Før du setter i gang, les følgende regler nøye:

Første frist for innlevering er fredag 9/11 klokken 17:00 (email eller postkassen min på astrofysisk, eller til gruppelærere). Obliger som leveres **før** dette tidspunktet vil jeg rette i helgen og gi en ny sjans hvis den ikke blir godkjent. OBS!OBS! Skriv emailadressen din på obligen slik at jeg kan kontakte deg hvis den ikke blir godkjent.

ABSOLUTT SISTE FRIST for innlevering er mandag 19/11 klokken 12:00. **Obliger som mottaes etter dette kan ikke bli evaluert og !!!De som leverer oblig til denne fristen og ikke får godkjent får ingen ny sjanse, dette er absolutt siste frist og sist sjanse!!!.**. Hvis noen skulle ha en god grunn til ikke å levere innen 19/11 må jeg ha beskjed **i god tid før 19/11**. Da administrasjonen må ha resultater av obligene i god tid før eksamen blir det kun mulig med en meget kort utsettelse for de som har god grunn til det og som ber om slik utsettelse i god tid før fristen 19/11.

Jeg anbefaler alle å gjøre alle oppgavene da dette er meget god eksamenstrening. Men kravene for å få godkjent er som følger:

- For de som fikk **godkjent +** på siste oblig: For dere er det nok å gjøre **to av de tre** kapitlene 2, 3, 4. For de som velger kapittel 2 som en av de to kapitlene er det nok å gjøre oppgave 1 til og med 13 samt oppgave 17 og 18 i kapittel 2. (Altså **enten** kap.2 (oppg.1 tom 13, 17, 18) samt kap.3 eller 4 **eller** bare kapittel 3 og 4.) Dere bør ha riktig tenktemåte/fremgangsmåte på de fleste av de oppgavene dere må gjøre.
- For de som fikk **godkjent** på siste oblig: Hele kapittel 2 må gjøres. I tillegg må enten hele kapittel 3 samt 3 valgfrie deloppgaver i kapittel 4 (oppg. 4.3 (a) regnes f.eks. som en deloppgave) **eller** hele kapittel 4 samt 3 valgfrie deloppgaver i kapittel 3 gjøres. Dere bør ha riktig tenktemåte/fremgangsmåte på de fleste av de oppgavene dere må gjøre.
- For de som fikk **godkjent -** på siste oblig: ALLE kapitlene og oppgavene må gjøres og de aller fleste med riktig fremgangsmåte/tenkemåte.

Begynn med obligen i dag, ikke vent, den absolutte fristen nærmer seg fort!!!

AST1100 Compulsory exercises, set 2

Must be received before noon (12:00), on the 19/11

During these exercises you may always assume that you know these numbers:

1 AU $= 150 \times 10^6 km$
Mass of Earth $= 6 \times 10^{24} kg$
Radius of Earth $= 6350 km$
Mass of Sun $= 2 \times 10^{30} kg$
Radius of Sun $= 696000 km$
Wavelength ranges for light : red 630-700nm, orange 590-630nm, yellow 560-590nm, green 490-560nm, blue 450-490nm, violet 400-450nm.

(this is the version for females!)

1 Chapter 1

Some theories predict the speed of light c to change with time. Here we will assume that a crazy version of these models turned out to be true. In this model, the speed of light is linked to the rate of expansion of the universe. At the moment the expansion of the universe is accelerating due to the presence of so-called dark energy (more about this in the lectures on cosmology). According to our crazy theory, the speed of light would fall rapidly when the expansion factor reaches a certain value. This story is about the night when this happened...

2 Chapter 2

You woke up at 7:00 in the morning and went for some breakfast. From the kitchen window you saw Mr. Early, one of your colleagues, passing the house in his car. The wall clock in your kitchen showed 7:30 as it always did when Mr. Early passed. You saw him looking a little bit confused and worried, but then he stared into your kitchen and looked at your kitchen clock. After that he seemed relaxed. The road to work was straight without any curves. The speed limit was 100 km/h, but as there were so many police controls in this area, everybody went with a velocity of exactly $v_0 = 99$ km/h. So did Mr. Early. You thought a little but about why he looked so worried in the beginning, but as you both hated each other you thought that it was just the discomfort of passing your house.

