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NOTE: There might be errors in the solution. If you find something which doens't look right, please let me know

Partial solutions to problems: Lecture 21

Problem 1

The normalization requirement

We have

$$n = \int_0^\infty n(p)dp = \int_0^\infty n\Big(\frac{1}{2\pi mkT}\Big)^{3/2} e^{-p^2/(2mkT)} 4\pi p^2 dp$$

$$= n\Big(\frac{1}{2\pi mkT}\Big)^{3/2} 4\pi \int_0^\infty e^{-p^2/(2mkT)} p^2 dp$$

substitute: $x = p^2/(2mkT)$, $p = \sqrt{2mkTx}$ and $dp = \frac{1}{2}\sqrt{2mkT/x}dx$. Dividing by n, the integral is then

$$1 = \Big(\frac{1}{2\pi mkT}\Big)^{3/2} 4\pi \int_0^\infty e^{-x} 2mkTx \frac{1}{2}\sqrt{2mkT/x}dx$$

summarizing terms:

$$= \Big(\frac{1}{2\pi mkT}\Big)^{3/2} 2\pi \Big(2mkT\Big)^{3/2} \int_0^\infty e^{-x} x^{1/2} dx$$

using that $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$, we obtain

$$= \Big(\frac{1}{2\pi mkT}\Big)^{3/2} \Big(2mkT\pi\Big)^{3/2} = 1$$

and we see that the distribution is already normalized, as required. We continue by deciding on P:

Obtaining P

Now, P is found by

$$P = \frac{1}{3}\int_0^\infty pvn(p)dp$$

written out, this becomes

$$P = \frac{1}{3}\int_0^\infty \frac{p^2}{m} n(p)dp = \frac{1}{3m}\int_0^\infty n\Big(\frac{1}{2\pi mkT}\Big)^{3/2} 4\pi e^{-p^2/(2mkT)} p^4 dp$$

summarizing terms:

$$P = \frac{4\pi}{3m}\Big(\frac{1}{2\pi mkT}\Big)^{3/2} \int_0^\infty n e^{-p^2/(2mkT)} p^4 dp$$

performing the substitute as in the previous section gives:

$$P = \frac{4\pi n}{3m}\Big(\frac{1}{2\pi mkT}\Big)^{3/2}\int_0^\infty e^{-x}(2mkTx)^2\frac{1}{2}\sqrt{2mkT/x}dx$$

Summarizing again:

$$P = \frac{2\pi n}{3m}\Big(\frac{1}{2\pi mkT}\Big)^{3/2}(2mkT)^{5/2}\int_0^\infty e^{-x}x^{3/2}dx$$

using that $\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{4}\sqrt{\pi}$, we find

$$P = \frac{\pi^{3/2}n}{2m}\Big(\frac{1}{2\pi mkT}\Big)^{3/2}(2mkT)^{5/2} = nkT$$

Problem 2

1. We have

   $n_e = n_p =$ percentage of protons in nucleus×total mass / mass of hydrogen $= \dfrac{Z}{A}\times\dfrac{\rho}{m_H}$

2.
   $$\frac{3}{2}kT < \frac{h^2}{8m_e}\Big(\frac{3n_e}{\pi}\Big)^{2/3}$$
   $$\frac{3}{2}kT < \frac{h^2}{8m_e}\Big(\frac{3Z\rho}{\pi Am_H}\Big)^{2/3}$$
   $$\frac{12}{h^2}kTm_e < \Big(\frac{3Z\rho}{\pi Am_H}\Big)^{2/3}$$
   $$\Big(\frac{12}{h^2}kTm_e\Big)^{3/2} < \frac{3Z\rho}{\pi Am_H}$$

   or

   $$\rho > \frac{\pi Am_H}{3Z}\Big(\frac{12}{h^2}kTm_e\Big)^{3/2}$$

3. This is done by direct insertion: $\rho > 7.03\cdot 10^8 kg/m^3$.

4. We have

   $$M = \frac{4}{3}\pi R^3\rho$$

   or

   $$R = \Big(\frac{3M}{4\pi\rho}\Big)^{1/3}$$

   with the density found in the previous question we obtain $R \sim 8790km$

5. Inserting the mass of the Earth instead of the mass of the Sun, we obtain $R \sim 126.6$ km.

Problem 3

In this exercise, we're asked to derive the expression for the mean kinetic energy of a particle in a degenerate gas. This gas no longer follows the normal M.B-distribution, which we have used in earlier exercises.

