

AST1100 Lecture Notes

1 - 2 Celestial Mechanics

1 Kepler's Laws

Kepler used Tycho Brahe's detailed observations of the planets to deduce three laws concerning the motion of the planets:

1. The orbit of a planet is an ellipse with the Sun in one of the foci.
2. A line connecting the Sun and the planet sweeps out equal areas in an equal time intervals.
3. The orbital period around the Sun and the semimajor axis (see figure 4 for the definition) of the ellipse are related through:

$$P^2 = a^3, \quad (1)$$

where P is the period in years and a is the semimajor axis in AU (astronomical units, 1 AU = the distance between the Earth and the Sun).

Whereas the first law describes the shape of the orbit, the second law is basically a statement about the orbital velocity: When the planet is closer to the Sun it needs to have a higher velocity than in a point far away in order to sweep out the same area in equal intervals. The third law is a mathematical relation between the size of the orbit and the orbital period. As an example we see that when the semimajor axis doubles, the orbital period increases by a factor $2\sqrt{2}$ (do you agree?).

The first information that we can extract from Kepler's laws is a relation between the velocity of a planet and the distance from the Sun. When the distance from the Sun increases, does the orbital velocity increase or decrease? If we consider a nearly circular orbit, the distance traveled by the planet in one orbit is $2\pi a$, proportional to the semimajor axis. The mean velocity can thus be expressed as $v_m = 2\pi a/P$ which using Kepler's third

law simply gives $v_m \propto a/(a^{3/2}) \propto 1/\sqrt{a}$ (check that you understood this!). Thus, the mean orbital velocity of a planet decreases the further away it is from the Sun.

When Newton discovered his law of gravitation,

$$\vec{F} = \frac{Gm_1m_2}{r^2}\vec{e}_r,$$

he was able to deduce Kepler's laws from basic principles. Here \vec{F} is the gravitational force between two bodies of mass m_1 and m_2 at a distance r and G is the gravitational constant. The unit vector in the direction of the force is denoted by \vec{e}_r .

2 General solution to the two-body problem

Kepler's laws is a solution to the *two-body problem*: Given two bodies with mass m_1 and m_2 at a positions \vec{r}_1 and \vec{r}_2 moving with speeds \vec{v}_1 and \vec{v}_2 (see figure 1). The only force acting on these two masses is their mutual gravitational attraction. How can we describe their future motion as a function of time? The rest of this lecture will be devoted to this problem.

In order to solve the problem we will now describe the motion from the rest frame of mass 1: We will sit on m_1 and describe the observed motion of m_2 , i.e. the motion of m_2 with respect to m_1 . (As an example this could be the Sun-Earth system, from the Earth you view the motion of the Sun). The only force acting on m_2 (denoted \vec{F}_2) is the gravitational pull from m_1 . Using Newton's second law for m_2 we get

$$\vec{F}_2 = -G\frac{m_1m_2}{|\vec{r}|^3}\vec{r} = m_2\ddot{\vec{r}}, \quad (2)$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$ the vector pointing from m_1 to m_2 . Overdots describe derivatives with respect to time,

$$\begin{aligned} \dot{\vec{r}} &= \frac{d\vec{r}}{dt} \\ \ddot{\vec{r}} &= \frac{d^2\vec{r}}{dt^2} \end{aligned}$$

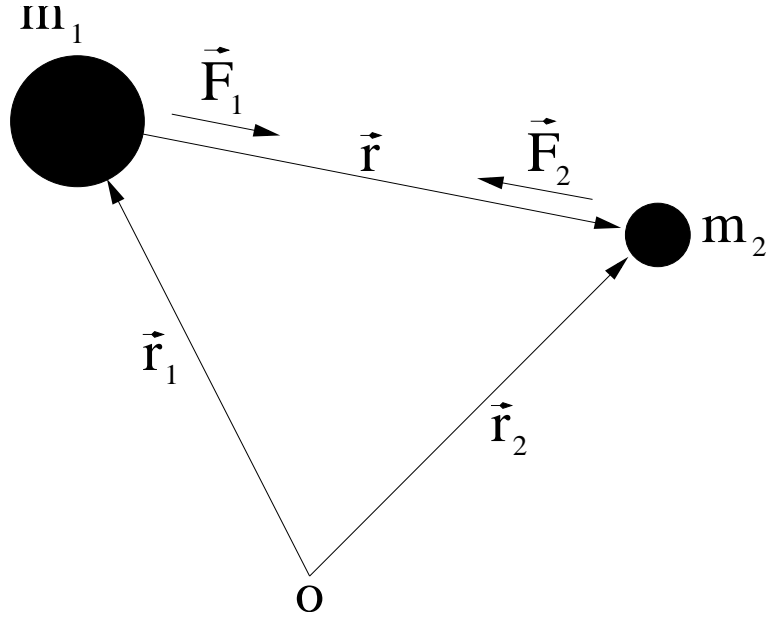


Figure 1: The two-body problem.

Sitting on m_1 , we need to find the vector $\vec{r}(t)$ as a function of time. This function would completely describe the motion of m_2 and be a solution to the two-body problem (do you see this?).

Using Newton's third law, we have a similar equation for the force acting on m_1

$$\vec{F}_1 = -\vec{F}_2 = G \frac{m_1 m_2}{|\vec{r}|^3} \vec{r} = m_1 \ddot{\vec{r}}_1. \quad (3)$$

Subtracting equation (3) from (2), we can eliminate \vec{r}_1 and \vec{r}_2 and obtain an equation only in \vec{r} which is the variable we want to solve for,

$$\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = -G \frac{m_1 + m_2}{|\vec{r}|^3} \vec{r} \equiv -m \frac{\vec{r}}{r^3}, \quad (4)$$

where $r = |\vec{r}|$ and $m = G(m_1 + m_2)$. This is the equation of motion of the two-body problem,

$$\ddot{\vec{r}} + m \frac{\vec{r}}{r^3} = 0. \quad (5)$$

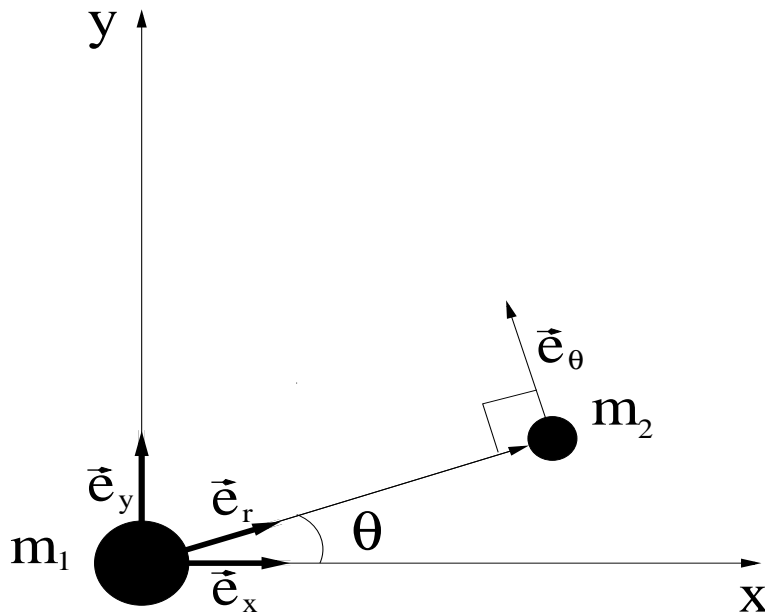


Figure 2: Geometry of the two-body problem.

