{eq}}$$

6 Recombination

When the temperature of the primordial plasma drops below the ionisation energy of hydrogen the process $e^- + p^+ \to H + \gamma$ starts to form neutral atoms. This is called recombination.

6.1 Saha approximation

In the Saha approximation this process satisfies

$$\frac{n_e n_p}{n_H n_\gamma} = \left(\frac{n_e n_p}{n_H n_\gamma}\right)_{\text{eq}}$$

Defining the free electron fraction $X_e = \frac{n_e}{n_b}$ we get the Saha equation

$$\frac{X_e^2}{(1-X_e)} = \frac{1}{n_b}\left(\frac{k_b T m_e}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{\epsilon_0}{k_b T}}$$

In this derivation we assume $n_e = n_p$ (Universe is neutral) and $n_\gamma = n_\gamma^{\text{eq}}$ (photons stay in equilibrium throughout). The ionosation energy of hydrogen is $m_e + m_p - m_H = \epsilon_0$. We naively expect recombination to happen when $T \sim \epsilon_0$, but due to the large photon-to-baryon ration the prefactor to the exponential on the RHS is very lagre which delays recombination until the temperature has dropped to ~ 0.3 eV (redshift of about 1200).

If we include Helium then we also need to take into account the processes

$$e^- + \text{He}^{++} \rightleftharpoons \text{He}^+ + \gamma$$
$$e^- + \text{He}^{+} \rightleftharpoons \text{He}^0 + \gamma$$

6.2 Peebles equation

This is the full Boltzmann equation (our master equation) with a small tweek. It takes the form

$$\frac{dX_e}{dx} = \frac{C_r}{H}\left((1-X_e)\beta - n_b \alpha^{(2)} X_e^2\right)$$

where $\alpha^{(2)} = \langle\sigma v\rangle$, $\beta = \langle\sigma v\rangle \left(\frac{k_b T m_e m_p}{2\pi m_H}\right)^{3/2} e^{-\frac{m_e+m_p-m_H}{k_b T}}$ and where the cross-section follows from atomic physics. The electron in the atom can be in many different energy and spin states (but will over time decay down to the ground state) and depending on the energy of the photon we can end up in any one of these states. For recombination we cannot recombine directly into the ground state of hydrogen. This process will emit a photon that has energy larger than the ground state (quantum number $n = 1$) energy of hydrogen, 13.6 eV, so even though we have created a new hydrogen atom this photon will likely ionize another atom so the net effect will be close to zero. This process is therefore too inefficient to recombine the Universe. Electrons therefore only efficiently recombine to the excited states of hydrogen, from which they decay quickly down to the first excited state with quantum number $n = 2$. From this first excited state, electrons can reach the ground state in two ways: 1) decay from the $2p$ state by emitting a photon. This photon will likely hit another hydrogen atom in its ground state. However, cosmological redshifting systematically decreases the photon frequency, so there is a small chance that it escapes reabsorption if it gets redshifted far enough before encountering another hydrogen atom. 2) decay from the 2s state by emitting two photons. This process is very slow (for atomic physics), with a rate of $\Lambda_{2s\to 1s} = 8.22/\text{s}$. What complicates things further is that atoms in the first excited state may also be reionized before they reach the ground state. When this is the case, it is as if the recombination to the excited state did not happen in the first place. To account for this possibility, Peebles defines a factor C_r as the probability that an atom in the first excited state reaches the ground state through either of the two pathways described above before being photoionized.

6.3 Optical depth and visibility functions

Light that travels through a medium can be absorbed by the medium. If we have a source emiting an intensity I_0 then an observer will observe an intensity $I = I_0 e^{-\tau}$. The (dimensionless) quantity τ is called the optical depth. If $\tau \ll 1$ then the medium does nothing (we say its optically thin) and if $\tau \gg 1$ then we will see nothing (the medium is optically thick). The optical depth for Thompson scattering in cosmology is defined as

$$\tau(\eta) = \int_\eta^{\eta_0} n_e \sigma_T a d\eta'$$

and quantifies the probability for scattering a photon between some previous time and today (the number of scatterings of photons by electrons per unit time is $c n_e \sigma_T$). Another important quantity is the so-called visibility function $g(t) = \tau' e^{-\tau}$ This function is a true probability distribution, and describes the probability density for a given photon to last time having scattered at time t.

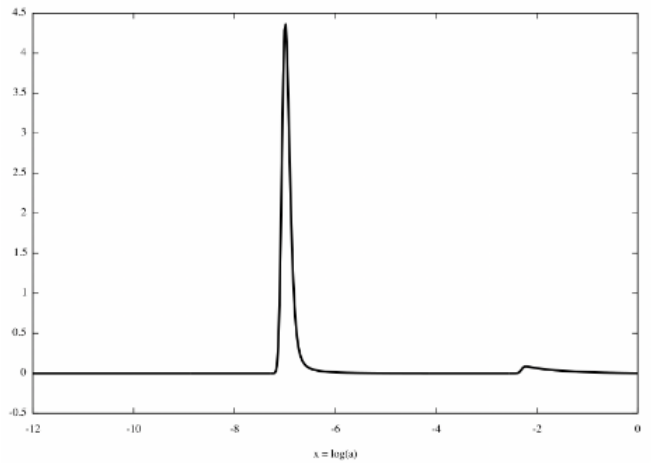


Figure 2: Quantity from milestone 2.

