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Partial solutions to problems: Lecture 9-10

We have not inserted numbers here, but leave this for the reader.

Problem 1

This exercise is self-contained and will not be presented here.

Problem 3

A Lorentz transformation is denoted $c_{\mu\nu}$, where μ and ν runs through $0-3$. Thus $c_{\mu\nu}$ is a 4×4 matrix, where the μ and ν specifies which element of the matrix one is working with. For instance, c_{12} would correspond to the element located at the 2st row, 3rd column. A Lorentz transformation (matrix) operates on a vector (in 4-dimensional flat Minkowski space-time) as such:

$$c_{\mu\nu}x_\nu = x'_\mu \tag{0.1}$$

Here, Einstein's summation convention was used: $\sum_{\mu=0}^{\mu=3} x_\mu x_\mu \equiv x_\mu x_\mu$. In matrix form, equation 0.2 is nothing but

$$\begin{pmatrix} c_{00} & c_{01} & c_{02} & c_{03} \\ c_{10} & c_{11} & c_{12} & c_{13} \\ c_{20} & c_{21} & c_{22} & c_{23} \\ c_{30} & c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$

We now define $D_\mu = A_\mu + B_\mu$, where A and B are 4-vectors. To show D_μ is a four-vector, we must show that it transforms as equation 0.2.

$$c_{\mu\nu}D_\nu = c_{\mu\nu}(A_\nu + B_\nu) = c_{\mu\nu}A_\nu + c_{\mu\nu}B_\nu = A'_\mu + B'_\mu = D'_\mu \tag{0.2}$$

Thus the sum of two 4-vectors is a 4-vector.

Problem 4

1. In the rest frame of the neutron, $v = 0$ such that $P_\mu(n) = (m_n, 0)$.

2. In the rest frame of the neutron, $p'_p = \gamma'_p m_p v'_p$ and $E'_p \ = \gamma'_p m_p$. The 4-vector is then

$$P_\mu(p) = (\gamma'_p m_p, \gamma'_p m_p v'_p) = \gamma'_p m_p (1, v'_p).$$

Here v'_p is the velocity of the proton from the neutron frame and $\gamma'_p = 1/\sqrt{1-(v'_p)^2}$.

3. In the rest frame of the neutron, $p'_e = \gamma'_e m_e v'_e$ and $E'_e = \gamma'_e m_e$, such that $P_\mu(e^-) = \gamma'_e m_e (1, v'_e)$. Here v'_e is the velocity of the electron from the neutron frame and $\gamma'_e = 1/\sqrt{1-(v'_e)^2}$.

4. We use conservation of momentum:

$$P'_\mu(n) = P'_\mu(p) + P'_\mu(e^-)$$

Inserting, we find

$$\begin{bmatrix} m_n \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma'_p m_p \\ \gamma'_p m_p v'_p \end{bmatrix} + \begin{bmatrix} \gamma'_e m_e \\ \gamma'_e m_e v'_e \end{bmatrix}$$

Conservation of energy (P_0, first line) then gives

$$m_n = \gamma'_p m_p + \gamma'_e m_e,$$

while the second line gives

$$\gamma'_p m_p v'_p = -\gamma'_e m_e v'_e.$$

Squaring the second line and writing it in terms of γ-factors:

$$(\gamma'_p)^2 m_p^2 - m_p^2 = (\gamma'_e)^2 m_e^2 - m_e^2$$

Solve for γ'_e from the first equation:

$$\gamma'_e m_e = m_n - \gamma'_p m_p$$

Insert in the second equation to obtain

$$\gamma'_p = \frac{m_n^2 + m_p^2 - m_e^2}{2 m_p m_n}$$

From which we easily find that $v'_p = 0.001262$. Going back to the first equation we then find that $v'_e = -0.9183016$ (where did we get the minus sign from?)

5. We now transform between the lab frame (where nothing is at rest) and the neutron rest frame. We use that $P_\mu(e^-) = c_{\mu\nu}P'_\nu(e^-)$ (note that the prime is now on the right hand side, meaning that we need to use $-v_n$ instead of v_n, why?). In matrix form for the electron,

$$P_\mu = c_{\mu\nu}P'_\nu(e^-) = \begin{pmatrix} \gamma_n & v_n\gamma_n \\ v_n\gamma_n & \gamma_n \end{pmatrix} \begin{bmatrix} 1 \\ v'_e \end{bmatrix} \gamma'_e m_e = \begin{bmatrix} \gamma_n + v'_e v_n \gamma_n \\ v_n\gamma_n + v'_e\gamma_n \end{bmatrix} \gamma'_e m_e$$

where v_n is the neutron velocity in the lab frame and $\gamma_n = 1/\sqrt{1-v_n^2}$. Inserting numbers we have $E_e = 1.481 \times 10^{-30}$kg and $p_e = -1.168 \times 10^{-30}$kg. In exactly the same way we find $E_p = 1.187 \times 10^{-26}$kg and $p_p = 1.175 \times 10^{-26}$kg.