We are looking for a solution of this equation with respect to $\vec{r}(t)$, this would be the solution to the two-body problem predicting the movement of m_2 with respect to m_1 .

To get further, we need to look at the geometry of the problem. We introduce a coordinate system with m_1 at the origin and with \vec{e}_r and \vec{e}_θ as unit vectors. The unit vector \vec{e}_r points in the direction of m_2 such that $\vec{r} = r\vec{e}_r$ and \vec{e}_θ is perpendicular to \vec{e}_r (see figure 2). At a given moment, the unit vector \vec{e}_r (which is time dependent) makes an angle θ with a given fixed (in time) coordinate system defined by unit vectors \vec{e}_x and \vec{e}_y . From figure 2 we see that (do you really see this? Draw some figures to convince yourself!)

$$\begin{aligned}\vec{e}_r &= \cos \theta \vec{e}_x + \sin \theta \vec{e}_y \\ \vec{e}_\theta &= -\sin \theta \vec{e}_x + \cos \theta \vec{e}_y\end{aligned}$$

The next step is to substitute $\vec{r} = r\vec{e}_r$ into the equation of motion (equation

5). In this process we will need the derivatives of the unit vectors,

$$\begin{aligned}\dot{\vec{e}}_r &= -\dot{\theta} \sin \theta \vec{e}_x + \dot{\theta} \cos \theta \vec{e}_y \\ &= \dot{\theta} \vec{e}_\theta \\ \dot{\vec{e}}_\theta &= -\dot{\theta} \cos \theta \vec{e}_x - \dot{\theta} \sin \theta \vec{e}_y \\ &= -\dot{\theta} \vec{e}_r\end{aligned}$$

Using this, we can now take the derivative of $\vec{r} = r\vec{e}_r$ twice,

$$\begin{aligned}\dot{\vec{r}} &= \dot{r}\vec{e}_r + r\dot{\vec{e}}_r \\ &= \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \\ \ddot{\vec{r}} &= \ddot{r}\vec{e}_r + \dot{r}\dot{\vec{e}}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta + r\dot{\theta}\dot{\vec{e}}_\theta \\ &= (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\vec{e}_\theta.\end{aligned}$$

Substituting $\vec{r} = r\vec{e}_r$ into the equation of motion, we thus obtain

$$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\vec{e}_\theta = -\frac{m}{r^2}\vec{e}_r.$$

Equating left and right hand sides, we have

$$\ddot{r} - r\dot{\theta}^2 = -\frac{m}{r^2} \quad (6)$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad (7)$$

The vector equation (equation 5) has thus been reduced to these two scalar equations. Go back and check that you understood the transition.

The last of these equations indicates a constant of motion, something which does not change with time (why?). What constant of motion enters in this situation? Certainly the angular momentum of the system should be a constant of motion so let's check the expression for the angular momentum vector \vec{h} (note that h is defined as angular momentum per mass, $(\vec{r} \times \vec{p})/m_2$ (remember that m_1 is at rest in our current coordinate frame)):

$$|\vec{h}| = |\vec{r} \times \dot{\vec{r}}| = |(r\vec{e}_r) \times (\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta)| = r^2\dot{\theta}.$$

So equation (7) just tells us that the magnitude of the angular momentum $h = r^2\dot{\theta}$ is conserved, just as expected.

To solve the equation of motion, we are left with solving equation (6). In order to find a solution we will

1. solve for r as a function of angle θ instead of time t . This will give us the distance of the planet as a function of angle and thus the orbit.
2. Make the substitution $u(\theta) = 1/r(\theta)$ and solve for $u(\theta)$ instead of $r(\theta)$. This will transform the equation into a form which can be easily solved.

In order to substitute u in equation (6), we need its derivatives. We start by finding the derivatives of u with respect to θ ,

$$\begin{aligned}\frac{du(\theta)}{d\theta} &= \dot{u} \frac{dt}{d\theta} = -\frac{\dot{r}}{r^2} \frac{1}{\dot{\theta}} = -\frac{\dot{r}}{h} \\ \frac{d^2u(\theta)}{d\theta^2} &= -\frac{1}{h} \frac{d}{d\theta} \dot{r} = -\frac{1}{h} \ddot{r} \frac{1}{\dot{\theta}}.\end{aligned}$$

In the last equation, we substitute \ddot{r} from the equation of motion (6),

$$\frac{d^2u(\theta)}{d\theta^2} = \frac{1}{h\dot{\theta}} \left(\frac{m}{r^2} - r\dot{\theta}^2 \right) = \frac{m}{h^2} - \frac{1}{r} = \frac{m}{h^2} - u,$$

where the relation $h = r^2\dot{\theta}$ was used twice. We thus need to solve the following equation

$$\frac{d^2u(\theta)}{d\theta^2} + u = \frac{m}{h^2}$$

This is just the equation for a harmonic oscillator (if you have not encountered the harmonic oscillator in other courses yet, it will soon come, it is simply the equation of motion for an object which is attached to a spring in motion) with known solution:

$$u(\theta) = \frac{m}{h^2} + A \cos(\theta - \omega),$$

where A and ω are constants depending on the initial conditions of the problem. Substituting back we now find the following expression for r

$$r = \frac{p}{1 + e \cos f}, \tag{8}$$

where $p = h^2/m$, $e = (Ah^2/m)$ and $f = \theta - \omega$. This is the general solution to the two-body problem. We recognize this expression as the general expression for a conic section.

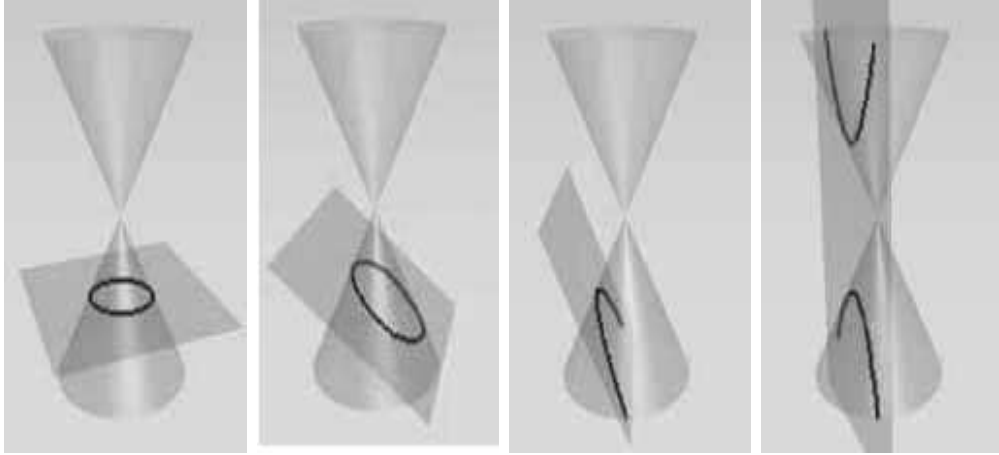


Figure 3: Conic sections: **Circle:** $e=0, p=a$, **Ellipse:** $0 \leq e < 1$, $p = a(1 - e^2)$, **Parabola:** $e = 1$, $p = 2a$, **Hyperbola:** $e > 1$ and $p = a(e^2 - 1)$