7 Background questions

- What does the distribution function $f(t, \vec{x}, \vec{p})$ represent physically and how can we compute the number density from f? (2025)
- List two interactions (between different species like electrons, protons, photons, etc) that are relevant for this course and why its important in the Universe history. (2025)
- What is the visibility function and what is the physical interpretation of this? Why it is practically zero at very early times? (2024)
- What do the Fermi-Dirac and Bose-Einstein distributions describe? What does the Maxwell distribution describe? (2024)
- What do we mean by decoupling and freeze-out?

8 Project related questions

What quantity is shown in figure 2 and what is its physical interpretation. What events do the two peaks correspond to? How would this curve change if we increased the temperature of the CMB today? Sketch the evolution of the optical depth τ and discuss some qualitative changes you would expect if we vary 1) the CMB temperature today T_{CMB0} and 2) the baryon density Ω_{b0}. In figure 3 we show the evolution of the free electron fraction $X_e(z)$ when Helium is included. This function is seen to drop to a lower value at three different times. What is the physical reason for these drops?

9 Problem: The Saha equation (2024)

For a gas consisting of photons, protons and electrons in thermodynamic equilibrium the Saha equation holds

$$\frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}},$$

where n_X is the density of species X, and $n_X^{(0)}$ is the equilibrium density. You can assume as known that the photon density equals the equilibrium density during thermodynamic equilibrium, $n_\gamma = n_\gamma^{(0)}$.

a) We assume $n_e = n_p$, what is the assumption behind this equality? Define the free electron fraction to be

$$X_e = \frac{n_e}{n_e + n_H}.$$

and show that the Saha equation may be written on the form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

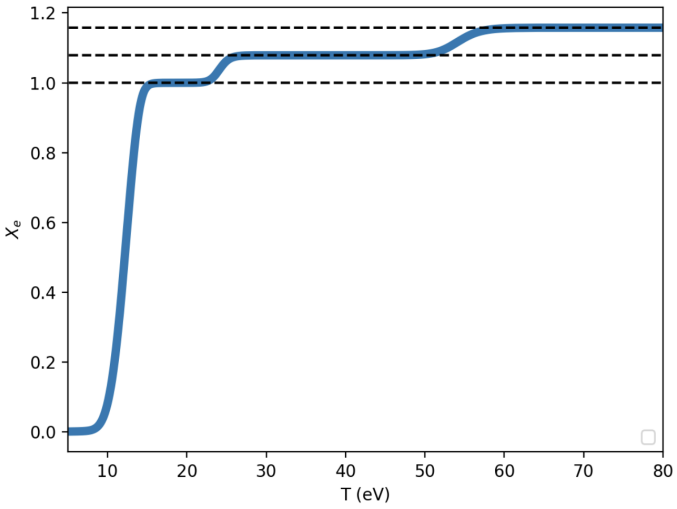


Figure 3: Evolution of $X_e(z)$ including helium.

b) The equilibrium density of (massive) species X is given by

$$n_X^{(0)} = \left(\frac{kTm_X}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{m_X c^2}{kT}}.$$

Show that the full Saha equation for the electron density is given by

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_e + n_H}\left(\frac{kTm_e}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{\epsilon_0}{kT}}.$$

What is ϵ_0 and what assumption(s) regarding m_p and m_H are used here?

c) When do we naively expect recombination to happen? In our Universe recombination happens much later than our simple guess, what is the reason for this?

10 Perturbations

Perturbations to the metric can be decomposed into terms that transform as scalars, vectors and tensors respectively under coordinate transformations. For linear perturbations the SVT decomposition theorem say that scalar-, vector- and tensor-perturbations evolve independently. We focus mainly on scalar perturbations in this course. Tensor perturbations correspond to gravitational waves.

The SVT decomposition of the perturbations gives us a great simplification: we only have 4 free functions to work with instead of 10. However the metric perturbations as we wrote them above are not uniquely defined, but depend on our choice of coordinates, the so-called gauge choice. When we wrote down the perturbed metric, we implicitly chose a specific timeslicing of the spacetime and defined specific spatial coordinates on these time slices. Choosing the Newtonian gauge the metric is then given by

$$ds^2 = -(1+2\Psi)dt^2 + (1+2\Phi)a^2(dx^2+dy^2+dz^2)$$

where Φ, Ψ are the gravitational potentials. When we work in linear perturbation theory we only keep terms up first order in the perturbed quantities. We also choose to work in Fourier space throughout:

$$\hat{f}(\vec{k}) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} f(\vec{x})$$

Transforming a linear equation to Fourier space is as simple as replacing f by $\hat{f}$ and changing $\frac{d}{dx^i} \to ik^i$.

10.1 Perturbed Boltzmann equation

The Boltzmann equation is a function of spacetime position x^μ and momentum $P^\mu = dx^\mu/d\lambda$. The momentum satisfy $g_{\mu\nu}P^\mu P^\nu = -m^2$. The 3-momentum p is defined via $p^2 \equiv g_{ij}P^iP^j$. The direction $\hat{p}^i$ is defined to be a unit-vector along P^i. As our coordinates we choose $(t, x^i, E, \hat{p}^i)$ where $E = \sqrt{m^2+p^2}$. We can compute $P^0 = E(1-\Psi)$ and $P^i = \frac{p\hat{p}^i}{a}(1-\Phi)$.

