Solution to the twin paradox

Magnus Axelsson

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Problem 1.1

In the Earth frame (which we will refer to as the Laboratory system) the spaceship travels from Earth to Rigel a distance of 800 light years which takes $\Delta t = 800/0.99995 = 800.04$ years. Due to time dilation the same trip takes a factor of γ less in the spaceship frame. Thus I measure that the time on my wristwatch is

$$\Delta t' = \frac{\Delta t}{\gamma} = \Delta t\sqrt{1-v^2} = 800.04\,\text{yrs}\,\sqrt{1-(0.99995)^2} = 8\,\text{yrs} \tag{1}$$

(where v is the speed of the spaceship measured in the Laboratory frame).

Problem 1.2

Since we are ignoring the time it takes to perform measurements (this might take a day or so, which is completely negligent compared to the time it took to travel to Rigel). Due to symmetry considerations we therefore conclude that the return trip to Earth from Rigel takes the same time as the trip to Rigel from Earth. This conclusion is valid in both the Laboratory and the spaceship frame.

Problem 1.3

Summary: The trip takes a total of 1600.08 years in the Earth frame. In the spaceship my wristwatch measures that the same trip took 16 years.

Problem 1.4

We now switch coordinate systems and consider the spaceship frame as the Laboratory system with coordinates (x, t), and the Earth frame as the moving system with coordinates (x', t'). In the spaceship the trip took $\Delta t = 8$ years. By virtue of the Lorentz transformations we now find that an observer in the Earth frame measures the trip to take

$$\Delta t' = \frac{\Delta t}{\gamma} = \frac{8\text{yrs}}{100} = 0.08\,\text{yrs} \tag{2}$$

An observer in the Earth frame finds that Rigel arrives at the spaceship's position after 0.08 years!

Problem 1.5

As in section **2** we appeal to symmetry. Therefore the 'trip' takes 16 years in the spaceship frame, and 0.16 years in the Earth frame.

Problem 1.6

It appears to be a paradox, but we will shortly find that really isn't! Do you know why this isn't really a paradox? What have we forgotten?

Problem 1.7

The previous calculations are not all valid and this is because we have violated one of the first principles of **special** relativity! The principle of relativity states that

> The laws of physics are equal in all **inertial** reference frames.

For the spaceship to attain a speed comparable to the speed of light it must accelerate, thus introducing fictional forces. Therefore the spaceship frame is no longer inertial. Since we are constantly changing frame of reference the Lorentz transformations (which are deduced for constant speed) are no longer valid thus making our previous calculations invalid.

Problem 1.8

To rectify the situation we introduce three different frames of reference (see figure in the problem text).
The Earth frame (Laboratory system), with coordinates (x, t),
The outgoing elevator (spaceship) frame, coordinates (x', t'),
The returning elevator frame, coordinates (x'', t'').
We define event A to be the departure of the spaceship from Earth. We then have:

$$(x_A, t_A) = (0, 0) \quad \text{and} \quad (x'_A, t'_A) = (0, 0) \tag{3}$$

Event B is me in the spaceship arriving at Rigel a distance L_0 from Earth:

$$(x_B, t_B) = (L_0, L_0/v) \quad \text{and} \quad (x'_B, t'_B) = (0, t'_B) \tag{4}$$

From these calculations we see that $t_B = L_0/v$.

Problem 1.9

We use the Lorentz transformation to obtain

$$t'_B = \frac{t_B}{\gamma} = \frac{L_0}{v}\sqrt{1 - v^2} = 8\,\text{yrs} \tag{5}$$

In the spaceship we still find that the trip to Rigel takes 8 years.

Problem 1.10

Event B' is defined as the event when an observer in another box in the same elevator looks out of his window and checks what time the clocks on Earth show. It is assumed that this observer has a clock which is synchronized with mine and hence $t'_{B'} = t'_B$ (where t'_B is the time my clock measures when I arrive at Rigel. For event B we thus have

$$(x_{B'}, t_{B'}) = (0, t_{B'}) \tag{6}$$
$$(x'_{B'}, t'_{B'}) = (x'_{B'}, t'_B) \tag{7}$$

Invariance of the line element between event A (where all coordinates are zero in both reference frames) and B' gives

$$\Delta s^2_{AB'} = \Delta s'^2_{AB'} \Longleftrightarrow t^2_{B'} = t'^2_B - x'^2_{B'} \tag{8}$$

with the result

$$t_{B'} = \sqrt{t'^2_B - x'^2_{B'}} \tag{9}$$

In the previous section we found an expression for t'_B so it remains to find an expression for $x'_{B'}$. This is not so hard. The observer in the outgoing elevator moving with speed v finds that the distance between Earth and Rigel is length contracted by a factor of γ. Thus

$$x'_{B'} = -\frac{L_0}{\gamma} = -L_0\sqrt{1-v^2} \tag{10}$$

The minus sign arises since I have chosen the positive x-axis to point from Earth to Rigel, but this does not matter since we need the square of this quantity. If we insert these factors we find

$$\begin{aligned} t_{B'} &= \sqrt{(L_0/v)^2(1-v^2) - L_0^2(1-v^2)} = \sqrt{(L_0/v)^2(1-v^2)^2} \\ &= (L_0/v)(1-v^2) = \frac{L_0}{v} - vL_0 \end{aligned} \tag{11}$$

which was to be shown.

