## Exercises for AST1100, Cosmology and the Big Bang

On large scales the Universe is homogenous and isotropic and we may describe its evolution by considering a spherical surface of sufficient size centered on any origin. The total energy of this surface, mass $m$, radius $R_s$ may be written

$$\frac{1}{2}mv^2 - \frac{GM_s m}{R_s} = E \qquad (1)$$

where $M_s$ is the mass inside the spherical surface.

1. Compute the mass inside the spherical surface as a function of radius $R_s$ and density $\rho$.

2. In writing the above Newtonian expression in relativistic form we write the total energy
$$E \equiv -mr_s^2\kappa c^2/2R_0^2,$$
where we have introduced the comoving coordinate $r_s = R_s/a(t)$. Explain the meaning of the various symbols in these expressions.

3. Use Hubble's law to write the Hubble parameter as a function of the scale factor $a$ and its time derivative.

4. Show that the equation (1) can be written
$$(H^2 - \frac{8\pi G}{3}\rho) = -\frac{\kappa c^2}{R_0^2 a(t)^2}. \qquad (2)$$

5. What is the critical density $\rho_c = 3H^2/8\pi G$? What characterizes Universes with $\kappa < 0$ og $\kappa > 0$?

6. Given $H_0 = 70$ km/s/Mpc, what is the critical density of today's Universe? What energy density does this correspond to?

7. The cosmic microwave background has a temperature $T_{\rm cmb} = 2.725$ K. The average baryon density is $n_{\rm bary,0} = 0.23$ m$^{-3}$. What energy density of these components of the Universe? How do these compare with the critical energy density? Compute the number density of photons in the CMB today. Compute the value of baryon-to-photon ratio $\eta$.

8. According to Einstein's general theory of relativity we must include all forms of energy when we measure or compute the Universe's density. What forms of matter/energy have we ignored when only considering the baryon density and the CMB? Formulate Friedmann's equation with terms included.

9. Photons and other relativistic particles are "stretched out" with the expansion of the universe such that

$$1 + z = \frac{\lambda_1}{\lambda_2} = \frac{a(t_1)}{a(t_2)} \tag{3}$$

where $z$ is the red-shift and $\lambda_{1,2}$ are the wavelengths at times $t_1$ and $t_2$. Explain with the help of Wiens law $\lambda_{\max} T = 0.0029$ mK, the connection between the temperature of the CMB, its red-shift and the expansion of the Universe.

10. Explain why the energy density contained in radiation was much more important than cold matter early in the history of the Universe.

11. Using $\Omega_{\rm m,0} = 0.3$ and $\Omega_{\rm r,0} = 8.4 \times 10^{-5}$ compute at what red-shift these components where equal. What was the expansion factor at this time? Why don't the given matter and radiation fractions correspond to the energy density you computed for photons and baryons?

12. Measurements of the size of the largest fluctuations in the CMB indicate that the Universe is flat ($\kappa = 0$). Use this, along with the convention that $a(t = t_0) = 1$ where $t_0$ is today, to show that the how the Friedmann equation 2 predicts how a matter dominated Universe will expand with time.

13. Compute the age of a matter dominated Universe $t_0$ when the Hubble constant is $H_0 = 70$ km/s/Mpc.

14. In its earliest phases the Universe was radiation dominated. What is the relation between time $t$ and the expansion factor $a$ in the Universe's early phases assuming the Universe is flat? What is the relation between the energy density and time? Temperature and time?

15. Consider now the case where the Universe is dominated by a cosmological constant $\Lambda$. What is the relation between the expansion factor and time in this Universe?

16. Measurements show that is likely that $\Omega_{\Lambda,0} \approx 0.7$ today. How long will it take before the average distance between distant galaxies has grown by a factor $e$?