You said goodbye to your husband and went to your car. You felt that something weren't like it used to be this morning. You couldn't tell what it was, but you just had this weird feeling that things looked different. You assumed that you still weren't properly awake. You started the car, accelerated and immediately reached the usual speed of 99 km/h. At this moment you looked at your wristwatch and found it to be wrong. The kitchen clock showed 8:00 and you set your watch to 8:00 as well (this was the moment when you had already reached the speed of $v_0 = 99$ km/h and just left your house).

1. As usual you passed the city hall which was located 10 km from your house. You thought it went very fast and actually looking at your wristwatch it showed 8:00:51. You was a bit shocked, usually it took 6 minutes to reach the city hall.....you didn't understand that you could have been driving that fast. Then you looked at the clock at the city hall which showed 8:06:04 and you relaxed.....'I really need to buy a new wristwatch' you expressed. You were unaware of it at this time, but now we know better, the speed of light has changed. We will now try to find out what the speed of light was now. To do so we need to use conventional units, meaning that we measure (for the moment) distance in meters and time in seconds. Remember that in this case, the spacetime interval for Lorentz geometry can be written

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2.$$

Use the invariance of the spacetime interval between two events, the event that you left your house and the event that you pass the city hall, to show that the speed of light is currently 100 km/h.

2. From now on we will go back to the usual habit of measuring distance and time in the same units so that $c = 1$. Only when we want the final result we convert to 'normal' units, meters for distance and seconds for time. Check that you remember how to do the conversion. We will also measure energy and momentum in units of mass. Remember that the conversion factor is c^2 to go from units of mass to units of J and c to go from units of mass to normal units of momentum. On the way to work, you see that there has been a traffic accident. You see that it is the car of Mr. Early. A bit further down the street you see the body of Mr. Early. The ambulance was already there, so you continued. You looked at the situation and tried to imagine what had happened. It looked like he had hit a car coming directly towards him also with the speed of $v_0 = -99$ km/h measured from the ground. We will now use four-vectors to find the velocity v_0' that Mr. Early measured the approaching car to have. Write an expression for the four-velocity (in the frame of reference of the ground) of each of the cars expressed in v_0 and $\gamma_0 = 1/\sqrt{1 - v_0^2}$.

3. Construct a four-vector in the frame of the ground which we call V_μ which is simply
$$V_\mu = V_\mu(\text{car1}) + V_\mu(\text{car2})$$
the sum of the four-vectors of the two cars. Show that the scalar product of this four-vector with itself is
$$V_\mu V^\mu = \frac{4}{1 - v_0^2}$$
and explain what it means that the result of a scalar product is *scalar*.

4. In the frame of reference of Mr. Early just before the collision, write the four-velocities of each of the cars expressed in terms of the velocity v_0' (v_0' is the velocity of the approaching car as measured from the frame of reference of Mr. Early).

5. Again construct a four vector V_μ' which is simply the sum of the two four-vectors in the frame of Mr. Early. Show that
$$V_\mu'(V')^\mu = 2(1 + \gamma'),$$

where $\gamma' = 1/\sqrt{1-(v_0')^2}$.

6. Use the properties of scalars to show that

$$v_0' = \frac{2v_0}{1+v_0^2}.$$

   We have just seen the power of the scalars and why we always try to form scalar products with four-vectors in order to solve problems. In this case however, there was a simpler way to arrive at this answer using properties of four-vectors, can you see how?

7. It seemed from the traces that the two cars got attached to each other in the collision and therefore moved together as one big object with a velocity U. Mr. Early went through the window and continued forwards in the the same direction as his car had before the collision. The mass of Mr. Early was m and the velocity with which he went out of the window was v. The approaching car had mass M and the car of Mr. Early had twice the mass of the approaching car. Write expressions for the four-momenta of the two cars before and after the collision $P_\mu^{\text{before},1}$, $P_\mu^{\text{before},2}$, $P_\mu^{\text{after},1}$ and $P_\mu^{\text{after},2}$and of Mr. Early after the collision P_μ^{Early} in the frame of reference of the ground.

