The following people have participated in creating these solutions: Nicolaas E. Groeneboom, Magnus Pedersen Lohne, Karl R. Leikanger *NOTE: There might be errors in the solution. If you find something which doens't look right, please let me know.*

Partial solutions to problems: Lecture 13-14

Problem 1

1. Full range given in following answer
2. Full range given in following answer
3. Using the HR-diagram (figure 1 in the lecture notes), the luminosity of a $G0$ star ranges between 0.8 and 60 L_{sun}, but these numbers are all approximate. In the same range, the absolute magnitude M would be between 2 and 6.
4. Recall that it is possible to decide the distance r to a star from the difference between apparent (m) and absolute (M) magnitude:

$$M - m = -5\log_{10}(\frac{r}{10pc})$$

solving for r gives

$$r = 10pc \cdot 10^{\frac{m-M}{5}}$$

With an apparent magnitude $m = 1$, we find that the range of distance for a $G0$ star becomes

$$r_{min} = 10pc \cdot 10^{\frac{1-6}{5}} = 1pc$$

$$r_{max} = 10pc \cdot 10^{\frac{1-2}{5}} = 6pc$$

which isn't very accurate.

Problem 2

1. For small values of θ, the diameter is given as $D = d \cdot \theta = 200pc \cdot 3.5' \approx 0.2pc$. The radius is then $d = D/2 = 0.1pc$.
2. The volume of a sphere is given as $V = \frac{4}{3}\pi r^3$, so assuming a uniformly distributed mass density ρ we obtain

$$M = \rho \cdot V = 3 \cdot 10^{-17} kg/m^3 \cdot \frac{4}{3}\pi(0.1pc)^3 \approx 3.62 \cdot 10^{30} kg$$

which approximately is 1.8 solar masses.

3. The mass of hydrogen is $m_h = 1.71 \cdot 10^{-27}$kg, while the mean molecular weight is assumed to be $\mu = 1$ (that is, there are only hydrogen atoms in the cloud). The Jean mass is defined as

$$M_J = \Big(\frac{5kT}{G\mu m_H}\Big)^{3/2} \Big(\frac{3}{4\pi\rho}\Big)^{1/2}$$

and describes the mass threshold for whether a molecular cloud will collapse to a more compact object ($M > M_J$) or not ($M < M_J$). Inserting the values (where T=10K), we obtain a Jeans mass of $M_J \approx 4.2 \cdot 10^{31}$kg, or 21 solar masses. This is more than the result obtained in 13.2.2, so this cloud will not collapse (alone) and form a protostar.

4. Recall that the condition for a cloud to collapse is that $2K < |U|$, where U is the potential energy and K kinetic energy. If a supernova in the vicinity contributes to compressing the gas, the gravitational attraction becomes stronger. This is because the mass density increases while the radius of the cloud decreases, thus U grows. But why would K on average not grow? Increasing the mass density should decrease the jeans mass ($M_J \propto \frac{1}{\sqrt{\rho}}$). It is therefore plausible that a supernova could contribute to the creation of protostars.

5. See the last answer.

6. The spiral shaped pressure wave will compress the gas at the tops of the wave and thus increase the probability for star birth in these areas.

Problem 3

1. The volume V of a sphere as function of radius r is given as $V(r) = \frac{4}{3}\pi r^3$. The total mass is the mass density times the volume, so

$$M(r) = \frac{4}{3}\pi r^3 \rho$$

if we assume ρ to be constant.