The Boltzmann equation for a given species can then be expanded as

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial t}\frac{dt}{d\lambda} + \frac{\partial f}{\partial x^i}\frac{dx^i}{d\lambda} + \frac{\partial f}{\partial E}\frac{dE}{d\lambda} + \frac{\partial f}{\partial \hat{p}^i}\frac{dp^i}{d\lambda} = C$$

The term $dE/d\lambda = -p^2\left[H - H\Psi + \frac{\partial\Phi}{\partial t} + \frac{E}{p}\frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right]$ can be computed using the geodesic equation. The last term is generally a second order perturbation and can be ignored (the distribution function does not depend on direction in a FRLW universe and things move in straight lines in the background).

The equations we get out from the Boltzmann equation are 1) a **continuity equation** describing conservation of particle number 2) an **Euler equation** describing momentum conservation and 3) equations for higher order multipoles (relevant only for relativistic species).

10.1.1 Photons

The relevant interaction is Thompson scattering $e^- + \gamma \rightleftharpoons e^- + \gamma$. The distribution function is given by $f = \frac{1}{e^{p/T}-1}$ and we perturb the local photon temperature $T = \overline{T}(1+\Theta(t, x^i, p, \hat{p}^i))$. Simplification: Thompson scattering does not change the momentum of the photons (much) only the direction so we can ignore the p dependence of Θ. The Boltzmann equation gives us

$$\frac{\partial\Theta}{\partial t} + \frac{\partial\Theta}{\partial x^i}\frac{\hat{p}^i}{a} + (\frac{\partial\Phi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a}) = n_e\sigma_T[\Theta_0 - \Theta + \hat{p}\cdot\vec{v}_b]$$

The equation in Fourier space becomes

$$\frac{\partial\Theta}{\partial t} + \frac{ik\mu}{a}\Theta + (\frac{\partial\Phi}{\partial t} + \frac{ik\mu}{a}\Psi) = n_e\sigma_T[\Theta_0 - \Theta + i\mu v_b]$$

where $\mu = \vec{k}\cdot\hat{p}/k$. To deal with the directional dependence we expand photons in multipoles

$$\Theta(t,k,\mu) = \sum \frac{2\ell+1}{i^\ell}\Theta_\ell(t,k)P_\ell(\mu)$$

Multiplying the photon equation above by $P_\ell(\mu)$ and integrating over μ we obtain the system

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi', \tag{16}$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{17}$$

$$\Theta_\ell' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1} + \tau'\left[\Theta_\ell - \frac{1}{10}\Pi\delta_{\ell,2}\right], \qquad 2 \le \ell \tag{18}$$

Remember the interpretation of the low multipoles: $\delta_\gamma = 4\Theta_0$ and $v_\gamma = -3\Theta_1$.

10.1.2 Cold Dark Matter

Dark matter has no interactions with the normal standard model particles. The collision term is zero and the Boltzmann equation reads

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{P^i}{P^0} + \frac{\partial f}{\partial E}\frac{1}{P_0}\frac{dE}{d\lambda} = 0$$

We take the moment (integrate it over all momenta) of the equation above to get the continuity equation

$$\frac{\partial n}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}(nv^i) + 3n[H + \frac{\partial \Phi}{\partial t}] = 0$$

We now perturb the number density directly by taking $n = \overline{n}(1+\delta)$. Taking the second moment we get an equation of the velocity

$$\frac{\partial v^i}{\partial t} + Hv^i = -\frac{1}{a}\frac{\partial \Psi}{\partial x^i}$$

Since CDM is non-relativistic we have no sizable quadrupole moment and the Boltzmann hierarchy truncates here. The velocity field is irrotational in linear theory so in Fourier space we can define a scalar velocity v via $\vec{v} = iv\frac{\vec{k}}{k}$. The full system is then

$$\delta'_{\rm CDM} = \frac{ck}{\mathcal{H}}v_{\rm CDM} - 3\Phi' \tag{19}$$

$$v'_{\rm CDM} = -v_{\rm CDM} - \frac{ck}{\mathcal{H}}\Psi \tag{20}$$

10.1.3 Baryons

For baryons (electrons and protons) the relevant iteractions are Coloumb scattering

$$e^- + p^+ \to e^- + p^+$$

and Thompson scattering

$$e^- + \gamma \to e^- + \gamma$$

We also have Thompson scattering with protons but their large mass compared to electrons means this interaction can be ignored (the cross-section of this process satisfy $\sigma \propto 1/m^2$). Coloumb scattering is so effective that it always keeps electrons and protons tightly coupled. We can therefore define the baryons overdensity and velocity as

$$\delta_p = \delta_e \equiv \delta_b$$

$$\vec{v}_p = \vec{v}_e \equiv \vec{v}_b$$

The continuity equation for baryons is the same as for CDM as none of the interactions change the particle number. For the Euler equation we get a contribution from the momentum transfer between electrons and photons

$$\delta'_b = \frac{ck}{\mathcal{H}}v_b - 3\Phi' \tag{21}$$

$$v'_b = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R(v_b - v_\gamma) \tag{22}$$

10.1.4 Neutrinos

The treatment of neutrinos is the same as for photons with the exception that there are no interactions so we can simply take the photon equations and put the cross-section to zero to get the neutrino equations.

10.2 Einstein equations

Evaluating the Einstein tensor for the perturbed metric we find two independent equations (here in Fourier space)

$$\Phi' = \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi + \frac{1}{2}\left[\Omega_{\rm CDM}(a)\delta_{\rm CDM} + \Omega_b(a)\delta_b + \Omega_\gamma(a)\delta_\gamma + \Omega_\nu(a)\delta_\nu\right] \tag{23}$$

$$\Phi + \Psi = -\frac{12\mathcal{H}^2}{c^2k^2}\left[\Omega_\gamma(a)\Theta_2 + \Omega_\nu(a)\mathcal{N}_2\right] \tag{24}$$

The first is the **Poisson equation**, the GR equivalent of the Newtonian $\nabla^2\Phi = 4\pi G\rho$. The second one relates the two gravitational potentials.