3 Conic sections

Conic sections are curves defined by the intersection of a cone with a plane as shown in figure 3. Depending on the inclination of the plane, conic sections can be divided into three categories with different values of p and e in equation (8),

1. the ellipse, $0 \leq e < 1$ and $p = a(1 - e^2)$ (of which the circle, $e = 0$, is a subgroup),
2. the parabola, $e = 1$ and $p = 2a$,
3. the hyperbola, $e > 1$ and $p = a(e^2 - 1)$.

In all these cases, a is defined as a positive constant $a \geq 0$. Of these curves, only the ellipse represents a bound orbit, in all other cases the planet just passes the star and leaves. We will discuss the details of an elliptical orbit later. First, we will check which conditions decides which trajectory an object will follow, an ellipse, parabola or hyperbola. Our question is thus: If we observe a planet or other object close to a star, is it in orbit around the star or just passing by? For two masses to be gravitationally bound, we expect that their total energy, kinetic plus potential, would be less than

zero, $E < 0$. Clearly the total energy of the system is an important initial condition deciding the shape of the trajectory.

We will now investigate how the trajectory $r(\theta)$ depends on the total energy. In the exercises you will show that the total energy of the system can be written as

$$E = \frac{1}{2}\hat{\mu}v^2 - \frac{\hat{\mu}m}{r}, \quad (9)$$

where $v = |\dot{\vec{r}}|$, the velocity of m_2 observed from m_1 (or vice versa) and $\hat{\mu} = m_1m_2/(m_1 + m_2)$. We will now try to rewrite the expression for the energy E in a way which will help us to decide the relation between the energy of the system and the shape of the orbit. We will start by rewriting the velocity in terms of its radial and tangential components using the fact that $\vec{v} = \dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$

$$v^2 = v_r^2 + v_\theta^2 = \dot{r}^2 + (r\dot{\theta})^2, \quad (10)$$

decomposed into velocity along \vec{e}_r and \vec{e}_θ (check that you got this!). We need the time derivative of r . Taking the derivative of equation (8),

$$\dot{r} = \frac{pe \sin f}{(1 + e \cos f)^2} \dot{\theta},$$

we get from equation (10) for the velocity

$$v^2 = \dot{\theta}^2 \frac{p^2 e^2 \sin^2 f}{(1 + e \cos f)^4} + r^2 \dot{\theta}^2.$$

Next step is in both terms to substitute $\dot{\theta} = h/r^2$ (where did this come from?) and then using equation (8) for r giving

$$v^2 = \frac{h^2 e^2 \sin^2 f}{p^2} + \frac{h^2 (1 + e \cos f)^2}{p^2}.$$

Collecting terms and remembering that $\cos^2 f + \sin^2 f = 1$ we obtain

$$v^2 = \frac{h^2}{p^2} (1 + e^2 + 2e \cos f).$$

We will now get back to the expression for E . Substituting this expression for v as well as r from equation (8) into the energy expression (equation 9), we obtain

$$E = \frac{1}{2}\hat{\mu} \frac{h^2}{p^2} (1 + e^2 + 2e \cos f) - \hat{\mu}m \frac{1 + e \cos f}{p} \quad (11)$$

Total energy is conserved and should therefore be equal at any point in the orbit, i.e. for any angle f . We may therefore choose an angle f which is such that this expression for the energy will be easy to evaluate. We will consider the energy at the point for which $\cos f = 0$,

$$E = \frac{1}{2}\hat{\mu}\frac{h^2}{p^2}(1 + e^2) - \frac{\hat{\mu}m}{p}$$

We learned above (below equation 8) that $p = h^2/m$ and thus that $h = \sqrt{mp}$. Using this to eliminate h from the expression for the total energy we get

$$E = \frac{\hat{\mu}m}{2p}(e^2 - 1).$$

If the total energy $E = 0$ then we immediately get $e = 1$. Looking back at the properties of conic sections we see that this gives a parabolic trajectory. Thus, masses which have just too much kinetic energy to be bound will follow a parabolic trajectory. If the total energy is different from zero, we may rewrite this as

$$p = \frac{\hat{\mu}m}{2E}(e^2 - 1).$$

We now see that a negative energy E (i.e. a bound system) gives an expression for p following the expression for an ellipse in the above list of properties for conic sections (by defining $a = \hat{\mu}m/(2|E|)$). Similarly a positive energy gives the expression for a hyperbola. We have shown that the total energy of a system determines whether the trajectory will be an ellipse (bound systems $E < 0$), hyperbola (unbound system $E > 0$) or parabola ($E = 0$). We have just shown Kepler's first law of motion, stating that a bound planet follows an elliptical orbit. In the exercises you will also show Kepler's second and third law using Newton's law of gravitation.

4 The elliptical orbit

We have seen that the elliptical orbit may be written in terms of the distance r as

$$r = \frac{a(1 - e^2)}{1 + e \cos f}.$$

In figure (4) we show the meaning of the different variables involved in this equation:

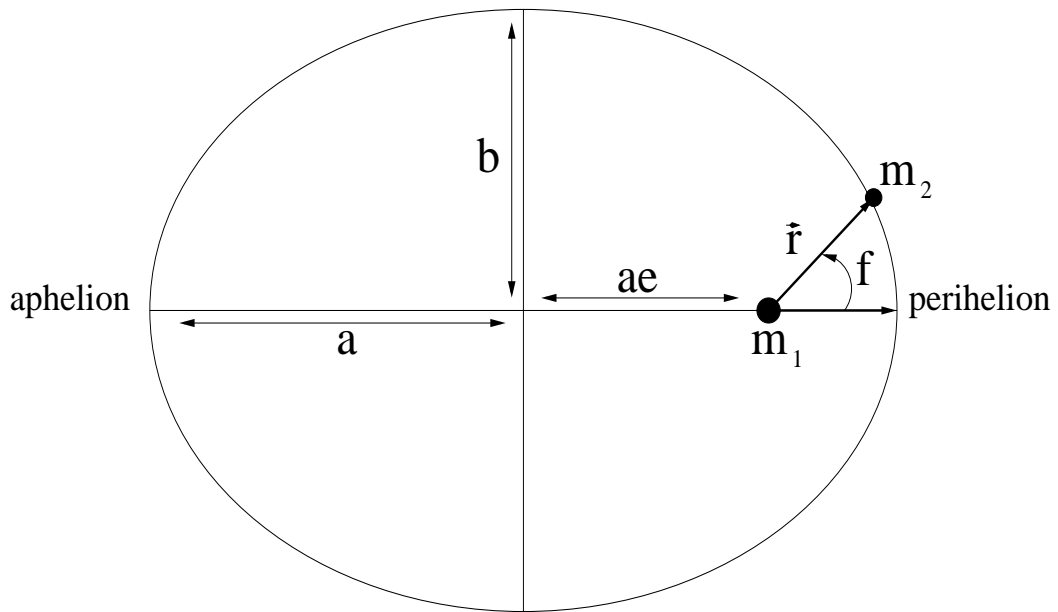


Figure 4: The ellipse.