2. The hydrostatic equation reads

$$\frac{dP}{dr} = -\rho G \frac{M(r)}{r^2} = -\frac{4}{3}\pi G \rho^2 r$$

where the $M(r)$ from 13.3.1 was inserted. We start by fluffing around with differentials:

$$\frac{dP}{dr} = \frac{dP}{dr}\frac{dT}{dT} = \frac{dT}{dr}\frac{dP}{dT}$$

The pressure is given as $P = \rho kT/(\mu m_H)$. Then

$$\frac{dP}{dr} = \frac{dT}{dr}\frac{dP}{dT} = \frac{dT}{dr}\frac{d}{dT}\left(\frac{\rho kT}{\mu m_H}\right) = \frac{dT}{dr}\frac{\rho k}{\mu m_H}$$

Insert this expression into the hydrostatic equation and obtain

$$\frac{dT}{dr} = -\frac{4}{3}\pi G\rho r\frac{\mu m_H}{k} \tag{0.1}$$

3. We now integrate this solution from 0 to r. Letting

$$C = \pi G\frac{\mu m_H}{k}$$

equation 0.1 becomes

$$\frac{dT}{dr} = -\frac{4}{3}C\rho \cdot r$$

integrating with regards to r from 0 to R gives

$$T(R) - T_C = -\frac{4}{3}C\rho \cdot \int_0^R r = -\frac{2}{3}C\rho R^2$$

such that

$$T_C = \frac{2}{3}C\rho R^2 + T(R)$$

4. Assuming the Sun to be spherical with a homogeneous (homogeneous means that $\rho(\vec{x}) \equiv \rho_0$ is constant) density, the total mass is expressed as

$$M = V \cdot \rho = \frac{4}{3}\pi r^3 \cdot \rho$$

solving for ρ

$$\rho = M\frac{3}{4\pi R^3} \approx 1.4 \cdot 10^3 kg/m^3$$

We now use this ρ for estimating the core temperature of the sun:

$$T_C = T(R) + \frac{2}{3}R^2\pi G\rho\frac{\mu m_H}{k} \approx 11.5\,\text{million K}$$

where $R = 700\,000$km, k the Boltzmann-constant, $\mu = 1$ (assuming only protons populate the sun), $T(R) \approx 0$ as the surface temperature is way lower than the core temperature, m_H is the proton mass and G the gravitational constant. The "real" temperature when accounting for varying density ρ is $\sim$ 15 million K. Pretty hot, that is.

5. The *pp*-chain dominates as the core temperature $T_C < 20$ million K.

6. We already saw that
$$\rho = M\frac{3}{4\pi R^3}$$
inserting into
$$T_C = \frac{2}{3}C\rho R^2 + T(R)$$
we find
$$T_C = \frac{2}{3}CR^2M\frac{3}{4\pi R^3} \propto \frac{M}{R}$$

7. The temperature in the core T_C is proportional to
$$T_C \propto \frac{M}{R}$$
so if the temperature increases by a factor of 10, then for a constant mass M the radius has to be decreased by a factor 10.

8. This is a nice exercise, as one has to utilize all previous knowledge from this exercise. It is basically just a repetition of things already done, but with a different pressure P. Return to the fact that
$$\frac{dP}{dr} = -\rho\frac{GM}{r^2} = \frac{dP}{dT}\frac{dT}{dr} \tag{0.2}$$
where now $P = \frac{1}{3}aT^4$ is pure good old relativistic radiation pressure. Then
$$\frac{dP}{dT} = \frac{4}{3}aT^3$$
inserting this back into 0.2 to obtain
$$\frac{dT}{dr} = -\rho\frac{GM}{r^2}\left(\frac{dP}{dT}\right)^{-1} = -\rho\frac{GM}{r^2}\frac{3}{4aT^3}$$
Separating the r and T on each side, we obtain a separable differential equation:
$$T^3dT = -\rho GM\frac{3}{4a}\frac{1}{r^2} = -\frac{\pi G}{a}r\rho^2dr$$
where we used that the mass $M = \frac{4}{3}\pi r^3\rho$. Integrating both sides gives
$$\int_{T_C}^{T(R)} T^3dT = -\rho^2\frac{\pi G}{a}\int_0^R rdr$$
such that
$$\frac{1}{4}\Big(T_C^4 - T(R)^4\Big) = \rho^2\frac{\pi G}{2a}R^2$$
Solving for T_C alone gives
$$T_C^4 = T(R)^4 + \rho^2\frac{2\pi G}{a}R^2$$
Take the 4th root on both sides, and Voilà! We're done.