8. We will now try to find the velocity v with which Mr. Early went through the window. We will do the calculation step by step

   (a) Use conservation of four-momentum to find a set of two equations with two unknowns, U and v.

   (b) Convince yourself that this set of equations is really dirty, it will involve solving several second order equations and you do abolutely not feel like doing it. This is a problem often encountered in research: You are looking for a simple number which doesn't actually interest you that much and you find that you need to do a lot of work to get out this tiny stupid number which you are not even sure you will need for the further work anyway. What do you do? A solution which often works very well and often gives you a very good approximation to the stupid little number you are looking for is the following: Use intuition and see if it is possible to find an approximation which will make this set of equations a lot easier to solve. You do not need to be sure whether this is a good approximation or not, we will check this when we have the final results. This also works often on exams if you do not know how to solve a problem exactly. In this way you might get an approximate answer. Here I would think in the following way: Mr. Early has very little mass compared to the cars. Also, the total momentum in the forward direction is a lot larger than in the opposite direction. My guess is that Mr. Early will get a very high speed such that $v \approx 1$. Make sure that you understand that this does NOT always mean that $1 - v^2 \approx 0$. With this in mind, subtract the two equations you have from each other and you find that you have a first order equation only in U (you might need some reorganizing to realize that you only have U to the first order here, not to the second). Solve this easy equation to find an expression for U.

(c) Time has come to insert some numbers. The weight $M = 1000$kg, the weight of Mr. Early was 70 kg. Find a number for U. In which direction did the cars move after the collision?

(d) Knowing U, you can find a number for $1/\sqrt{1 - U^2}$ as well. And then you can use one of your original equations to find a number for v. What is the velocity v?

(e) Hold on, stop, we're not done yet. We made an assumption, remember? The results we have got are only correct if the assumption was correct. To check this, insert your numbers for U and v on the left and right hand side in your two equations. Do you get the same number on either side in both equations? If so, we're done.

9. The distance to your work is 100 km. Just before arriving, you look out of the window of your car and at the large clock at the building of the company where you work. What time is it?

10. Then you look at your wristwatch. What time does it show? (use invariance of the spacetime interval or simply the formula for time dilution, your choice)

11. At this moment you understood what was going on. You remembered from the time you studied at the university from compulsory exer-

cises in AST1100 that these things could happen if the speed of light changed. So you concluded that in this night the speed of light had changed to 100 km/h and this was the reason for all the strange things happening. Your colleague Mrs. Late lives further away from work but along the same straight road. Mrs. Late drove, as everybody else, with a velocity of 99 km/h and she always arrived later than you did. When Mrs. Late passed your house she always waved her hands and made a (suspiciously) large smile to your husband. You had been suspecting something between Mrs. Late and your husband for a long time and you always wondered whether she arrived late at work because she used to stop at your place for a while on her way to work. At the same time as you arrived at work (in your frame of reference while your car was still moving at 99 km/h), Mrs. Late passed your house. What time did she read on the kitchen clock? Use invariance of the spacetime interval: Set up the position and time of the two events (you looking at the company clock and Mrs. Late looking at your kitchen clock) in both frames of reference.

12. Now make the previous calculation in a different way: Assume that the frame of reference of the moving car is the laboratory frame and that your house with the kitchen clock is the moving frame. Use the formula for time dilution to find the time that Mrs. Late reads on your kitchen clock. Check that you get the same answer as in the previous question.

13. If you have done previous calculations correct, the following is the case: In your frame of reference you arrive at work at the same time as Mrs. Late passes you house. You look at the clock outside your company and finds that it shows a different time than Mrs. Late reads on the kitchen clock in your house, even thought the two of you looked at these clocks at the same time (seen from you frame). The kitchen clock and the company clock are synchronized and are both at rest in the ground frame. Explain this apparent discrepancy.

14. When Mrs. Late arrived at work you said:

    'You are late today. You didn't stop somewhere on the way, did you?'