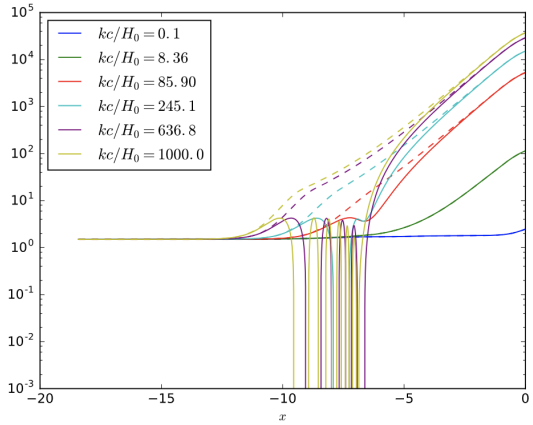


Figure 4: Milestone 3 plot, showing the evolution of some perturbations for 4 different k values.

11 Background questions

- Why do we solve the perturbation equations in Fourier space? (2025)
- The metric in the conformal Newtonian gauge reads

$$ds^2 = -(1+2\Psi)dt^2 + (1+2\Phi)a^2(dx^2+dy^2+dz^2)$$

  What do Ψ and Φ represent physically in this expression? (2016)
- How many independent equations do Einstein's field equations correspond to in the conformal Newtonian gauge, and how many do they correspond to in the most general metric? (2016)
- Why is it acceptable to set the curvature potential, Φ, to 1 at early times when solving the Einstein-Boltzmann equations? (2013)
- Write down the Boltzmann equation on schematic form, and expand the left-hand side into partial derivatives using t, x, E and $\hat{p}$ as free variables. Why can we neglect the term depending on $\hat{p}$? (2012)

12 Project related questions

What quantities are shown in figure 4. Give a brief explanation for the evolution that you see in the figure. Describe the different epochs (of time) visible in the figure, and what the epochs correspond to physically. Explain the shape of the solid curves and the dotted curves and why they are different. Explain why the curves corresponding to different values of k behave differently. (2018)

When does it mean that a perturbation "enters the Horizon"? Write down the condition that determines when this happens.

Figure 5 shows the curvature potential, Φ plotted for a few different values of k as a function of time x. Which main phases can you see here? Explain physically the general behaviour of these functions. Why are the functions flat after $x \approx -6$? In the standard ΛCDM model will they remain flat to $x = 0$? If not, why? (2010)

13 Problem: Interactions between protons, electrons and photons (2022)

a) The two key fluid equations are the continuity and Euler equation:

$$\frac{dn}{dt} + \nabla(n\vec{v}) = 0, \tag{25}$$

$$\frac{d\vec{v}}{dt} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla\Psi - \frac{\nabla P}{\rho}, \tag{26}$$

where n is the number density, $\vec{v}$ its velocity, Ψ the gravitational potential and P the fluid pressure. What conservation laws lie behind each of the equations?

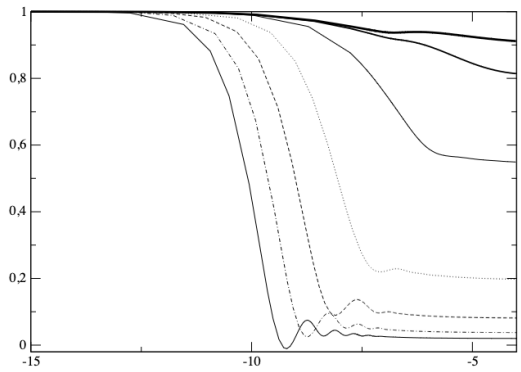


Figure 5: Milestone 3 plot.

b) The most important particles for the study of the CMB are protons p^+, electrons e^- and photons γ and the key interaction is Thompson scattering:

$$e^- + \gamma \rightarrow e^- + \gamma$$

In the early Universe this leads to so-called tight-coupling between baryons and photons. What does this mean?

c) This interaction leaves its mark on the continuity and Euler equations for photons and baryons in the τ' terms:

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi', \tag{27}$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{28}$$

$$\delta_b' = \frac{ck}{\mathcal{H}}v_b - 3\Phi', \tag{29}$$

$$v_b' = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b). \tag{30}$$

What does having the interaction term in the equations above represent physically? Why does this interaction not affect the continuity equation?

d) The most important interaction between electrons and protons is $e^- + p^+ \rightarrow e^- + p^+$. We don't have any interaction term in the equations above, as we did for Thompson scattering. What is the role of this interaction and what assumptions does it allow us to make?

e) Why don't we include the effect of photons scattering of protons, $p^+ + \gamma \rightarrow p^+ + \gamma$, in our equations?

14 From perturbations to observables

14.1 Initial conditions

The leading theory is that the initial fluctuation are set up by inflation where quantum fluctuations gets stretched to macroscopic scales by the rapid expansion and becomes classical. The perturbations gets frozen in when the mode leaves the horizon during inflation and stays constant until it reenters the horizon in the late Universe. The simplest inflationary models predicts the initial fluctuations to be that of a near gaussian random field characterized by a power-spectrum

$$\frac{k^2 P(k)}{2\pi^2} = A_s (k/k_*)^{n_s - 1}$$

A_s is the primordial amplitude and is determined by when inflation happened and n_s is the spectral index and is determined by how long inflation happened (the longer the closer to unity).