Problem 4

1. We are now given a variable (and much more realistic) mass density of a star which is dependent on r and the radius R:

$$\rho(r) = \frac{\rho_C}{1 + \left(\frac{r}{R}\right)^2}$$

The mass inside a spherical shell of radius r is given as $M = \int \rho \cdot dV$, where the volume element $dV = 4\pi r^2 dr$. Then

$$M(r) = \int_0^r \rho(r) 4\pi r^2 dr = 4\pi \int_0^r \frac{\rho_C r^2}{1 + \left(\frac{r}{R}\right)^2} dr$$

Substituting $x = r/R$ gives $r = xR$ and $dr = Rdx$, such that

$$M(r) = 4\pi \int_0^r \frac{\rho_C x^2 R^2}{1 + x^2} R dx = 4\pi \rho_C R^3 \int_0^x \frac{x^2}{1 + x^2} dx$$

Using the fact that

$$\int_0^x \frac{x^2}{1 + x^2} dx = x - \arctan x$$

the mass is expressed as

$$M(r) = 4\pi \rho_C R^3 \Big(\frac{r}{R} - \arctan \frac{r}{R}\Big)$$

2. The hydrostatic equilibrium is expressed as

$$\frac{dP}{dr} = -\rho(r) \frac{GM}{r^2} = -4\pi \frac{\rho_C}{1 + \left(\frac{r}{R}\right)^2} \rho_C R^3 \Big(\frac{r}{R} - \arctan \frac{r}{R}\Big) \frac{G}{r^2} \qquad (0.3)$$

We use the ideal gas law $P = \rho(r) k T(r)/(\mu m_H)$, and take the derivative with respect to r. Then

$$\frac{dP}{dr} = \frac{d}{dr}\Big(\frac{T(r)\rho(r)k}{\mu m_h}\Big)$$

Inserting this expression into 0.3 and move the constants μ, m_H and k to the right hand side yields

$$\frac{d}{dr}\Big(\rho(r)T(r)\Big) = -\frac{\mu m_H}{k} 4\pi \frac{\rho_C^2}{1 + \left(\frac{r}{R}\right)^2} R^3 \Big(\frac{r}{R} - \arctan \frac{r}{R}\Big) \frac{G}{r^2}$$

puh.

3. This is again a separable differential equation, so we separate the $r's$ and the $T's$ on each side:

$$\rho(r)T(r) - \rho_C T_C = -\int_0^r \frac{\mu m_H}{k} 4\pi \frac{\rho_C^2}{1+\left(\frac{r}{R}\right)^2} R^3 \left(\frac{r}{R} - \arctan\frac{r}{R}\right)\frac{G}{r^2} dr$$

Ni-ice. Now move the constants outside the integral:

$$\rho(r)T(r) - \rho_C T_C = -\left(\frac{\mu m_H}{k}\rho_C^2 4\pi R^3 G\right)\int_0^r \frac{1}{1+\left(\frac{r}{R}\right)^2}\left(\frac{r}{R} - \arctan\frac{r}{R}\right)\frac{1}{r^2} dr$$

and use the same substitution as in exercise 13.4.1:

$$\rho(r)T(r) - \rho_C T_C = -\left(\frac{\mu m_H}{k}\rho_C^2 4\pi R^2 G\right)\int_0^x \frac{1}{1+x^2}\left(x - \arctan x\right)\frac{1}{x^2} dx$$

where one of the R's in the denominator disappeared due to the change of variable. Including the $1/x^2$, we split the integral into two parts:

$$\rho(r)T(r) - \rho_C T_C = -\left(\frac{\mu m_H}{k}\rho_C^2 4\pi R^2 G\right)\int_0^x \left(\frac{1}{(1+x^2)x} - \frac{\arctan x}{(1+x^2)x^2}\right) dx \tag{0.4}$$

