**The following people have participated in creating these solutions: Nicolaas E. Groeneboom, Magnus Pedersen Lohne, Karl R. Leikanger** *NOTE: There might be errors in the solution. If you find something which doens't look right, please let me know*

# Partial solutions to problems: Lecture 18

## Problem 1

We use equation (3) in the lecture notes:

$$\frac{dr}{dt} = \pm(1-2M/r)\sqrt{1-(1-2M/r)\frac{b^2}{r^2}}$$

and square:

$$\left(\frac{dr}{dt}\right)^2 = (1-2M/r)^2\left(1-(1-2M/r)\frac{b^2}{r^2}\right)$$

Then switching to shell-coordinates

$$\frac{dr_{shell}}{dt_{shell}} = \frac{(1-2M/r)^{-1/2}dr}{(1-2M/r)^{1/2}dt} = \frac{dr}{(1-2M/r)dt}$$

such that

$$\left(\frac{dr_{shell}}{dt_{shell}}\right)^2 = 1-(1-2M/r)\frac{b^2}{r^2}$$

Divide by $b^2$ to obtain the desired equation.

We see that this equation is on the form of equation (4) in the previous lecture notes (on relativistic orbits). We identify $A = B = 1/b^2$, $x = dr_{\text{shell}}/dt_{\text{shell}}$ and $V^2(x) = (1-2M/r)/r^2$.

## Problem 2

1. We differentiate the potential

$$V(r) = \sqrt{\frac{1-2M/r}{r^2}}$$

and find the extremal points:

$$\frac{d}{dr}V(r) = \frac{1}{2\sqrt{\frac{1-2M/r}{r^2}}}\cdot\left(-\frac{2}{r^3}+\frac{6M}{r^4}\right) = 0$$

such that $\frac{d}{dr}V(r) = 0$ if

$$\frac{2}{r^3} = \frac{6M}{r^4}$$

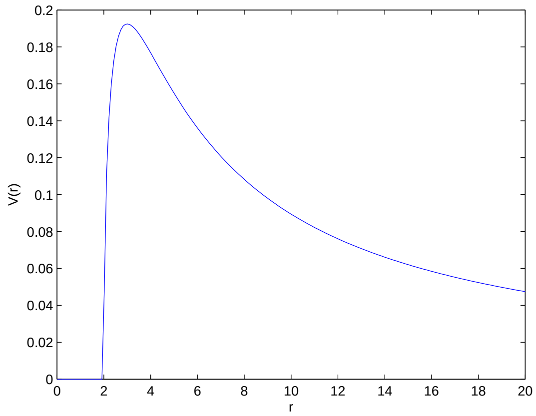


Figure 1: Potential V(r) for $M = 1$ with a maximum at $r = 3$.

for $r = 3M$.

As seen from figure 1, this extremal point is a maximum and not a minimum. As energy states stabilize towards minima, any perturbation from the maximum of the potential will rapidly decay towards lower energy states. Think of trying to balance a ball on top of the potential maximum - any perturbation to the ball will make it fall down either way. However, in a minimum, the ball would just roll back and forth in the potential. As this potential describes the orbits of light around a heavy object, we conclude that there are no stable orbits for light.

2. See the text to find the explanation for why $r = 3M$ is called the light sphere.

3. The critical point is when $1/b^2 \propto V(r)$ is larger or smaller than the peak in figure 1. At the maximum, the value of $V(r_{crit})$ is

$$V(r_{crit}) = V(3M) = \frac{1}{3M}\sqrt{1 - \frac{2M}{3M}} = \frac{1}{3\sqrt{3}M} = \frac{1}{b_{crit}}$$

# Problem 3

1. The situation is depicted in figure 2 with an enlargement of the triangle ABC in figure 3.

   The angular shift on the sky is given by $\alpha$, the deflection of light is $\Delta\phi$. First we observe that as the distance to the star goes to infinity $\gamma \to 90°$. The star is much more distant than the Sun so it is a good approximation to set $\gamma \approx 90°$. Then we see from the figure that