- a is the **semimajor axis**
- b is the **semiminor axis**
- e is the **eccentricity** defined as $e = \sqrt{1 - (b/a)^2}$
- m_1 is located in the **principal focus**
- the point on the ellipse closest to the principal focus is called **perihelion**
- the point on the ellipse farthest from the principal focus is called **aphelion**
- the angle f is called the **true anomaly**

The eccentricity is defined using the ratio b/a . When the semimajor and semiminor axis are equal, $e = 0$ and the orbit is a circle. When the semimajor axis is much larger than the semiminor axis, $e \rightarrow 1$.

5 Center of mass system

In the previous section we showed that seen from the rest frame of one of the masses in a two-body system, the other mass follows an elliptical / parabolic / hyperbolic trajectory. How does this look from a frame of reference which is not at rest with respect to one of the masses? We know that both masses m_1 and m_2 are moving due to the gravitational attraction from the other. If we observe a distant star-planet system, how does the planet **and** the star move with respect to each other? We have only shown that sitting on either the planet or the star, the other body will follow an elliptical orbit.

An elegant way to describe the full motion of the two-body system (or in fact an N-body system) is to introduce **center of mass coordinates**. The center of mass position \vec{R} is located at a point on the line between the two masses m_1 and m_2 . If the two masses are equal, the center of mass position is located exactly halfway between the two masses. If one mass is larger than the other, the center of mass is located closer to the more massive body. The center of mass is a weighted mean of the position of the two masses:

$$\vec{R} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2, \quad (12)$$

where $M = m_1 + m_2$. We can similarly define the center of mass for an N-body system as

$$\vec{R} = \sum_{i=1}^N \frac{m_i}{M}\vec{r}_i, \quad (13)$$

where $M = \sum_i m_i$ and the sum is over all N masses in the system. Newton's second law on the masses in an N-body system reads

$$\vec{F} = \sum_{i=1}^N m_i \ddot{\vec{r}}_i, \quad (14)$$

where \vec{F} is the total force on all masses in the system. We may divide the total force on all masses into one contribution from internal forces between masses and one contribution from external forces,

$$\vec{F} = \sum_i \sum_{j \neq i} \vec{f}_{ij} + \vec{F}_{\text{ext}},$$

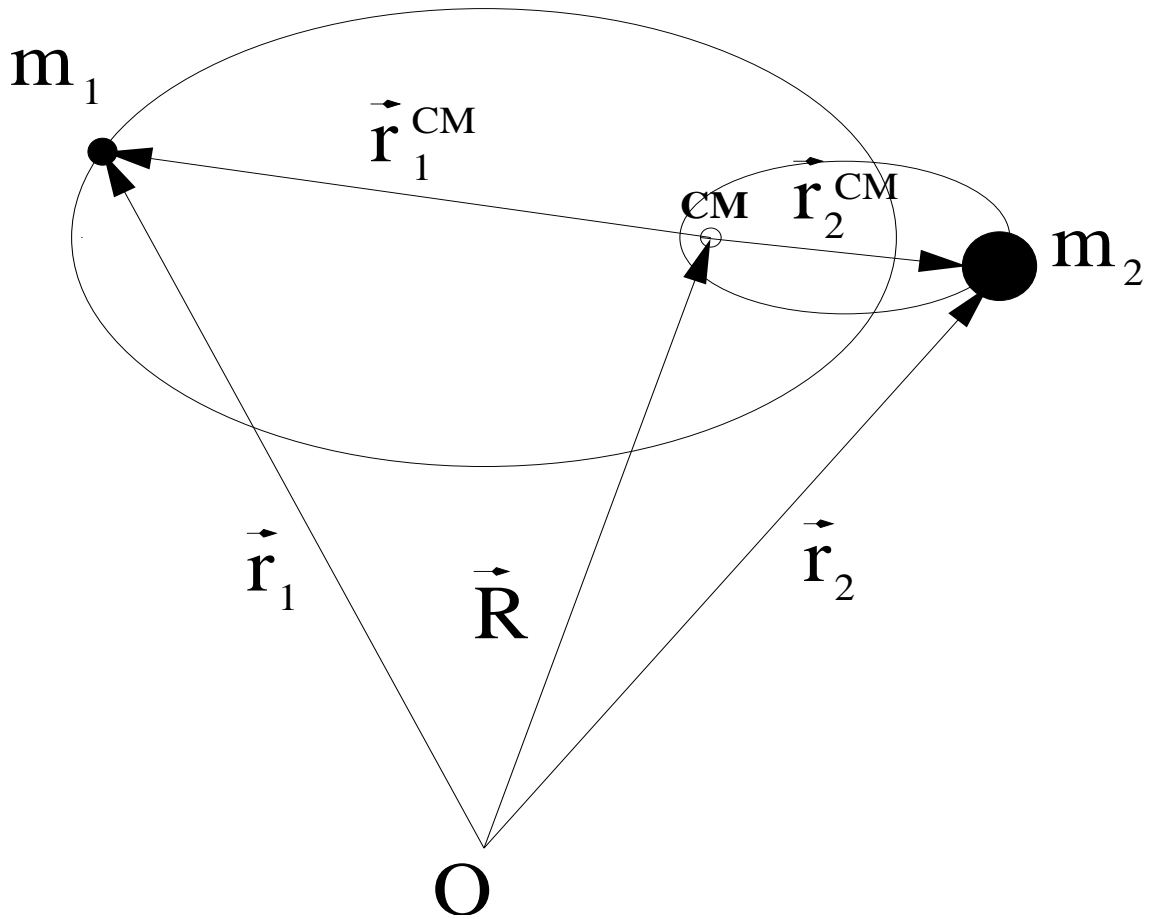


Figure 5: The center of mass system: The center of mass (CM) is indicated by a small open circle. The two masses m_1 and m_2 orbit the center of mass in elliptical orbits with the center of mass in one focus of both ellipses. The center of mass vectors \vec{r}_1^{CM} and \vec{r}_2^{CM} start at the center of mass and point to the masses.

where \vec{f}_{ij} is the gravitational force on mass i from mass j . Newton's third law implies that the sum over all internal forces vanish ($\vec{f}_{ij} = -\vec{f}_{ji}$). The right side of equation (14) can be written in terms of the center of mass coordinate using equation (13) as

$$\sum_{i=1}^N m_i \ddot{\vec{r}}_i = M \ddot{\vec{R}},$$

giving

$$M \ddot{\vec{R}} = \vec{F}_{\text{ext}}.$$

(Check that you followed this deduction!). If there are no external forces on the system of masses ($\vec{F}_{\text{ext}} = 0$), this equation tells us that the center of mass position does not accelerate, i.e. if the center of mass position is at rest it will remain at rest, if the center of mass position moves with a given velocity it will keep moving with this velocity. We may thus divide the motion of a system of masses into the motion of the center of mass and the motion of the individual masses with respect to the center of mass.