1. Let's summarize: We have a relation between $n(\vec{p})$(the number density per volume per momentum space volume for particles with momentum $\vec{p}$) and $n(p)$ (the number density per real space volume for particles with absolute momentum p). This relation is given by $n(p)dp = 4\pi p^2 n(\vec{p})dp$, where we obtain the real-space volume element by integrating a sphere over the momentum-space for a fixed absolute momentum. We're now asked to find a relation between $n(p)$ and $n(E)$. We know that

$$E = \frac{p^2}{2m}$$

such that

$$p = \sqrt{2mE}$$

and

$$dp = \frac{1}{2\sqrt{2mE}} \cdot 2mdE = \sqrt{\frac{m}{2E}}dE$$

$$\frac{dp}{dE} = \sqrt{\frac{m}{2E}}$$

Now, we switch from $n(p)$ to $n(E)$ using the chain rule:

$$n(E) = n(p)\frac{dp}{dE} = n(p)\sqrt{\frac{m}{2E}}$$

and insert for $n(p) = 4\pi p^2 n(\vec{p})$:

$$n(E) = 4\pi p^2 n(\vec{p})\sqrt{\frac{m}{2E}} = 4\sqrt{2}\pi m^{3/2}\sqrt{E}n(\vec{p})$$

where we substituted $p^2 = (2mE)$. We now insert for

$$n(\vec{p}) = \frac{2}{h^3}\frac{1}{e^{(p^2-p_F^2)/(2mkT)}+1}$$

$$n(E) = \frac{8\sqrt{2}\pi m^{3/2}}{h^3}\sqrt{E}\frac{1}{e^{(p^2-p_F^2)/(2mkT)}+1}$$

Rewriting, we find

$$n(E) = 4\pi\left(\frac{2m}{h^2}\right)^{3/2}\sqrt{E}\frac{1}{e^{(p^2-p_F^2)/(2mkT)}+1} = \frac{g(E)}{e^{(p^2-p_F^2)/(2mkT)}+1}$$

2. We continue by finding the mean kinetic energy of a particle in a degenerate gas:

$$\langle E\rangle = \int_0^\infty P(E)EdE$$

First, remember that the probability distribution is given by $n(E)$, but a probability distribution needs to be normalized such that

$$P(E) = Nn(E)$$

where N is found by

$$\int_0^\infty P(E)dE = N\int_0^{E_f} n(E)dE = 1$$

where the E_f-limit is because $n(E) = 0$ for $E > E_F$. The next thing we do is an approximation: in this energy range, the $e^{(p^2-p_F^2)/(2mkT)}$ in $n(E)$ is much less than 1. We can then approximate $n(E) \approx g(E)$, and the integral becomes surprisingly simple. But first we need to normalize the distribution. For simplicity we define $K = 4\pi\left(\frac{2m}{h^2}\right)^{3/2}$. Then

$$1 = N\int_0^{E_f} g(E)dE = NK\int_0^{E_f} E^{1/2}dE = NK\frac{2}{3}E_F^{3/2} = 1$$

such that $N = 3/2(KE_F^{3/2})$. The expectation value is thus

$$\langle E\rangle = N\int_0^{E_f} g(E)EdE = \frac{3}{2}E_F^{-3/2}\int_0^{E_f} E^{3/2}dE$$

$$\langle E\rangle = \frac{3}{2}E^{-3/2}\frac{2}{5}E_F^{5/2} = \frac{3}{5}E_F$$