4. We magically use that

$$\int_0^x \frac{1}{x(x^2+1)} dx = \ln x - \frac{1}{2}\ln(x^2+1)$$

and

$$\int_0^x \frac{\arctan x}{x^2(x^2+1)} dx = -\frac{1}{2}(\arctan x)^2 - \frac{1}{x}\arctan x + \ln x - \frac{1}{2}\ln(x^2+1) + 1$$

The extra $+1$ has a curious origin: in the limit when $x \to \infty$, then by L'hôpital's rule, $\lim_{x\to 0} \arctan(x)/x = 1$. Inserting these two fellows into equation 0.4, the logarithmic parts luckily cancel (as $\ln_{x\to 0} x = -\infty$!). Then:

$$\rho(r)T(r) - \rho_C T_C = -\left(\frac{\mu m_H}{k}\rho_C^2 4\pi R^2 G\right)\left(\frac{1}{2}\left(\arctan(\frac{r}{R})\right)^2 + \frac{R}{r}\arctan\frac{r}{R} - 1\right)$$

Rearranging terms and dividing by ρ_C results in

$$T_C = \frac{\rho(r)}{\rho_C}T(r) + \left(\frac{\mu m_H}{k}\rho_C 4\pi R^2 G\right)\left(\frac{1}{2}\left(\arctan(\frac{r}{R})\right)^2 + \frac{R}{r}\arctan\frac{r}{R} - 1\right)$$

inserting for $\rho(r)$ gives

$$T_C = \frac{1}{1+\left(\frac{r}{R}\right)^2} T(r) + \Big(\frac{\mu m_H}{k}\rho_C 4\pi R^2 G\Big)\Big(\frac{1}{2}\big(\arctan(\frac{r}{R})\big)^2 + \frac{R}{r}\arctan\frac{r}{R} - 1\Big) \tag{0.5}$$

which is the end result.

5. What happens when the arctan's $r \propto x \to \infty$? From basic arithmetic's, we know that $\lim_{x\to\pi/2}\tan(x) = \infty$, so $\lim_{x\to\infty}\arctan x = \pi/2$. Inserting this into equation (0.5) one obtains

$$T_C = \Big(\frac{\mu m_H}{k}\rho_C 4\pi R^2 G\Big)\Big(\frac{1}{2}(\frac{\pi}{2})^2 - 1\Big)$$

where the 1st and 3rd terms disappear as $\lim_{x\to\infty} 1/(1+x^2) = 0$.

6. From

$$\rho(r) = \frac{\rho_C}{1+\left(\frac{r}{R}\right)^2}$$

it is easy to see that the density $\rho(r) = \frac{1}{2}\rho_C$ when $r = R$. We now need to decide what this R is. The core stops where $r = 0.2R_{sun}$, and at this point $\rho(r) = \frac{1}{10}\rho_C$. Then

$$\frac{1}{10}\rho_C = \frac{\rho_C}{1+\left(\frac{0.2R_{sun}}{R}\right)^2}$$

where R is the point that the density is halved. Inverting both sides and removing ρ_C yields

$$10 = 1 + (\frac{0.2R_{sun}}{R})^2$$

such that

$$\sqrt{9} = \frac{0.2R_{sun}}{R}$$

or

$$R = \frac{0.2R_{sun}}{3} \approx 0.067R_{sun}$$

7. We now use the approximation in the core:

$$T_C = \Big(\frac{\mu m_H}{k}\rho_C 4\pi R^2 G\Big)\Big(\frac{1}{2}(\frac{\pi}{2})^2 - 1\Big)$$

where R was given in 13.4.6. Solve for ρ_C:

$$\rho_C = \frac{T_C k}{\mu m_H G R^2 4\pi(\pi^2/8 - 1)} \approx 2.9 \cdot 10^5 kg$$

or 200 times the mean density (assuming $1400kg/m^3$), about a factor two wrong. Not bad!