We now return to the two-body system assuming that no external forces act on the system. The center of mass moves with constant velocity and we decide to deduce the motion of the masses with respect to the center of mass system, i.e. the rest frame of the center of mass. We will thus be sitting at the center of mass which we define as the origin of our coordinate system, looking at the motion of the two masses. When we know the motion of the two masses with respect to the center of mass, we know the full motion of the system since we already know the motion of the center of mass position.

Since we take the origin at the center of mass location, we have $\vec{R} = 0$. Using equation (12) we get

$$0 = \frac{m_1}{M} \vec{r}_1^{\text{CM}} + \frac{m_2}{M} \vec{r}_2^{\text{CM}},$$

where CM denotes position in the center of mass frame (see figure 5). Using that $\vec{r} = \vec{r}_2 - \vec{r}_1 = \vec{r}_2^{\text{CM}} - \vec{r}_1^{\text{CM}}$ we obtain

$$\vec{r}_1^{\text{CM}} = -\frac{\hat{\mu}}{m_1} \vec{r}, \quad (15)$$

$$\vec{r}_2^{\text{CM}} = \frac{\hat{\mu}}{m_2} \vec{r}, \quad (16)$$

$$(17)$$

The **reduced mass** $\hat{\mu}$ is defined as

$$\hat{\mu} = \frac{m_1 m_2}{m_1 + m_2}.$$

The relative motion of the masses with respect to the center of mass can be expressed in terms of \vec{r}_1^{CM} and \vec{r}_2^{CM} as a function of time, or as we have seen before, as a function of angle f . We already know the motion of one mass with respect to the other,

$$|\vec{r}| = \frac{p}{1 + e \cos f}.$$

Inserting this into equations (15) and (16) we obtain

$$\begin{aligned} |\vec{r}_1^{\text{CM}}| &= \frac{\hat{\mu}}{m_1} |\vec{r}| = \frac{\hat{\mu} p}{m_1 (1 + e \cos f)} \\ |\vec{r}_2^{\text{CM}}| &= \frac{\hat{\mu}}{m_2} |\vec{r}| = \frac{\hat{\mu} p}{m_2 (1 + e \cos f)} \end{aligned}$$

For a bound system we thus have

$$\begin{aligned} |\vec{r}_1^{\text{CM}}| &= \frac{\frac{\hat{\mu}}{m_1} a (1 - e^2)}{1 + e \cos f} \equiv \frac{a_1 (1 - e^2)}{1 + e \cos f} \\ |\vec{r}_2^{\text{CM}}| &= \frac{\frac{\hat{\mu}}{m_2} a (1 - e^2)}{1 + e \cos f} \equiv \frac{a_2 (1 - e^2)}{1 + e \cos f} \end{aligned}$$

We see from these equations that for a gravitationally bound system, *both* masses move in elliptical orbits with the center of mass in one of the foci (how do you see this?). The semimajor axis of these two masses are given by

$$\begin{aligned} a_1 &= \frac{\hat{\mu} a}{m_1}, \\ a_2 &= \frac{\hat{\mu} a}{m_2}, \\ a &= a_1 + a_2 \end{aligned}$$

where a_1 and a_2 are the semimajor axis of m_1 and m_2 respectively and a is the semimajor axis of the elliptical orbit of one of the masses seen from

the rest frame of the other. Note that the larger the mass of a given body with respect to the other, the smaller the ellipse. This is consistent with our intuition: The more massive body is less affected by the same force than is the less massive body. The Sun moves in an ellipse around the center of mass which is much smaller than the elliptical orbit of the Earth. Figure (5) shows the situation: the planet and the star orbit the common center of mass situated in one common focus of both ellipses.

6 Exercise to be presented on the blackboard: Lagrangian points

The problem of finding the orbits of three masses moving under the influence of their mutual gravitational field is known as ‘the three-body problem’. In its full generality it can only be treated numerically, and the solutions are complicated and very often chaotic. However, there are special cases that can be solved analytically and that are also of great practical interest in astronomy. The most famous cases are perhaps the so-called ‘Lagrangian points’, configurations where the three bodies maintain their relative distances at all times. These solutions can appear in situations where the mass of one of the objects is negligible in comparison with the other two. For a system like that it can be proved that there are five Lagrangian points. In this exercise we will find two of them for the system consisting of the Sun, the Earth, and an artificial satellite. We will make the following assumptions:

1. The mass m_S of the satellite is much smaller than the mass of the Sun (M) and the Earth (m).
2. The mass of the Earth is much smaller than the mass of the Sun.
3. The Earth and the satellite move in circular orbits.
 - a) What does assumption 1 allow us to say about the motion of the Sun and the Earth?
 - b) Which approximation does assumption 2 allow us to make? (Hint: Where will the center of mass be?)

We will first consider the situation where the satellite is located somewhere between the Sun and the Earth. The first Lagrangian point, L1, is the

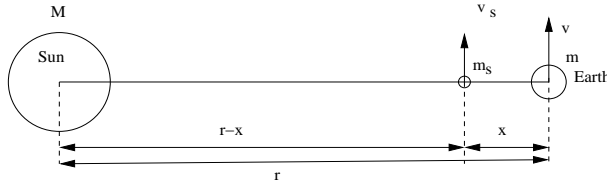


Figure 6: The satellite at the first Lagrangian point, L1, seen from above the orbital plane.

point where the satellite moves so that it is always on the line connecting the Sun and the Earth, see figure 1.

Our goal is to find the distance x between the satellite and the Earth.

- c) Ignore the satellite and show that Kepler's 3rd law under the assumptions 1-3 can be written as

$$T^2 = \frac{4\pi^2}{GM} r^3,$$

where T is the Earth's orbital period.

- d) The satellite moves in a circular path centered on the Sun with speed v_S . Show that

$$v_S^2 = \frac{GM}{r-x} - \frac{Gm(r-x)}{x^2}. \quad (18)$$

- e) Show that we also have

$$v_S^2 = \frac{4\pi^2(r-x)^2}{T_S^2},$$

where T_S is the orbital period of the satellite.

- f) What is the relationship between T and T_S when the satellite is at L1?

- g) Show that equation (18) can be simplified to

$$\frac{1}{(r-x)^3} - \frac{\frac{m}{M}}{x^2(r-x)} = \frac{1}{r^3}. \quad (19)$$

- h) We introduce the notation $y = \frac{m}{M}$. What is the numerical value of y ?

- i) We will now solve equation (19) for x . Show first that it can be rewritten as

$$\frac{1}{\left(1 - \frac{x}{r}\right)^3} - \frac{y}{\left(\frac{x}{r}\right)^2 \left(1 - \frac{x}{r}\right)} = 1.$$

If y were exactly equal to zero, what would the solution of this equation be?

