Exercises for AST1100, Interstellar Matter & the Milky Way

1. A rough estimate of the mass fraction of dust in the Galaxy proceeds as follows: Approximate a spherical dust grain of radius R to have a cross sectional area πR^2 for blocking visual light. (This "shadow" formula breaks down if R is comparable to or smaller than the wavelength of visual light, but is good enough for our purposes here.) A photon mean-free-path l is defined to be the length between successive encounters with dust grains. If the number density of dust grains is n, show that

$$l = 1/n\pi R^2$$

   If a beam of photons from a star travels a distance l towards an observer, the beam will suffer an extinction in intensity by a factor of $e = 2.718\ldots$ Extinction observations indicate that at the position of the Sun in the Galaxy, $l \approx 1$ kpc. Given the estimate $R = 10^{-7}$ m, calculate the value of n in units of m^{-3}. Solid material typically has a mass density of 2000 $\mathrm{kg/m^3}$. Estimate the mass m of a typical interstellar dust grain. Let V be the volume in which one typically finds one solar mass of stars in the Galaxy. If $V = 10$ pc^3, what is the mass $M = mVm$ of dust grains in the same volume? What, then is the mass fraction of dust compared to stars at the solar position in the Galaxy?

2. A Strömgren sphere is the part of a H II region where the ultraviolet output of the central star is able to keep a balance between recombination and ionization. To calculate the size of the Strömgren sphere, idealize the problem by considering a pure hydrogen gas of uniform density which surrounds a single hot star. Let $N_\star$ be the number of ultraviolet photons beyond the Lyman limit (photons that can ionize hydrogen in the ground state) which leave the star per unit time. Assume that each such photon will ultimately ionize one and only one hydrogen atom. Let R be the number of recombinations of protons and electrons into hydrogen atoms per unit time and unit volume. In a steady state, the total number of recombinations in the Strömgren sphere of radius r must balance the the total number of ionizations:

$$R(4\pi r^3/3) = N_\star$$

   Given R and $N_\star$, this equation would allow us to find r. To obtain R, let us note that recombination at interstellar densities is a two-body process (involving for each recombination one proton and one electron). Thus the number of recombinations per unit volume R must be proportional to the product of the number densities protons and electrons $n_\mathrm{p}n_\mathrm{e}$. The proportionality factor is denoted by α, and is called the "recombination coefficient". Thus

$$R = \alpha n_\mathrm{p} n_\mathrm{e} = \alpha n_\mathrm{e}^2$$

where we require $n_\mathrm{p} = n_\mathrm{e}$ for overall charge neutrality.

(a) Find the Strömgren radius r.

(b) The recombination coefficient is a function of the temperature of the hydrogen plasma. For temperatures characteristic of Galactic H II regions, $\alpha \approx 3 \times 10^{-13}\mathrm{cm}^3\mathrm{s}^{-1}$. Assume $n_\mathrm{e} = 10\ \mathrm{cm}^{-3}$; compute r when $N_\star = 3 \times 10^{49}\ \mathrm{s}^{-1}$ (O5 V star), $N_\star = 4 \times 10^{46}\ \mathrm{s}^{-1}$ (B0 V star), $N_\star = 10^{39}\ \mathrm{s}^{-1}$ (G2 V star). Convert your answer to light-years. What types of main sequence stars have appreciable H II regions?

3. We have proved a virial theorem of form $2U + W = 0$, where U and W are the thermal and self gravitational energies of the gas mass. For a gaseous sphere which is partially confined by an external pressure P_ext, the right hand side of the corresponding equation is not zero, but equal to the radius r times the product of the surface area $4\pi r^2$ and the external pressure P_ext.

$$2U + W = 4\pi r^3 P_\mathrm{ext}$$

For and isothermal non-degenerate gas, $U = 3NkT/2$, where N equals the total number of particles in the cloud. For a total mass M and a mean molecular weight m, $N = M/m$. The self gravitational energy of the cloud can be expected to be of the form $W = -\alpha GM^2/r$, where α is a number of order unity whose value depends on how concentrated the center of the globule is.

(a) Eliminate r in favor of $V = 4\pi r^3/3$ and show that the virial theorem can be written

$$P_\mathrm{ext} = \frac{MkT}{mV} - \frac{\beta GM^2}{V^{4/3}}$$

where $\beta \equiv (4\pi/3)^{1/3}\alpha/3$. Further define the non-dimensional volume $v = V(GMm/kT)^{-3}$ and the non-dimensional pressure $p = P_\mathrm{ext}[G^3M^2(m/kT)^4]$, and show that the above equation becomes

$$p = \frac{1}{v} - \frac{\beta}{v^{4/3}}.$$

(b) Plot p versus v under the assumption $\beta = 0.45$, and show that p has a maximum value of 1.1 at $v = 0.22$.

(c) The external pressure on a globule in an H II region might be on the order $P_\mathrm{ext} = 10^{-16}\mathrm{N/m}^2$. What is the critical mass M required to produce gravitational collapse in a globule of temperature $T = 10$ K under these circumstances? What is the radius r of such a globule? Assume the globule to be made of pure molecular hydrogen.

4. A star of luminosity L at a distance r_0 will have an apparent brightness $f_0 = L/4\pi r_0^2$. All stars of the same luminosity but with distances $r < r_0$ will have apparent brightnesses greater than f_0. Suppose that the number

density of stars of luminosity L, $n(L, S, \Omega, r)$, where S is some stellar attribute, Ω the solid angle, and r the distance, does not depend on r, or Ω. Show that the number of stars of intrinsic brightness L which have $f > f_0$ is given by

$$N_L(f > f_0) = n(L)4\pi r_0^3/3 = \frac{n(L)L^{3/2}}{3(4\pi)^{1/2}} f_0^{-3/2}$$

Show that the amount of light in an infinite universe of constant stellar number density and no interstellar extinction arriving at Earth from a cone of solid angle Ω diverges exponentially as r tends to infinity (or equivalently $f_0 \to 0$).

5. Approximately how many times has the Sun circled the center of the Galaxy since the star's formation?

6. (a) From the data given in table 1 of the lecture notes, and using a typical temperature of 15 K for hydrogen in the interstellar medium, estimate the thermal energy density of hydrogen gas in the disk of the Galaxy. For this problem assume that the disk has a radius of 8.0 kpc and a height of 160 pc.

   (b) A typical magnetic field strength in the Galaxy's spiral arms is 4×10^{-10} T. Using

   $$E_m = \frac{B^2}{2\mu_0}$$

   estimate the energy density of the magnetic field in the spiral arms. Compare your answer with the thermal energy density of the gas. Would you expect the magnetic field to play a significant role in the structure of the Galaxy?