To derive the initial conditions in the early radiation era we consider the Einstein-Boltzmann system in the limit where $k/\mathcal{H} \ll 1$. For example the Einstein equations (here ignoring neutrinos) in this limit reads

$$\Phi' \approx \Psi + 2\Theta_0 \tag{31}$$

$$\Phi + \Psi \approx 0 \tag{32}$$

where we have ignored matter contributions (as its small compared to radiation), terms $k^2/\mathcal{H}^2\Phi \ll \Phi$ and the photon quadrupole (small during tight coupling). The Θ_0 equation in the same limit (ignoring the $k/\mathcal{H}\Theta_1$ term) reads

$$\Theta_0' \approx -\Phi'$$

Taking the derivative of the Poisson equation and using the equations above we get

$$\Phi'' + 3\Phi' \approx 0 \tag{33}$$

$$\Theta_0 \approx -\frac{1}{2}\Phi \tag{34}$$

with solution $\Phi =$ constant (we also have $\Phi \propto e^{-3x} = 1/a^3$ but this is unphysical as it blows up as $a \to 0$). This shows that Φ is constant outside the horizon. To relate the initial conditions for the different density contrasts we need an assumption

- **Adiabatic initial conditions:** the numberdensity in every species is the same everywhere initially: $\frac{n_i}{n_j} = \frac{\overline{n}_i}{\overline{n}_j}$ or simply $\frac{\delta_i}{1+w_i} = \frac{\delta_j}{1+w_j}$. This implies $4\Theta_0 = 4\mathcal{N}_0 = \delta_b = \delta_{\rm CDM}$.

To derive initial conditions for the velocity (e.g. for CDM) we start with

$$v_{\rm CDM}' = -v_{\rm CDM} - \frac{ck}{\mathcal{H}}\Psi$$

Velocities are initially zero and grow of order $k/\mathcal{H}$ so we cannot ignore the last term. Using that Ψ is constant initially we can integrate this equation ($\mathcal{H} \propto 1/a$ in the radition era) to get $v_{\rm CDM} = -\frac{ck}{2\mathcal{H}}\Psi$ where we have discarded an unphysical $1/a$ solution.

15 Entering the horizon

The first key event in the evolution of the perturbations is when the mode "enters the horizon". This happens when the wavelength of the mode drops below the size of the horizon

$$k\eta \sim 1 \text{ or equivalently } k/\mathcal{H} \sim 1$$

16 Line of sight integration

Instead of expanding the photon equation

$$\frac{\partial(\Theta + \Psi)}{\partial\eta} + \left(ik\mu - \frac{d\tau}{d\eta}\right)(\Theta + \Psi) + \frac{\partial(\Phi - \Psi)}{\partial\eta} = -\frac{d\tau}{d\eta}[\Theta_0 + \Psi + i\mu v_b]$$

in multipoles we can first formally integrate it over $[\eta_{\rm ini}, \eta_0]$ by introducing the integrating factor $e^{ik\mu(\eta-\eta_0)-\tau}$ to get

$$(\Theta+\Psi)_{\rm today} - (\Theta+\Psi)_{\rm ini} e^{ik\mu(\eta_{\rm ini}-\eta_0)} e^{-\tau_{\rm ini}} = \int_{\eta_{\rm ini}}^{\eta_0} d\eta\, g e^{ik\mu(\eta-\eta_0)}[\Theta_0+\Psi] \tag{35}$$

$$+ \int_{\eta_{\rm ini}}^{\eta_0} d\eta\, \frac{\partial(\Psi-\Phi)}{\partial\eta} e^{-\tau} e^{ik\mu(\eta-\eta_0)} \tag{36}$$

$$+ \int_{\eta_{\rm ini}}^{\eta_0} d\eta\, g e^{ik\mu(\eta-\eta_0)} i\mu v_b \tag{37}$$

where we have introduced the visibility function $g = \frac{d\tau}{d\eta} e^{-\tau}$. The term $e^{-\tau_{\rm ini}} \ll 1$ can be ignored. We can now take multipoles of this expression using the definition of the spherical Bessel function $j_\ell(x) = \frac{i^\ell}{2}\int_{-1}^{1} P_\ell(x) e^{-ix\mu} d\mu$. Before we do this we need to get ridd of μ-factors that are not in the exponential. The way we do this is doing integration by parts on the exponential which corresponds to replacing $\mu \to -\frac{1}{ik}\frac{d}{d\eta}$ where the derivative acts on everything except the exponential. The end result is

$$\Theta_\ell^{\rm today}(k) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \left[g[\Theta_0+\Psi] + \frac{\partial(\Psi-\Phi)}{\partial\eta} e^{-\tau} - \frac{1}{k}\frac{d}{d\eta}(g v_b) \right] j_\ell(k(\eta_0-\eta))$$

The advantage of this expression is that it allows us to compute it for, say, $\ell = 1000$ without integrating a coupled system of 1000s of equations, we simply need to include $6-7$ multipoles in the integration to get the source-function accurate enough and then line of sight integration takes care of the rest.