- j) Next, introduce $z = \frac{x}{r}$, assume $z \ll 1$ (why is that a good idea?), and use the approximations

$$\begin{aligned} \frac{1}{(1-z)^3} &\approx 1 + 3z \\ \frac{1}{(1-z)} &\approx 1 + z \end{aligned}$$

to show that

$$3z^3 = y(1+z) \approx y.$$

Show that the satellite is located approximately 1.5 million kilometers from the Earth.

- k) You now have all the tools you need to find the second Lagrangian point, L2, where the satellite is located on the opposite side of the Earth, see figure 7. Do it! You may find the following approximations useful:

$$\begin{aligned} \frac{1}{(1+z)^3} &\approx 1 - 3z \\ \frac{1}{1+z} &\approx 1 - z \end{aligned}$$

- l) What type of satellite-borne observatories would be good to place at L1? What about L2? Look up ‘Lagrangian points’ on Wikipedia, and write a few words about some of the satellite experiments that are currently at L1 and L2 and what they are doing.

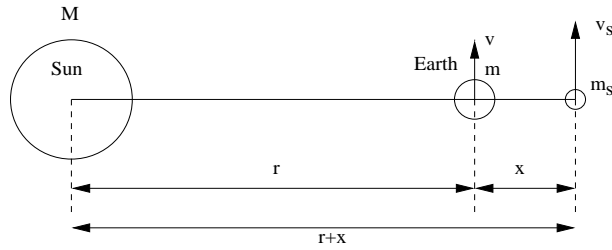


Figure 7: The satellite at the second Lagrangian point, L2, seen from above the orbital plane.

7 Problems

Problem 1 (20-45 min.)

The scope of this problem is to deduce Kepler's second law. Kepler's second law can be written mathematically as

$$\frac{dA}{dt} = \text{constant},$$

i.e. that the area A swept out by the vector \vec{r} per time interval is constant. We will now show this step by step:

1. Show that the infinitesimal area dA swept out by the radius vector \vec{r} for an infinitesimal movement dr and $d\theta$ is $dA = \frac{1}{2}r^2d\theta$.
2. Divide this expression by dt and you obtain an expression for dA/dt in terms of the radius r and the tangential velocity v_θ .
3. By looking back at the above derivations, you will see that the tangential velocity can be expressed as $v_\theta = h/r$.
4. Show Kepler's second law.

Problem 2 (20-45 min.)

The scope of this problem is to deduce Kepler's third law. Again we will solve this problem step by step:

1. In the previous problem we found an expression for dA/dt in terms of a constant. Integrate this equation over a full period P and show that

$$P = \frac{2\pi ab}{h}$$

(hint: the area of an ellipse is given by πab)

2. Use expressions for h and b found in the text to show that

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3 \quad (20)$$

3. This expression obtained from Newtonian dynamics differs in an important way from the original expression obtained empirically by Kepler (equation (1)). How? Why didn't Kepler discover it?

Problem 3 (15-30 min.)

1. How can you measure the mass of a planet in the solar system by observing the motion of one of its satellites? Assume that we know only the semimajor axis and orbital period for the elliptical orbit of the satellite around the planet. **hint 1:** Kepler's third law (the exact version). **hint 2:** You are allowed to make reasonable approximations.
2. Look up (using Internet or other sources) the semimajor axis and orbital period of Jupiter's moon Ganymede.
 - (a) Use these numbers to estimate the mass of Jupiter.
 - (b) Then look up the mass of Jupiter. How well did your estimate fit? Is this an accurate method for computing planetary masses?
 - (c) Which effects could cause discrepancies from the real value and your estimated value?

Problem 4 (70-90 min.)

1. Show that the total energy of the two-body system in the center of mass frame can be written as

$$E = \frac{1}{2}\hat{\mu}v^2 - \frac{GM\hat{\mu}}{r},$$

where $v = |d\vec{r}/dt|$ is the relative velocity between the two objects, $r = |\vec{r}|$ is their relative distance, $\hat{\mu}$ is the reduced mass and $M \equiv m_1 + m_2$ is the total mass. **hint:** make the calculation in the center of mass frame and use equation (15) and (16).

2. Show that the total angular momentum of the system in the center of mass frame can be written

$$\vec{P} = \hat{\mu}\vec{r} \times \vec{v},$$

3. Looking at the two expressions you have found for energy and angular momentum of the system seen from the center of mass frame: Can you find an equivalent two-body problem with two masses m'_1 and m'_2 where the equations for energy and momentum will be of the same form as the two equations which you have just derived? What are m'_1 and m'_2 ?

Problem 5 (optional 30-45 min.)

1. At which points in the elliptical orbit (for which angles f) is the velocity of a planet at maximum or minimum?
2. Using only the mass of the Sun, the semimajor axis and eccentricity of Earth's orbit (which you look up in Internet or elsewhere), can you find an estimate of Earth's velocity at aphelion and perihelion?
3. Look up the real maximum and minimum velocities of the Earth's velocity. How well do they compare to your estimate? What could cause discrepancies between your estimated values and the real values?
4. Use Python (or Matlab or any other programming language) to plot the variation in Earth's velocity during one year.

hint 1: Use one or some of the expressions for velocity found in section (3) as well as expressions for p and h found in later sections (including the above problems). **hint 2:** You are allowed to make reasonable approximations.

Problem 6 (optional 10-30 min.)

1. Find our maximum and minimum distance to the center of mass of the Earth-Sun system.
2. Find Sun's maximum and minimum distance to the center of mass of the Earth-Sun system.
3. How large are the latter distances compared to the radius of the Sun?

Problem 7: Numerical solution to the 2-body/3-body problem

In this problem you are first going to solve the 2-body problem numerically by a well-known numerical method. We will start by considering the ESA satellite Mars Express which entered an orbit around Mars in December 2003 (http://www.esa.int/esaMI/Mars_Express/index.html). The goal of Mars Express is to map the surface of Mars with high resolution images. When Mars Express is at an altitude of 10107km above the surface of Mars with a velocity 1166 m/s (with respect to Mars), the engines are turned off and the satellite has entered the orbit.

In this exercise we will use that the radius of Mars is 3400 km and the mass of Mars is 6.4×10^{23} kg. Assume the weight of the Mars Express spacecraft to be 1 ton.

1. **(4-5 hours)** A distance of 10107km is far too large in order to obtain high resolution images of the surface. Thus, the orbit of Mars Express need to be very eccentric such that it is very close to the surface of Mars each time it reaches perihelion. We will now check this by calculating the orbit of Mars Express numerically. We will introduce a fixed Cartesian coordinate system to describe the motion of Mars and the satellite. Assume that at time $t = 0$ Mars has position $[x_1 = 0, y_1 = 0, z_1 = 0]$ and Mars express has position $[x_2 = 10107+3400 \text{ km}, y_2 = 0, z_2 = 0]$ in this fixed coordinate system (see Figure 8).

