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NOTE: There might be errors in the solution. If you find something which doens't look right, please let me know

Partial solutions to problems: Lecture 19

Problem 1

1. We need to find the normalizing constant of the probability distribution $n(v)$. One of the requirements for $n(v)$ to be a probability distribution is that all the probabilities adds up to 1 :

$$\int_{-\infty}^{\infty} n_{\text{norm}}(v)dv = \frac{1}{N}\int_{-\infty}^{\infty} n(v)dv = 1$$

where N is the normalizing constant. We insert for the Maxwell-Boltzmann distribution:

$$n\Big(\frac{m}{2\pi kT}\Big)^{3/2} 4\pi \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{mv^2}{kT}} v^2 dv = N \tag{0.1}$$

Perform the substitution $x = \frac{1}{2}\frac{mv^2}{kT}$ such that

$$v^2 = \frac{2xkT}{m}$$

and hence

$$dv = \frac{kT}{vm}dx = \frac{kT\sqrt{m}}{m\sqrt{2xkT}}dx = \sqrt{\frac{kT}{2mx}}dx$$

When inserting this back into equation (0.1):

$$n\Big(\frac{m}{2\pi kT}\Big)^{3/2} 4\pi \int_{0}^{\infty} e^{-x}\frac{2xkT}{m}\sqrt{\frac{kT}{2mx}}dx = N$$

such that

$$n\Big(\frac{m}{2\pi kT}\Big)^{3/2} 4\pi \frac{2kT}{m}\sqrt{\frac{kT}{2m}}\int_{0}^{\infty} e^{-x}x^{\frac{1}{2}}dx = N$$

summarizing terms:

$$n\Big(\frac{m}{2\pi kT}\Big)^{3/2} 4\pi\sqrt{2}\Big(\frac{2kT}{m}\Big)^{3/2}\Gamma(\frac{3}{2}) = N$$

where

$$\Gamma(n) = n\Gamma(n-1) = \int_{0}^{\infty} e^{-x}x^{n-1}dx$$

is the **Gamma-function**. For $n \in \mathbb{N}$, we have that $\Gamma(n+1) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$ such that $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$. This function will become *very important* when working with statistical physics and quantum mechanics, so it's in general a good idea to get familiarized and friendly with it as soon as possible. The Γ-function doesn't bite.. too much.

Now, summarizing shows that most terms cancel and we end up with

$$N = n$$

which is not very surprising: integrating the number density per velocity over all velocities we should expect to find the total number density.

2. We proceeed by determining the average energy of the gas:

$$\langle E\rangle = \frac{1}{2}m\langle v^2\rangle = \frac{1}{2}m\int n_{\mathrm{norm}}(v)v^2dv$$

Inserting all values yields

$$\langle E\rangle = \frac{m}{2}\Big(\frac{m}{2\pi kT}\Big)^{3/2}4\pi\int_0^\infty e^{-\frac{1}{2}\frac{mv^2}{kT}}v^4dv$$

Performing the same substitution as in the previous exercise, then

$$\langle E\rangle = \frac{m}{2}\Big(\frac{m}{2\pi kT}\Big)^{3/2}4\pi\int_0^\infty e^{-x}\Big(\frac{2xkT}{m}\Big)^2\sqrt{\frac{kT}{2mx}}dv$$

or

$$\langle E\rangle = \frac{m}{2}\Big(\frac{m}{2\pi kT}\Big)^{3/2}4\pi\Big(\frac{2kT}{m}\Big)^2\sqrt{\frac{kT}{2m}}\Gamma(\frac{5}{2})$$

Using that $\Gamma(5/2) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{4}\sqrt{\pi}$, then

$$\langle E\rangle = \frac{m}{2}\Big(\frac{m}{2\pi kT}\Big)^{3/2}4\pi\Big(\frac{2kT}{m}\Big)^2\sqrt{\frac{kT}{2m}}\frac{3}{4}\sqrt{\pi}$$

Summarizing:

$$\langle E\rangle = \frac{3}{2}m\Big(\frac{m}{2\pi kT}\Big)^{3/2}\Big(\frac{2\pi kT}{m}\Big)^{3/2}\Big(\frac{kT}{m}\Big)$$

or (finally)

$$\langle E\rangle = \frac{3}{2}kT$$

Problem 2

1. We use supplied formula for *pp*-chain:

$$\epsilon_{pp} \approx \epsilon_{0,pp}X_H^2\rho T_6^4 \approx 0.001$$

where $\epsilon_{0,pp} = 1.08\cdot 10^{-12}$, $X_H = 0.33$, $\rho = 1.5\cdot 10^5$ and $T = 15.7$. Similarly,a

$$\epsilon_{cno} \approx 3.4\cdot 10^{-4}$$

and

$$\epsilon_{3\alpha} \approx 0$$

as $0.157^{41} \approx 0$.

2. We see that $\epsilon_{pp}/\epsilon_{cno} \approx 3$, which is way off the 1% expectation. This answer is wrong because we assumed that the temperature is constant in the core, which is not true: the high temperatures are only evident in the center of the core. This means that most of the energy is created at lower temperatures, where the *pp* chain dominates.

   The 3α is practically non-existing.

3. With $T = 13 \cdot 10^6 K$, we obtain a ratio $\epsilon_{pp}/\epsilon_{cno} \approx 65$, which is more closer to reality (that is, *pp* dominates CNO by approx 1.5%).

4. At what temperature T is $\epsilon_{pp} = \epsilon_{CNO}$? We equate:

$$\epsilon_{0,pp} X_H^2 \rho T_6^4 = \epsilon_{0,CNO} X_H X_{CNO} \rho T_6^{20}$$

   Solving for T_6:

$$T = \left(\frac{\epsilon_{0,pp} X_H}{\epsilon_{0,CNO} X_{CNO}} \right)^{\frac{1}{16}} \approx 17$$

   such that CNO dominates from 17 million K (assuming the temperature in the core is homogeneous).

5. The total energy L_{sun} emmitted from the sun must equal the total mass inside the core of the sun multiplied with the reaction rate:

$$L_{sun} = \frac{4}{3} \pi R^3 \cdot \rho \cdot \epsilon$$

   Solving for R:

$$R = \left(\frac{3 L_{sun}}{4 \pi \rho \epsilon} \right)^{1/3} \approx 0.15 R_{sun}$$

   when the values have been inserted.

6. Using the same equation as in the previous question, we obtain that $R \approx 0.6 R_{sun}$.