The leading contribution to the CMB power-spectrum comes from the first term. Approximating the visibility function by a delta-function at the last scattering surface, $g \approx \delta(\eta - \eta_{\rm rec})$, we get

$$\Theta_\ell(k) \approx (\Theta_0+\Psi)_{\rm rec} j_\ell(k\eta_0)$$

17 The CMB power-spectrum

We measure the CMB temperature perturbations as function of direction on the sky, i.e. we measure $\Theta(\vec{x}_{\text{our position}}, \hat{p}, t_{\rm today})$. We can expand this in spherical harmonics

$$\Theta(\vec{x},\hat{p},t) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}(\vec{x},t) Y_{\ell m}(\hat{p})$$

for which the information about the fluctuations are contained in the $a_{\ell m}$'s. From this we can compute the CMB power-spectrum. This is formally defined via

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell$$

where the LHS is an average over all different realisations of the initial conditions our Universe could have. This leads to

$$C_\ell = 4\pi \int_0^\infty \frac{dk}{k} |\Theta_\ell(k)|^2 \cdot A_s (k/k_*)^{n_s-1}$$

Now we only have one Universe to measure and the C_ℓ can be estimates from measuring the $a_{\ell m}$'s of our Universe and computing $\hat{C}_\ell = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} |a_{\ell m}|^2$. This leads to an effect called **cosmic variance**: for low ℓ's we have very few m's to estimate C_ℓ so there is a fundamental uncertainty in our estimate (that is due to us only having one single Universe to measure this in) that we cannot get around.

18 From Θ_ℓ to C_ℓ

From the LOS integral we know the leading contribution to the CMB power-spectrum is

$$\Theta_\ell(k) \approx (\Theta_0+\Psi)_{\rm rec} j_\ell(k\eta_0)$$

The spherical bessel function $j_\ell(x)$ peaks when $x \approx \ell$ so C_ℓ gets its main contribution from $(\Theta_0+\Psi)_{\rm rec}$ at scales $k\eta_0 \approx \ell$. Said differently: a feature in $(\Theta_0+\Psi)_{\rm rec}$ at some scale k gets mapped into C_ℓ at $\ell \approx k\eta_0$. Thus to understand C_ℓ we need to know $\Theta_0+\Psi$ at last scattering.

19 Acoustic oscillations

To find the value of $\Theta_0 + \Psi$ at last scattering we can take the time-derivative of the continuity equation for photons

$$\frac{\partial \Theta_0}{\partial \eta} = -k\Theta_1 - \frac{\partial \Phi}{\partial \eta}$$

and using the Euler equation

$$\frac{\partial \Theta_1}{\partial \eta} = \frac{k}{3}(\Theta_0 - 2\Theta_2 + \Psi) + \frac{\partial \tau}{\partial \eta}(\Theta_1 + \frac{v_b}{3})$$

to obtain

$$\frac{\partial^2(\Theta_0 + \Psi)}{\partial \eta^2} + \frac{k^2}{3}(\Theta_0 + \Psi) = F$$

Thus $\Theta_0 + \Psi$ is determined by an harmonic oscillator with a driving force $F = \frac{k^2}{3}(2\Theta_2) - k\frac{\partial \tau}{\partial \eta}(\Theta_1 + \frac{v_b}{3}) + \frac{\partial^2(\Psi - \Phi)}{\partial \eta^2}$.

19.1 Free oscillator

If we ignore the driving force then the solution is simply

$$\Theta_0 + \Psi = (\Theta_0 + \Psi)_{\rm ini} \cos(ks)$$

where $s = \int c_s d\eta$ with $c_s = c/\sqrt{3}$ is the distance a sound wave have travelled since the big-bang and the leading contribution to $\Theta_\ell(k)$ today is then

$$\Theta_\ell(k) \approx (\Theta_0 + \Psi)_{\rm ini} \cos(kr_s) j_\ell(k\eta_0)$$

where $r_s \approx 150$ Mpc is the sound-horizon. This gives us that the (leading contribution to) $|\Theta_\ell(k)|^2$ is maximum whenever $kr_s = \pi, 2\pi, 3\pi, \ldots$. Now using what we know about how a feature at a scale k gets mapped into C_ℓ we see that the CMB power-spectrum should have peaks for

$$\ell = \frac{n\pi\eta_0}{r_s}$$

Similary the (leading contribution to) $|\Theta_\ell(k)|^2$ is minimal whenever $kr_s = \pi/2, 3\pi/2, \ldots$ and we should have troughs for

$$\ell = \frac{(n + 1/2)\pi\eta_0}{r_s}$$

This gives the basic qualitative picture of the CMB power-spectrum. The contribution from the other terms in the source function are also relevant. For example the Doppler term depends on the baryon velocity which is approximately equal to the photon velocity. This is generally out of phase, i.e. if $\Theta_0 \propto \cos(ks)$ then $\Theta_1 \propto \sin(ks)$ so where the leading contibution vanishes then the Doppler contribution is at its maximum (and this is one thing that ensures that the troughs are not at zero amplitude). Going beyond the free oscillator there are a few effects you should know about.

19.2 Baryon loading

The effects of baryons are two-fold

- It lowers the speed of sound $c_s^2 = \frac{1}{3(1+R)}$ with $R = 3\Omega_b/4\Omega_\gamma$ and thereby also lowers the sound-horizon r_s.
- It changes the zero point of the oscillations and we now have

$$(\Theta_0 + \Psi)_{\rm rec} = (\Theta_0 + (1 + R)\Psi)_{\rm ini} \cos(ks) - (R\Psi)_{\rm rec}$$

The latter effect is called the baryon loading effect. Adding baryons is like taking a ball rolling on a spring on a horisontal surface and then suspending it vertically for which it will have to stretch further to be at rest. The effect on the CMB power-spectrum is that even peaks, corresponding to $\cos(kr_s) = +1$, gets a smaller amplitude while the odd peaks, $\cos(kr_s) = -1$, are enhanced (compared to the free oscillator). The relative height of the first three peaks can therefore be used to tell us how much baryons there is in our Universe.