The velocity of Mars express is only in the positive y-direction at this moment. In our fixed coordinate system, the initial velocity vectors are therefore $\vec{v}_1 = 0$ (for Mars) and $\vec{v}_2 = 1166 \frac{m}{s} \vec{j}$ (for Mars Express), where \vec{j} is the unitvector along the y-axis. There is no velocity component in

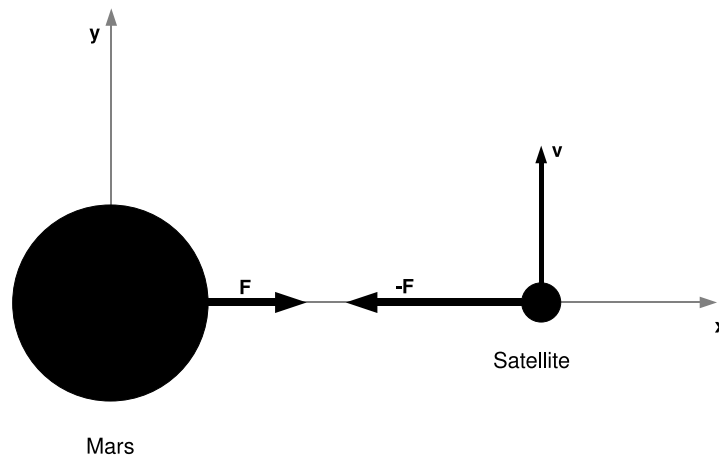


Figure 8: Mars and Mars Express at time $t = 0$.

the z -direction so we can consider the system as a 2-dimensional system with movement in the (x,y) -plane.

Use Newton's second law,

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F},$$

to solve the 2-body problem numerically. Use Euler's method for differential equations. Plot the trajectory of Mars Express. Do 10^5 calculations with timestep $dt = 1$ second. Is the result what you would expect?

Hints - Write Newton's second law in terms of the velocity vector.

$$m \frac{d\vec{v}}{dt} = \vec{F} \Rightarrow m \left(\frac{dv_x}{dt} \vec{i} + \frac{dv_y}{dt} \vec{j} \right) = F_x \vec{i} + F_y \vec{j}$$

Then we have the following relation between the change in the components of the velocity vector and the components of the force vector;

$$\frac{dv_x}{dt} = \frac{F_x}{m}$$

$$\frac{dv_y}{dt} = \frac{F_y}{m}$$

These equations can be solved directly by Euler's method and the given initial conditions. For each timestep (use a for- or while-loop), calculate the velocity $v_{x/y}(t+dt)$ (Euler's method) and the position $x(t+dt)$, $y(t+dt)$ (standard kinematics) for Mars and the space craft. A good approach is to make a function that calculates the gravitational force. From the loop you send the previous timestep position data (4 values) to the function which calculates and returns the components of the force (4 values, two for Mars and two for the space craft) with correct negative/positive sign. Collect the values in arrays and use the standard python/scitools plot-command.

2. **(60-90 min)** Mars Express contained a small lander unit called Beagle 2. Unfortunately contact was lost with Beagle 2 just after it should have reached the surface. Here we will calculate the path that Beagle 2 takes down to the surface (this is not the real path that was taken). We will assume that the lander does not have any engines and is thus moving under the influence of only two forces: the force of gravity from Mars and the force of friction from the Martian atmosphere. The friction will continuously lower the altitude of the orbit until the lander hits the surface of the planet. We will assume the weight of the lander to be 100 kg.

We will now assume that Mars Express launches Beagle 2 when Mars Express is at perihelion. We will assume that it adjusts the velocity of Beagle 2 such that it has a velocity of 4000 m/s (with respect to Mars) at this point. Thus we have 2-body problem as in the previous exercise. At $t = 0$, the position of Mars and the lander is $[x_1 = 0, y_1 = 0, z_1 = 0]$ and $[x_2 = -298-3400 \text{ km}, y_2 = 0, z_2 = 0]$ respectively (Figure 9). The initial velocity vector of the lander is $\vec{v}_1 = -4000 \frac{m}{s} \vec{j}$ with respect to Mars. Due to Mars' atmosphere a force of friction acts on the lander which is always in the direction opposite to the velocity vector. A simple model of this force is given by

$$\vec{f} = -k\vec{v},$$

where $k = 0.00016 \text{ kg/s}$ is the friction constant due to Mars' atmosphere. We will assume this to have the same value for the full orbit.

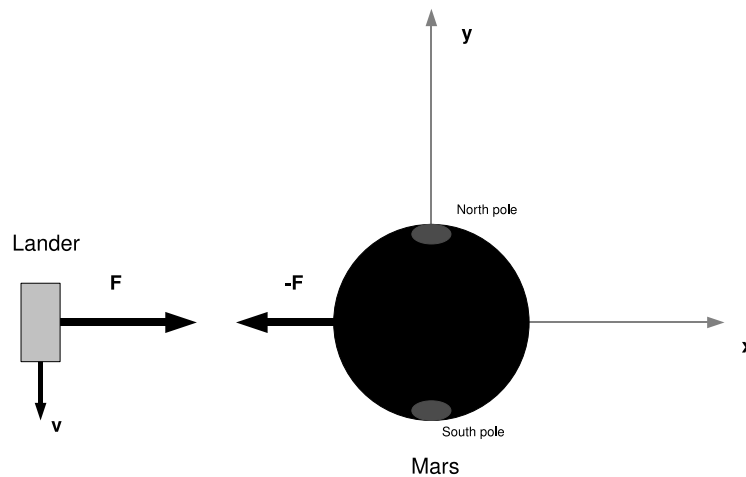


Figure 9: Mars and Beagle 2 at time $t = 0$.

Plot the trajectory that the lander takes down to the surface of Mars. Set $dt = 1$ second.

Hints - You can use most of the code from the previous exercise. First, we write Newton's second law in terms of the cartesian components;

$$\frac{dv_x}{dt} = \frac{F_x + f_x}{m}$$

$$\frac{dv_y}{dt} = \frac{F_y + f_y}{m}$$

The best approach is to make one more function that calculates the force of friction with the lander's velocity components as arguments. In this problem you should use a while-loop. For each evaluation, first call the gravitational function (as before) and then the friction function. Remember to send the space craft's velocity components from the previous timestep. In the friction function, first calculate the total force f , and then the components f_x and f_y (use simple trigonometry)

with correct positive/negative-sign by checking the sign of the velocity components. Then return the force components to the loop. For each evaluation (in the while-loop) check whether the spaceship has landed or not.

3. Use the trajectory of the previous exercise to check the landing site: Was the lander supposed to study the ice of the Martian poles or the rocks at the Martian equator? Use figure 9 to identify the position of the poles with respect to the geometry of the problem (the result does not have any relation with the objectives or landing site of the real Beagle 2 space craft)
4. **(90-120 min.)** Finally, we will use our code to study the 3-body problem. There is no analytical solution to the 3-body problem, so in this case we are forced to use numerical calculations. The fact that most problems in astrophysics consider systems with a huge number of objects strongly underlines the fact that numerical solutions are of great importance.