6. We use the expression for relativistic energy (using the previous result)

$$E_e = \frac{m_e}{\sqrt{1-v_e^2}}$$

Solving for v_e we obtain $v_e = 0.788922$ Similarly we obtain $v_p = 0.990025$

7. Using the formula for relativistic addition of velocities we have

$$v_e = \frac{v'_e + v_n}{1 + v'_e v_n}$$

using again the $-v_n$ as the speed of the neutron (check again that you understand why!). Similarly for the proton.

8. I don't like long and ugly calculations.

Problem 5

1. We let the electron move in the positive x-direction $v_e = v$ and the positron in the negative x-direction $v_p = -v$ such that

$$v'_p = \frac{v_p - v_e}{1 - v_p v_e} = \frac{-2v}{1+v^2}$$

2. $P_\mu(e) = \gamma m(1, v)$ and $P_\mu(p) = \gamma m(1, -v)$, where m is the electron/positron mass and $\gamma = 1/\sqrt{1-v^2}$.

3.

$$P'_\mu(e\pm) = c_{\mu\nu}P_\nu(e\pm) = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{bmatrix} 1 \\ \mp v \end{bmatrix} m\gamma = \begin{bmatrix} 1 \pm v^2 \\ -v \mp v \end{bmatrix} m\gamma^2$$

4. In general, $E^2 = p^2 + m^2$. Photons have zero mass, so $E = \pm p$. A four-vector is generally expressed $P_\mu = (E, p, 0, 0)$, such that the four-vector of a photon is always $P_\mu(\gamma) = (E, \pm E, 0, 0)$.

5. Conservation of four-vectors gives (omitting the y-z-directions)

$$P_\mu(e) + P_\mu(p) = P_\mu(\gamma_1) + P_\mu(\gamma_2),$$

inserting:

$$m\gamma(1, v) + m\gamma(1, -v) = (E_1, E_1) + (E_2, -E_2).$$

Thus

$$(2m\gamma, 0) = (E_1 + E_2, E_1 - E_2)$$

Momentum conservation gives $E_1 - E_2 = 0$, so $E_1 = E_2$.

6. The wavelength is given as $E = hc/\lambda$, so $\lambda = hc/E$. From the previous question we have $E = m\gamma$ such that $\lambda = hc/(m\gamma)$

7. A Lorentz boost (omitting y and z directions) is given by

$$c_{\mu\nu} P_\nu = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = \begin{bmatrix} P_0' \\ P_1' \end{bmatrix} = P_\mu'$$

When $P = (E, E)$ and $P' = (E', E')$,

$$\begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{bmatrix} E \\ E \end{bmatrix} = \begin{bmatrix} \gamma E - \gamma v E \\ -v\gamma E + \gamma E \end{bmatrix}$$

Conservation of energy gives

$$E' = \gamma E - \gamma v E = E\gamma(1 - v)$$

8. This is found by insertion of the electron velocity v:

$$E' = E\gamma(1 \pm v)$$

where E is the energy of the photons in the laboratory frame.

9. We start with

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda}{\lambda} = \frac{\lambda'}{\lambda} - 1 = \frac{E}{E'} - 1,$$

where we used $E = hc/\lambda$. Inserting the expression for energy,

$$\frac{\Delta\lambda}{\lambda} = \frac{E}{E'} - 1 = \frac{1}{\gamma(1 - v)} - 1 = \frac{\sqrt{1 - v^2}}{1 - v} - 1 = \sqrt{\frac{(1 - v)(1 + v)}{(1 - v)^2}} - 1 = \sqrt{\frac{1 + v}{1 - v}} - 1$$

which is the relativistic Doppler formula.

10. We Taylor expand the expression $f(v) = \sqrt{(1+v)/(1-v)}$ to first order, as v is very small (and hence v^2 even smaller).

$$f(v) \approx f(0) + f'(0) \cdot v$$

where $f(0) = 1$ is trivial. We differentiate f:

$$f'(v) = \frac{d}{dv}\sqrt{\frac{1+v}{1-v}} = \frac{1}{2\sqrt{\frac{1+v}{1-v}}}(\frac{1}{(1-v)^2} + \frac{1}{1-v} + \frac{v}{(1-v)^2})$$

letting $v = 0$,we find $f'(0) = 1$, such that we end up with

$$\frac{\Delta\lambda}{\lambda} = \sqrt{\frac{1+v}{1-v}} - 1 = f(v) - 1 \approx 1 + v - 1 = v$$

which is the non-relativistic Doppler effect.