Problem 1.11

If we insert $L_0 = 800\,\text{lyrs}$ and $v = 0.99995$ we find $t_{B'} = 0.08$ years. The observations are not simultaneous.

Problem 1.12

Event D is defined as when person P in the returning elevator frame (coordinates (x'', t'')) starts his journey towards Rigel a distance $2L_0$ away from Earth (seen from the Earth frame). Then $(x''_D, t''_D) = (0, 0)$. We want to know what time person P's clocks show when he arrives at Rigel. Since he arrives at Rigel at the same time as I do in my outgoing elevator (event B) we examine the line elements between events B and D. We have

$$\begin{aligned} \Delta s^2_{BD} &= \Delta t^2_{BD} - \Delta x^2_{BD} = (t_B - t_D)^2 - (x_B - x_D)^2 \\ &= (L_0/v - 0)^2 - (L_0 - 2L_0)^2 = (L_0/v)^2 - L_0^2 \end{aligned} \tag{12}$$

where we have used the fact that event D is simultaneous with event A in the Earth frame. For the line element in the returning elevator frame we find

$$\Delta s''^2_{BD} = \Delta t''^2_{BD} - \Delta x''^2_{BD} = (t''_B - t''_D)^2 - (x''_B - x''_D)^2 = t''^2_B \tag{13}$$

where we have used $x''_B = 0$ since we are measuring distances relative to the returning elevator frame (you should make sure you understand why this is so). Invariance then gives

$$t''^2_B = (L_0/v)^2 - L_0^2 \Rightarrow t''_B = (L_0/v)\sqrt{1-v^2} = \frac{L_0}{v\gamma} \tag{14}$$

Problem 1.13

An observer in the returning elevator frame at Earth's postion $x = 0$ looks out of his window and checks what time the Earth clocks show. This happens simultaneously with person P arriving at Rigel, in Person P's reference frame. Again we assume that person P and the observer have synchronized clocks. We label the observation of the Earth clocks event B". We will now look at the space and time intervals between events B" and D.
In the Earth system:

$$\begin{aligned} t_D &= t_A = 0, \quad t_{B''} = t_{B''} \\ &\Rightarrow \Delta t_{DB''} = t_{B''} \\ x_D &= 2L_0, \quad x_{B''} = 0 \\ &\Rightarrow \Delta x_{DB''} = 2L_0 \end{aligned} \tag{15}$$

In the returning elevator frame:

$$\begin{aligned} t''_D &= 0, \quad t''_{B''} = t''_B = L_0/(v\gamma) \\ &\Rightarrow \Delta t''_{DB''} = L_0/(v\gamma) \end{aligned} \tag{16}$$

$$\begin{aligned} x''_D &= 0, \quad x''_{B''} = L_0/\gamma \\ &\Rightarrow \Delta x''_{DB''} = L_0/\gamma \end{aligned} \tag{17}$$

Here $x''_{B''}$ is the observers position (he is at Earth) relative to person P at Rigel. In the Earth frame this distance is L_0 but since he is moving with speed v the distance gets contracted by a factor of γ. We could also have included a minus sign, but I haven chosen the axis to have positive direction towards Earth for this calculation (this will not have any influence on the calculations anyway...)

Problem 1.14

We calculate the line element in both reference frames:

$$\begin{aligned} \Delta s^2_{DB''} &= \Delta t^2_{DB''} - \Delta x^2_{DB''} = t^2_{B''} - (2L_0)^2 \\ \Delta s''^2_{DB''} &= \Delta t''^2_{DB''} - \Delta x''^2_{DB''} = (L_0/(v\gamma))^2 - (L_0/\gamma)^2 \end{aligned}$$

Setting them equal we find

$$\begin{aligned} t^2_{B''} &= (2L_0)^2 + (L_0/(v\gamma))^2 - (L_0/\gamma)^2 \\ &= 4L_0^2 + (L_0/v)^2(1 - v^2) - L_0^2(1 - v^2) \\ &= 4L_0^2 + \frac{L_0^2}{v^2} - 2L_0^2 + L_0^2 v^2 \\ &= (L_0/v)^2 + 2L_0^2 + L_0^2 v^2 && (18) \\ &= \left(\frac{L_0}{v} + L_0 v\right)^2 && (19) \end{aligned}$$

And finally

$$t_{B''} = \frac{L_0}{v} + L_0 v \tag{20}$$

Inserting numbers we find $t_{B''} = 1600$ years. Surprised?