About half of all the stars are binary stars, two stars orbiting a common center of mass. Binary star systems may also have planets orbiting the two stars. Here we will look at one of many possible shapes of orbits of such planets. We will consider a planet with the mass identical to the mass of Mars. One of the stars has a mass identical to the mass of the Sun (2×10^{30} kg), the other has a mass 4 times that of the Sun.

The initial positions are $[x_1 = -1.5 \text{ AU}, y_1 = 0, z_1 = 0]$ (for the planet), $[x_2 = 0, y_2 = 0, z_2 = 0]$ (for the small star) and $[x_3 = 3 \text{ AU}, y_3 = 0, z_3 = 0]$ (for the large star) (Figure 3). The initial velocity vectors are $\vec{v}_1 = -1 \frac{km}{s} \vec{j}$ (for the planet), $\vec{v}_2 = 30 \frac{km}{s} \vec{j}$ (for the small star) and $\vec{v}_3 = -7.5 \frac{km}{s} \vec{j}$ (for the large star).

Plot the orbit of the planet and the two stars in the same figure. Use timestep $dt = 400$ seconds and make 10^6 calculations. It should now be clear why it is impossible to find an analytical solution to the 3-body problem. Note that the solution is an approximation. If you try to change the size and number of time steps you will get slightly different orbits, small time steps cause numerical problems and large time steps is too inaccurate. The given time step is a good trade-off between the

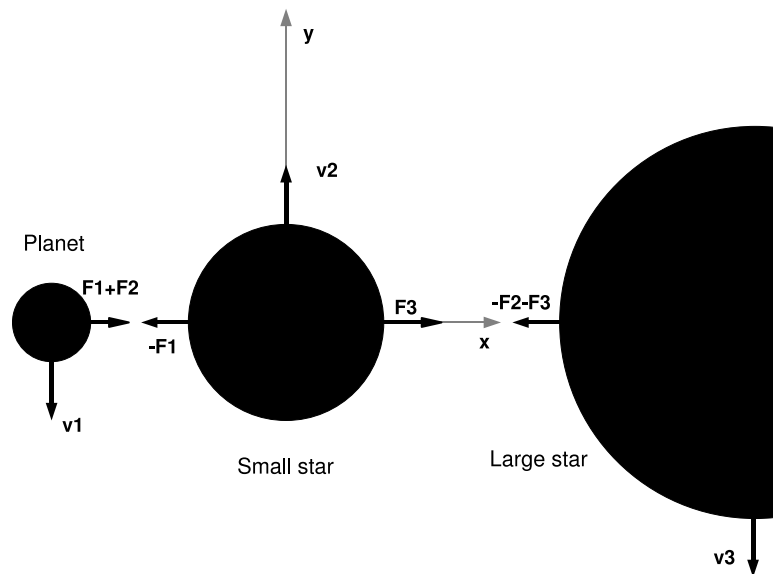


Figure 10: The binary star system with the planet at time $t = 0$.

two problems but does not give a very accurate solution. Accurate methods to solve this problem is outside the scope of this course. Play around and try some other starting positions and/or velocities.

Hints - There is really not much more code you need to add to the previous code to solve this problem. Declare arrays and constants for the three objects. In your for-/while-loop, calculate the total force components for each object. Since we have a 3-body problem we get two contributions to the total force for each object. In other words, you will have to call the function of gravitation three times for each time-evaluation. For each time step, first calculate the force components between the planet and the small star, then the force components between the planet and the large star, and finally the force components between the small and the large star. Then you sum up the contributions that belong to each object.

5. Look at the trajectory and try to imagine how the sky will look like at different epochs. If we assume that the planet has chemical conditions

for life equal to those on earth, do you think it is probable that life will evolve on this planet? Use your tracjectory to give arguments.

8 List of symbols

- P is the orbital period of a planet or star.
- a is the semimajor axis of an ellipse (see fig. 4). It is also used as a constant in describing the properties of the parabola and hyperbola. For circles, a is just the radius.
- b is the semiminor axis of an ellipse (see fig. 4).
- m_1 and m_2 are the masses of the two bodies in the two-body problem.
- \vec{r}_1 is the vector showing the position of mass m_1 from a given origin.
- \vec{r}_2 is the vector showing the position of mass m_2 from a given origin.
- \vec{r}_1^{CM} and \vec{r}_2^{CM} are the vectors \vec{r}_1 and \vec{r}_2 taken with the origin in the center of mass.
- \vec{R} is the vector pointing at the position of the center of mass from a given origin.
- \vec{r} is the vector $\vec{r} = \vec{r}_2 - \vec{r}_1$ pointing from mass m_1 to mass m_2 .
- $r = |\vec{r}|$.
- G is Newton's gravitational constant.
- \vec{F}_1 and \vec{F}_2 the gravitational forces on mass m_1 and m_2 .
- $\dot{\vec{r}}$ and $\ddot{\vec{r}}$ are the first and second time derivatives of the vector \vec{r} .
- $m = G(m_1 + m_2)$.
- \vec{e}_r is the unit vector pointing in the direction of \vec{r} , $\vec{r} = r\vec{e}_r$. The direction of the vector is time dependent (because \vec{r} is time dependent).
- \vec{e}_θ is the unit vector perpendicular to \vec{e}_r . The direction of the vector is time dependent (because \vec{e}_r is time dependent).

- \vec{e}_x and \vec{e}_y are unit vectors in a fixed cartesian coordinate system.
- θ is the angle between the x-axis in an arbitrarily defined coordinate system with unit vector \vec{e}_x and the direction vector from m_1 to m_2 given by \vec{e}_r .
- h is the spin per mass, i.e. $(\vec{r} \times \vec{p})/m_2$
- $r(\theta)$ is the distance $r = |\vec{r}|$ as a function of angle θ
- $u(\theta) = 1/r(\theta)$
- A is a constant in the solution of the equation of motion. It's value needs to be determined in each case depending on the initial distance between m_1 and m_2 as well as the initial velocities of these masses.
- ω is a constant in the solution of the equation of motion. It's value needs to be determined in each case depending on the initial direction of m_2 with respect to m_1 in a given coordinate system.
- $f = \theta + \omega$ is the angle between the line connecting m_1 and the point in the orbit where m_1 and m_2 are closest (called perihelion for the ellipse) and the vector \vec{r} .
- p is the variable describing a conic section. It is given by three different expressions depending on the conic section involved (see section 3).
- e is a constant entering the expression for conic sections. For an ellipse this is the eccentricity given by $e = \sqrt{1 - (b/a)^2}$. A circle has $e = 0$.
- v_r and v_θ are the components of the velocity vector along the unit vectors \vec{e}_r and \vec{e}_θ respectively.
- M is the sum of all masses in a system of bodies.
- \vec{f}_{ij} is the force from mass j on mass i .
- \vec{F}_{ext} is the sum of all external forces on a system of masses.
- $\hat{\mu}$ is the reduced mass $\hat{\mu} = (m_1 m_2)/(m_1 + m_2)$.
- a_1 and a_2 are the semimajor axes of the masses m_1 and m_2 about the common center of mass.