19.3 Radiation driving

This is due to the time-derivatives of the gravitational potentials in the driving force. In the radiation dominated era the gravitational potentials decays as they enter the horizon (radiation has pressure and therefore cannot cluster enough to fight the expansion of the Universe and keep the potentials up). Our oscillator starts its first compression phase as soon as it enters the horizon and when it turns around and starts its rarification phase the gravitational potentials are gone so it does not have to fight gravity on its way out and this leads to it getting a much higher amplitude. The effect of this is to drive up the ampltitude by a factor of 4-5. This enhancement only happens for scales that enter the horizon before matter-radiation equality. In our Universe this corresponds to $\ell \gtrsim 100$ in the CMB power-spectrum.

19.4 Diffusion damping

Photons scattering of electrons perform a random walk. This leads to photons in hot regions to diffuse into cold regions (and vice versa) and thereby washing out perturbations. The effect this has on $\Theta_\ell(k)$ is to supress it by a factor $e^{-k^2/k^2_{\rm damp}}$ which again leads to an exponential supression of the CMB power-spectrum at large $\ell \gtrsim 800$.

20 The matter power-spectrum

The matter power-spectrum is

$$P(k,t) = |\Delta_M(k,t)|^2 \cdot A_s \left(\frac{k}{k_{\rm pivot}}\right)^{n_s-1}$$

where $\Delta_M(k,x) \equiv \frac{2}{3}\left(\frac{ck}{H_0}\right)^2 \frac{\Phi(k,x)}{\Omega_{M0}a^{-1}}$ is the gauge invariant total matter density contrast. The evolution of Δ_M is as follows

- $\Delta_M \propto a^2$ outside the horizon in the radiation era.
- $\Delta_M \propto \log(a)$ inside the horizon in the radiation era.
- $\Delta_M \propto a$ in the matter era.

Due to the fact that perturbations hardly grow inside the horizon in the radiation era the small scale power-spectrum will be suppressed relative to the large-scale power-spectrum (i.e. for modes that enter in the radiation era). The largest scale that does not get any suppression corresponds to those that enter exactly when the matter era starts

$$k_{\rm eq} = \mathcal{H}(a_{\rm eq})$$

where $a_{\rm eq}$ is matter-radiation equality. This represents the peak of the matter power-spectrum and is a measure of the physical matter density $\Omega_{M0}H_0^2$. At larger scales the power-spectrum grows at k^{n_s} while at smaller scales the power-spectrum is suppressed by $(k_{\rm eq}/k)^4$ compared to this. The matter power-spectrum also has small oscillations (the baryon acoustic oscillations).

21 Parameter dependence

A few of the effects of varying the different parameters:

- Curvature Ω_{k0} - If the Universe is positively/negatively curved then hot and cold spots in the CMB appear larger/smaller than they really are. The main effect of Ω_{k0} is therefore to shift the CMB power-spectrum to the left/right.
- Primordial amplitude A_s - Controls the ampltude of the CMB and matter power-spectrum. Varying it shifts them both up and down.
- Spectral index n_s - Varying n_s "tilts" the CMB and matter power-spectrum.
- Optical depth at reionization τ - Increasing it leads to more photons scattering in the late Universe which suppresses the CMB power-spectrum by a factor $e^{-2\tau}$ for all but the smallest ℓ's. The amplitude of the CMB power-spectrum is therefore mainly set by $A_s e^{-2\tau}$.
- $\Omega_{M0}H_0^2$ - The lower the physical matter density the later matter-radiation equality and the radiation driving effect gets stronger which boosts the amplitude of the CMB power-spectrum for $\ell \gtrsim 100$.
- Ω_Λ - Varying the dark energy density we change the distance to the last scattering surface and thereby the peak locations. Increasing it also leads to a higher ISW effect which has a signature at low ℓ in the CMB power-spectrum.

22 Background questions

- Explain qualitatively how inflation gave rise to the fluctuations that seeded the subsequent growth of structure in the universe.
- What are the main advantages of line of sight integration over the traditional Boltzmann hierarchy approach? (2024)
- You are given an theoretical spectrum showing the Sachs-Wolfe plateau ($\ell < 30$) of the CMB power spectrum. C_ℓ is seen to increase as we go from large to small ℓ. What are the cosmological parameters and the related physical effects that can cause such a signal? (2025)
- Explain briefly the effects of changing the optical depth of reionizaiton on the CMB (including polarization) and matter power-spectrum. Give an example of a cosmological parameter the optical depth is degenerate with in the CMB power-spectrum. (2020)
- Explain the reason for why we get a peak in the matter power-spectrum based on how Δ_M grows in different regimes and give an expression for the wave-number $k_{\rm peak}$ it correspond to (you can assume that the only relevant forms of energy in the Universe is matter and radiation and you don't need to derive how Δ_M evolves in different regimes - its enough to explain how it evolves) (2020).