    Mrs. Late ignored your question and started telling you:

'As I passed your house on the way, I found that my watch was running very wrong, but I know that your kitchen clock is alway correct so I set my clock to the time that I read on your kitchen clock in the moment I passed your house'.

You looked at the wristwatch of Mrs. Late at the moment when she arrived at work. It showed 9:07. You thought a little bit....it would normally take about one hour to drive 100 km with a velocity of 100 km/h but today....and then you suddenly hit her in the face. Why? (the reason might not be what you think. Make some calculations and arguments to show what the wristwatch of Mrs. Late should have shown if she was telling the truth)

15. We found above that at the same time as you arrived at work (in your car-frame) your kitchen clock showed 8:01:12. So apparently, measured on your kitchen clock, it took you 1 minute and 12 seconds to arrive at work. When you drive from work the big clock on the company building shows 16:00. When you arrived at home (still sitting in the moving car) what time did the kitchen clock show? (This question is a trap: make the calculation both with the ground as the lab frame and with your car as the lab frame and make sure that you obtain the same result!)

16. What did your wristwatch show? (assume you set your wristwatch to 16:00 when you left work)

17. After arriving home, you had an argument with your husband about Mrs. Late. In the end you were so angry that you decided to leave him. You went directly out to your car, set off in the direction of your job. It was evening and you now had the Sun in front of you. Desperate and angry you accelerated and accelerated. Finally you reached a speed of 99.9999 km/h. This was fatal.......why? **hint:** Use an expression you derived in some exercises a few weeks ago to find the wavelength of photons arriving from the Sun in the frame of reference of your car.

18. Actually, this story should have had a tragic ending much much earlier. But, as we have learned from numerous cartoons, if you run outside a cliff and don't notice it, you can carry on for a long time. It is only when you discover that there is no ground below you, that you start falling. This principle can be carried over to the theory of relativity

without modifications. We will now see that if you had started doing some calculations from general relativity, the story would have went really bad already from the beginning. But fortunately you didn't, so it all went well until now...Your husband found out that he needed to teach Mrs. Late some theory of relativity so she didn't do any more stupid mistakes. He wanted to give her an exercise to calculate her Schwarzschild coordinate r, but this had a really tragic result. The circumference of the Earth is about 40000km. What is the Schwarzschild coordinate of the surface of the Earth measured in Earth masses? (remember the new speed of light when converting from meters to kg!) Why did the story end even worse now?

3 Chapter 3

Next night the constants of nature really went crazy. The speed of light changed back to its original value, but the gravitational constant had changed such that now $G = 3.15 \times 10^{-6}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$. This made the gravitational force a lot stronger, but by coincidence also some other constants of nature had changed so that for instance your muscular forces and binding forces between molecules were stronger. For this reason you did not feel your increased weight due to a higher gravitational constant. When doing general relativity you usually measure mass in meters and the constant converting from mass measured in kilograms to mass measured in meters is G/c^2

$$M(\mathrm{m}) = M(\mathrm{kg})\frac{G}{c^2}$$

Thus, the change in gravitational constant changed the ratio M/r appearing in the Schwarzschild line element.

You woke up the next morning, still remembering what had happened last night. Actually you wondered why you were still alive. You clearly remember something bad happening when you started to accelerate your car. Was everything just a dream? Going for a coffee your husband was there like always, he didn't say anything about Mrs. Late. Maybe it was all a dream. Then you looked out of the window and thought that it was a really nice sunrise although it was a little bit late for the sunrise you thought.

1. The Sun appears yellow because most photons leaving the surface of the Sun has a wavelength of about 580nm. What is the frequency and

wavelength of these photons at a large distance from the Sun $r \to \infty$? Use the Schwarzschild line element to argue and remember that G has changed! **hint:** The circumference of the Sun is 4.4 million km.

2. We assume that the Earth is so far away from the Sun so that we can use the previous result for wavelength and frequency of the photons when they arrive at Earth. When the photons arrive at the Earth assume that they have arrived from very far away with these numbers for wavelength and frequency. With what frequency and wavelength do we observe the photons from the Sun? In which color do you see the Sun (check table at the beginning) ? **hint:** The circumference of the Earth is about 40 000km.