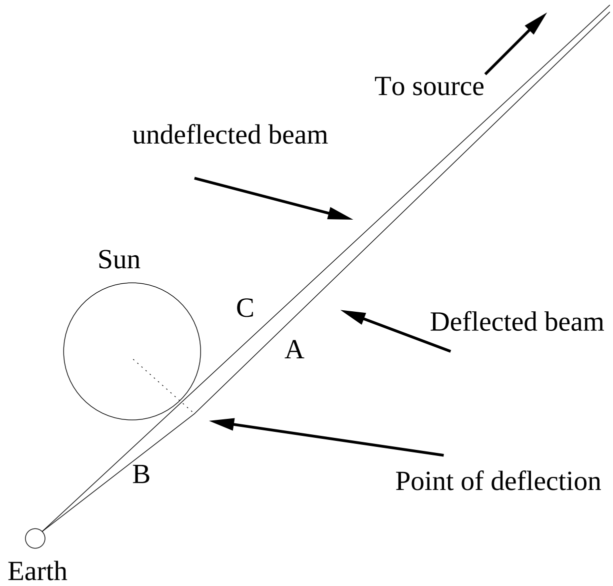


Figure 2: Deflection of light from a distant star by the Sun. 'Undeflected beam' refers to how a light beam from the star would have moved if the Sun had not been there to deflect it.

$\beta = 90° - \Delta\phi$. Using the small triangle on the left hand side in figure 3 we have that

$$\alpha + (90° - \Delta\phi) + 90° = 180°$$

giving $\Delta\alpha = \Delta\phi$.

2. Inserting numbers for the mass and radius of the Sun (assuming that the light passes very close to the solar surface) $\Delta\alpha = \Delta\phi = 4M/R \approx 1.7'$

3. Similarly for the moon we get $\alpha = 6.3 \times 10^{-8}$ arc seconds.

## Problem 4.6

Problems 4.1-4.5 should be possible to solve using the equations and figures which are given. Here we only give the solution to problem 4.6. We use that $d_S = 10^{10} ly, d_L = 10^9 ly$. The lensing formula is given as

$$\theta_E = \sqrt{\frac{4M(d_S - d_L)}{d_L d_S}}$$

solving for $M$:

$$M = \frac{\theta_E^2 d_L d_S}{4(d_S - d_L)} \times \frac{c^2}{G} = 1.35 \times 10^{15} M_\odot.$$

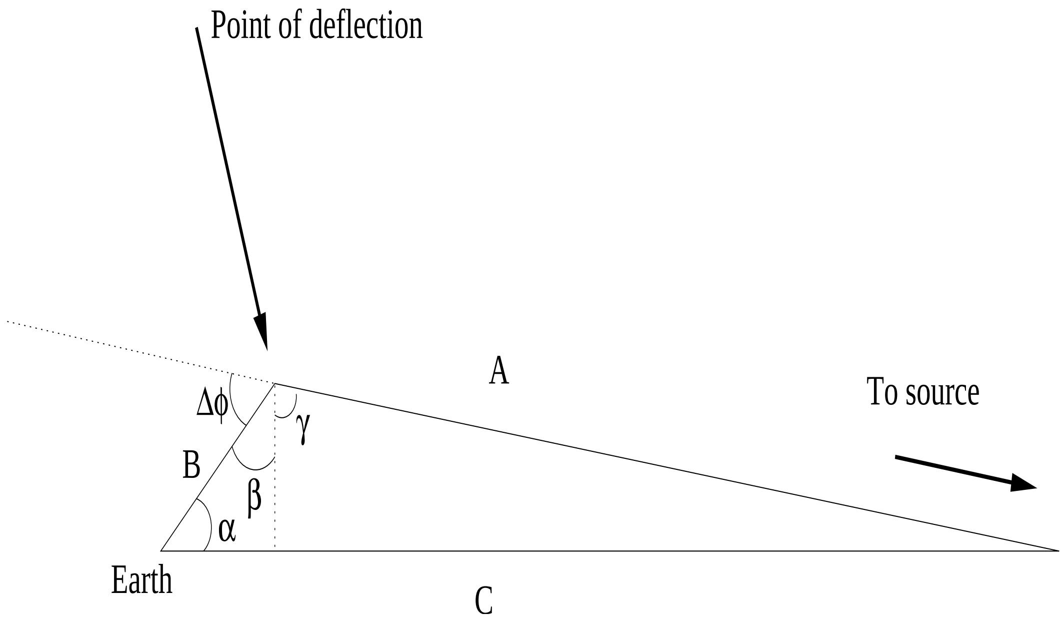


Figure 3: The triangle ABC in figure 2 enlarged.