23 Problem: From perturbations to observables (2025)

a) The evolution equation for the photon temperature perturbation $\Theta(k, \eta, \mu)$ is given by (in a flat Universe and ignoring only the quadrupolar correction Π):

$$\frac{\partial(\Theta + \Psi)}{d\eta} + (ik\mu - \frac{d\tau}{d\eta})(\Theta + \Psi) = \frac{d(\Psi - \Phi)}{d\eta} - \frac{d\tau}{d\eta}(\Theta_0 + \Psi + i\mu v_b)$$

What is the integrating factor[1] for this ODE? Derive the line of sight integral expression for $\Theta_\ell(k)$ today.

b) Line of sight integration gives rise to the following expression for Θ_ℓ:

$$\Theta_\ell(k, \eta_0) = \int_0^{\eta_0} \left[g(\Theta_0 + \Psi) + \left(\frac{d\Psi}{d\eta} - \frac{d\Phi}{d\eta} \right) e^{-\tau} - \frac{1}{k} \frac{d}{d\eta}(g v_b) \right] j_\ell(k\eta_0 - k\eta) d\eta$$

Explain what the different terms above represent physically. Use what you know about the visibility function to write down an approximate expression for the integral of the first term. Comment on the result, what does this tell us about how the anisotropies we observe today come from?

c) In the lectures we showed that in the absence of baryons we have the simple solution:

$$(\Theta_0 + \Psi)(k, \eta) \propto \cos(ks)$$

where $s(\eta) = \eta/\sqrt{3}$ is the distance a photon sound-wave have travelled since the Big Bang. How does this solution change (qualitatively) if we add the effect of baryons? What is the resulting effect on the CMB power-spectrum?

d) The CMB power-spectrum is given by

$$C_\ell = 4\pi \int \frac{dk}{k} |\Theta_\ell(k, \eta_0)|^2 \cdot A_s (k/k_{\rm pivot})^{n_s - 1}$$

Based on what you know about the behaviour of the spherical Bessel function explain how a feature on a Fourier scale k gets maps into C_ℓ. Use the results in the previous problems to give an expression for the location $\ell_{\rm peak}$ of the first peak of the CMB power-spectrum.

e) The calculations above ignore curvature. Explain the main effect of curvature on the CMB power-spectrum.

[1] An integrating factor for an ODE $Y' + AY = B$ is a factor C that allows us to write the ODE as $Y' + AY = \frac{1}{C}(YC)'$.

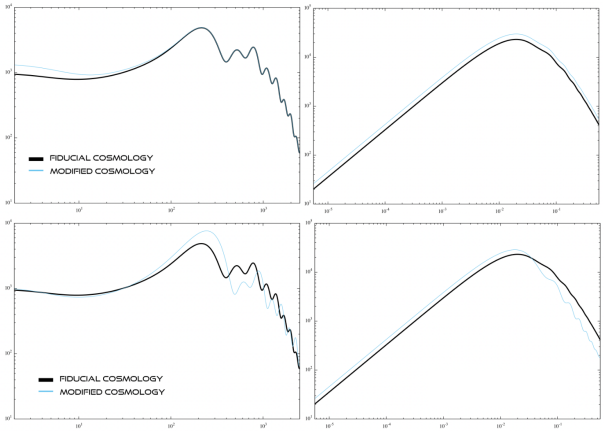


Figure 6: Left: The CMB power-spectrum. Right: The matter power-spectrum $P(k)$. In both plots the thick black line shows the result of the fiducial cosmology and in the blue curve two cosmological parameters have been changed.

24 Problem: Initial conditions (2024)

When we set the initial conditions for our system we start early in the radiation era when the modes of interest have a comoving wavelength much larger than the comoving size of the horizon. In this problem you will derive the appropriate expressions for Ψ, δ, δ_b, Θ_0 and v, relating all to Φ. You should neglect neutrinos and polarisation in this problem, and you need only consider adiabatic initial conditions.

a) What does the ratio $k/\mathcal{H} \approx k\eta$ describe and what can we say about the size of this quantity when we set the initial conditions? How can we approximate $\mathcal{H}^2$ in the radiation era? What can we say about the size of the photon quadrupole Θ_2 in the early radiation era?

b) Starting from the full set of perturbation equations (the ones you implemented in milestone 3), write down a simplified (as much as possible) set of equations for $\Theta_0, \delta_{\rm CDM}, \delta_b, \Phi$ and Ψ that is valid for very early times.

c) Derive a second-order differential equation for Φ. Show that this equation has two solutions. Which solution survives and is therefore cosmologically relevant?

d) What do we mean by adiabatic initial conditions and what does this assumption imply for the relation between the different density contrasts? Find the appropriate initial conditions for Θ_0, $\delta_{\rm CDM}$, δ_b and Ψ expressed relative to Φ.

e) Use the equation for $v'_{\rm CDM}$ to derive the initial conditions for $v_{\rm CDM}$.

25 Problem: Cosmological parameters (2022)

a) What is a so-called parameter degeneracy? Give an example of a parameter degeneracy in the CMB spectrum or matter power-spectrum and explain how varying each of the parameters changes the spectrum.

b) The density parameter combination $\Omega_M H_0^2$ is often used when deriving parameter constraints from observations of the CMB. What is the reason for working with this particular combination?

c) In Figure 6 we show the CMB and matter power-spectrum for the fiducial cosmology (Planck best-fit) in thick black and in the thin blue curve we show the result when varying two cosmological parameters. Which two cosmological parameters have been changed, and in what direction have they been changed, in the top plot of Figure 6? Explain your answer. If you cannot figure out which parameters it is, try to say something about what parameters it cannot be.

d) Which two cosmological parameters have been changed, and in what direction have they been changed, in the bottom plot of Figure 6? Explain your answer. If you cannot figure out which parameters it is, try to say something about what parameters it cannot be. (Hint: here we have done a dramatic change - one parameter is three times as large as it is in the fiducial cosmology)