3. You drive to work and find it reassuring that the clocks outside and your wristwatch seem to show the same time. You think that it was all a dream and that everything was normal. During this day, Mrs. Late was sent on a business trip to Australia. The airplane kept an altitude of $\Delta r_{\mathrm{shell}} = 10000$ m as measured locally by Earth observers. What was the coordinate r of the airplane measured in Schwarzschild coordinates? (**hint1:** The circumference of the Earth is about 40 000km, **hint2:** You need the mass of the Earth to find it).

4. If you did not manage to find an answer in the previous exercise, you can use $\Delta r = 10000m$ (altitude in Schwarzschild coordinates) in the following. The total one-way flight time was 20 hours. When Mrs. Late came back from Australia a few days later, you compared the time on your wristwatches. Not knowing anything about what had happened with the constants of nature you realized that one of your clocks was running slow. Which one?

5. Use the Schwarzschild line element to find a general expression for the time difference $\Delta t^{\mathrm{Earth}}/\Delta t^{\mathrm{Airplane}}$ between your clock and the clock of Mrs. Late expressed in terms of the mass M of the Earth, Schwarzschild coordinate r_{Earth} of the surface of the Earth as well as r_{Airplane} for the airplane, as well as the tangential velocities v_{Earth} and v_{Airplane} of you and the airplane with respect to the center of the Earth. Assume that Mrs. Late was in the airplane at a constant height with a constant velocity for the total duration of the flight.

6. Show how you can make a fast and easy estimate of whether the special relativistic or the general relativistic effect will be most important here. Which is most important? **hint1:** Assume the typical speed of an airplane is about 1000 km/h.**hint2:** If you use the correct expressions your computer might have some problems giving you a correct numerical answer. However if you use approximations (from the weekly exercises) for small effects, it will be easier to get a numerical answer)

7. Now, take only into account the effect which you found to be most important and find the time difference between your clocks in seconds. Even with such a huge increase in the gravitational constant, would you have noticed the effect gravitation has on time ?**hint:** Same hint as in the previous question, it is much easier to use an approximation when inserting the number in a computer.

4 Chapter 4

One night, the change in the gravitational constant got even larger by a factor 10. It was now $G = 3.15 \times 10^{-5}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$.

1. Calculate the Schwarzschild coordinate of the surface of the Sun in terms of the solar mass. The circumference of the Sun is 4.4×10^6km. What has happened to the Sun?

2. What is the Schwarzschild coordinate r of the orbit of the Earth, measured in solar masses? **hint:** The circumference of the orbit is about 6 AU.

3. People on Earth notice that something strange has suddenly happened to the Sun. Astronauts prepare to go in orbit around the Sun to study it closer. The rocket is launched towards the Sun at an angle $\theta = 179°$ with the vector pointing from the Sun to the Earth (see the figure). The velocity of the rocket is $v_{\text{shell}} = 0.99$. All the fuel of the rocket is used at launch. Afterwards the rocket is floating freely.

   (a) Use the general relativistic expression for angular momentum to show that angular momentum per mass of the rocket is

$$\frac{L}{m} = r\gamma_{\text{shell}}v_{\text{shell}} \sin\theta,$$

where r is the Schwarzschild coordinate of the point where the rocket is launched, $\gamma_{sh} = 1/\sqrt{1 - v_{\text{shell}}^2}$ and m is the mass of the rocket.

(b) Use the general relativistic expression for energy to show that energy per mass of the rocket is

$$\frac{E}{m} = \sqrt{1 - \frac{2M}{r}}\gamma_{\text{shell}}.$$

(c) Insert numbers in these expressions to obtain numbers for L/m (in units of solar masses $M_\odot$) and E/m.

(d) Make a computer code to plot the orbit of the rocket all the way to $r = 2M$. Check a similar exercise in the weekly exercises given earlier. Attach a plot of the rocket's trajectory. On the same plot, show the Schwarzschild radius. If you already did this exercise, you just need to use the same code again with different numbers. **hint: Use about 1000 steps with $\Delta\tau = 0.1M_\odot$.**

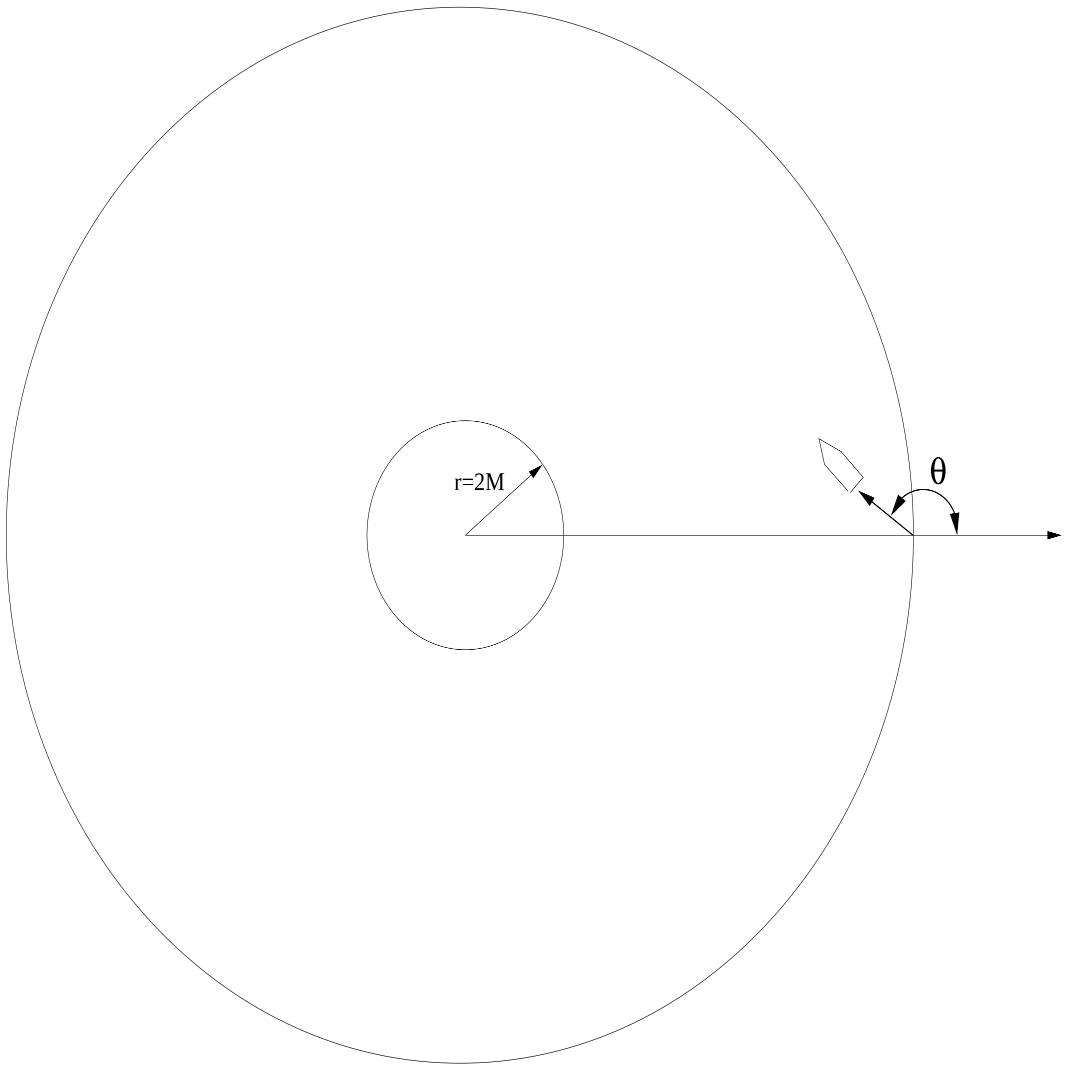

Figure 1: Rocket launched inwards at an angle θ. Note: Figure not to scale.