AST1100 Lecture Notes

13 - 14 Stars and stellar birth:

1 The Hertzsprung-Russell diagram revisited

We have already encountered the Hertzsprung-Russell (HR) diagram, the diagram where stars are plotted according to their temperature and luminosity. There are several versions of this diagram, differing mainly in the units plotted on the axes. The most used units on the x-axis are:

- Temperature
- B-V color index
- spectral classes

We have so far seen temperature on the x-axis. The temperature of a star is directly related to its color and one can therefore also use the $B - V$ color index (see the lecture on cosmic distances) on the x-axis. There is also another another possibility: *spectral classes*. Stars are classified according to their spectral class which consists of a letter and a number. This historical classification is based on the strength of different spectral lines found in the spectra of the stars. It turned out later that these spectral classes are strongly related to the temperature of the star: The temperature of the star determines the state of the different atoms and therefore the possible spectral lines which can be created.

The letters used in the spectral classification are, in the order of decreasing temperature, O, B, A, F, G, K, M. The warmest O stars have surface temperatures around 40 000K, the coldest M stars have surface temperatures down to about 2 500K. Each of these classes are divided into 10 subclasses using a number from 0 to 9. So the warmest F stars are called $F0$ and the coldest F stars are called $F9$.

Normally observational astronomers tend to use either spectral class or color index which are quantities related to the observed properties of the star. Theoretical astrophysicists on the other hand, tend to use temperature which is more important when describing the physics of the star.

Also the y-axis in an HR-diagram have different units. We have already seen luminosity and absolute magnitude which are two closely related quantities. In addition one can use *luminosity classes*. It turns out that stars which have the same spectral class but different luminosities also have some small differences in the spectral lines. These differences have been shown to depend on the luminosity of the star. There are 6 luminosity classes, numbered with Roman numerals from I to VI. The most luminous stars have luminosity class I. Using this classification, the Sun is a G2V star.

Before we start to discuss the diagram in more detail, let us try to understand what it is telling us. We know that the flux of a star with temperature T can be expressed using the Stefan-Boltzmann law as $F = \sigma T^4$. To obtain the luminosity L, we need to integrate this flux over the full area $4\pi R^2$ of the surface of the star giving (why?, check that you understand this!),

$$L = 4\pi R^2 \sigma T^4.$$

Looking at the HR-diagram (see figure 1), we see that there are some spectral classes for which there are stars with many different luminosities. For instance stars with spectral class K0 have a range in luminosity from 0.5 to 1000 solar luminosities. If we fix T in the relation above (remember: fixed T means fixed spectral class) , we see than higher luminosity simply means larger radius. So for a fixed temperature, the higher the star is located in the HR-diagram the larger radius it has. This also means that we can find lines of constant radius in the diagram. Fixing the radius to a constant we get

$$R^2 = \frac{L}{4\pi\sigma T^4} = \text{constant},$$

so that for stars located along lines following $L \propto T^4$ in the diagram, the radius is the same. In figure 2 some of these lines have been plotted. Note that these lines go from the upper left to the lower right, a bit similar to the main sequence. So main sequence stars are stars which have a certain range of radii. The fact that most of the stars are located on the main sequence means that the physics of stars somehow prohibits smaller and larger radii (look at the figure again and check that you understand) . We will come to this in some more detail later.

Now it is clear why the stars which are situated above the main sequence are called giants or super giants and the stars well below the main sequence are called dwarfs. Main sequence stars usually have radii in the range $0.1R_\odot$

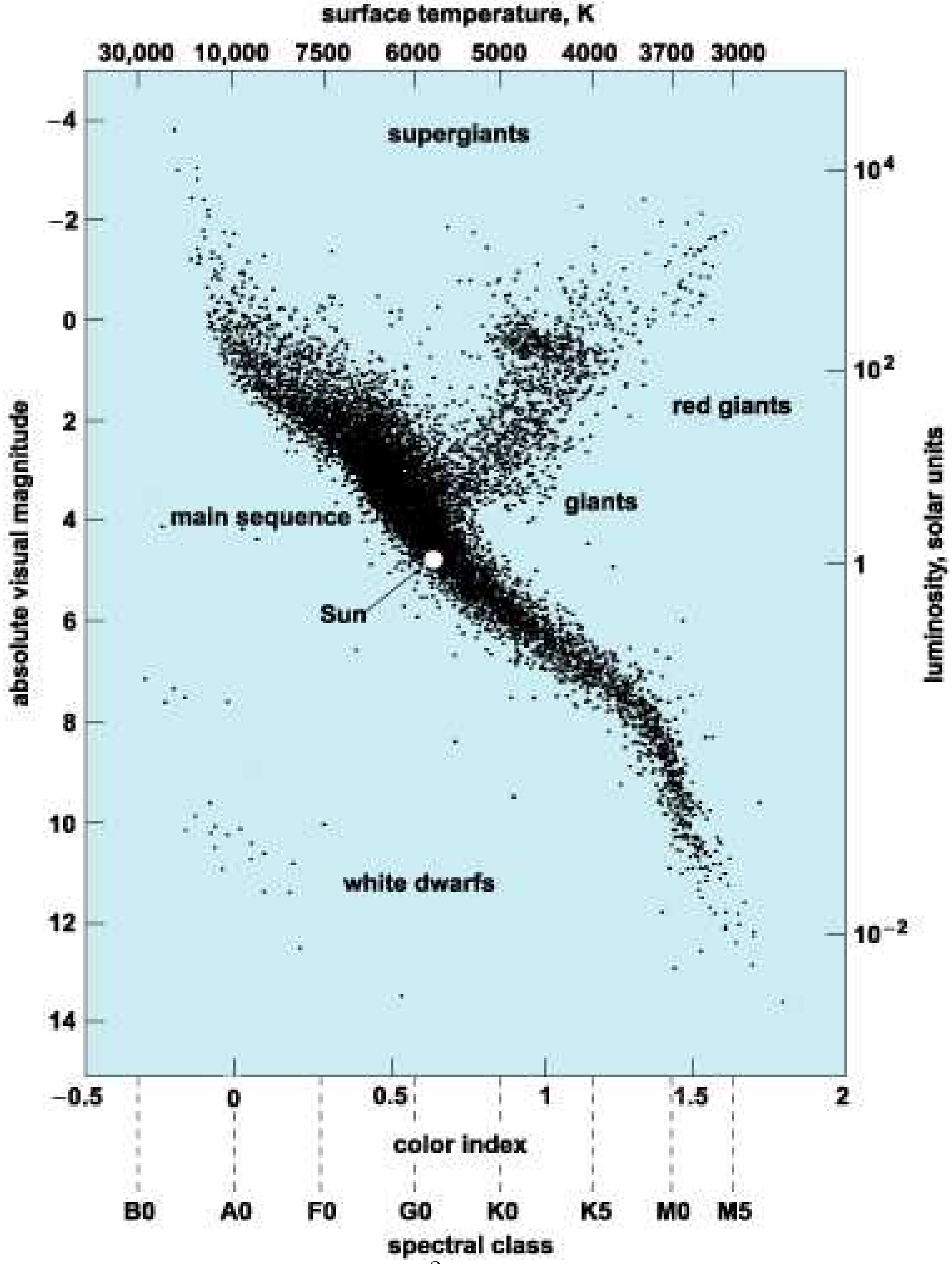

Figure 1: HR-diagram. From http://www.answers.com/topic/hertzsprung-russell-diagram?cat=technology

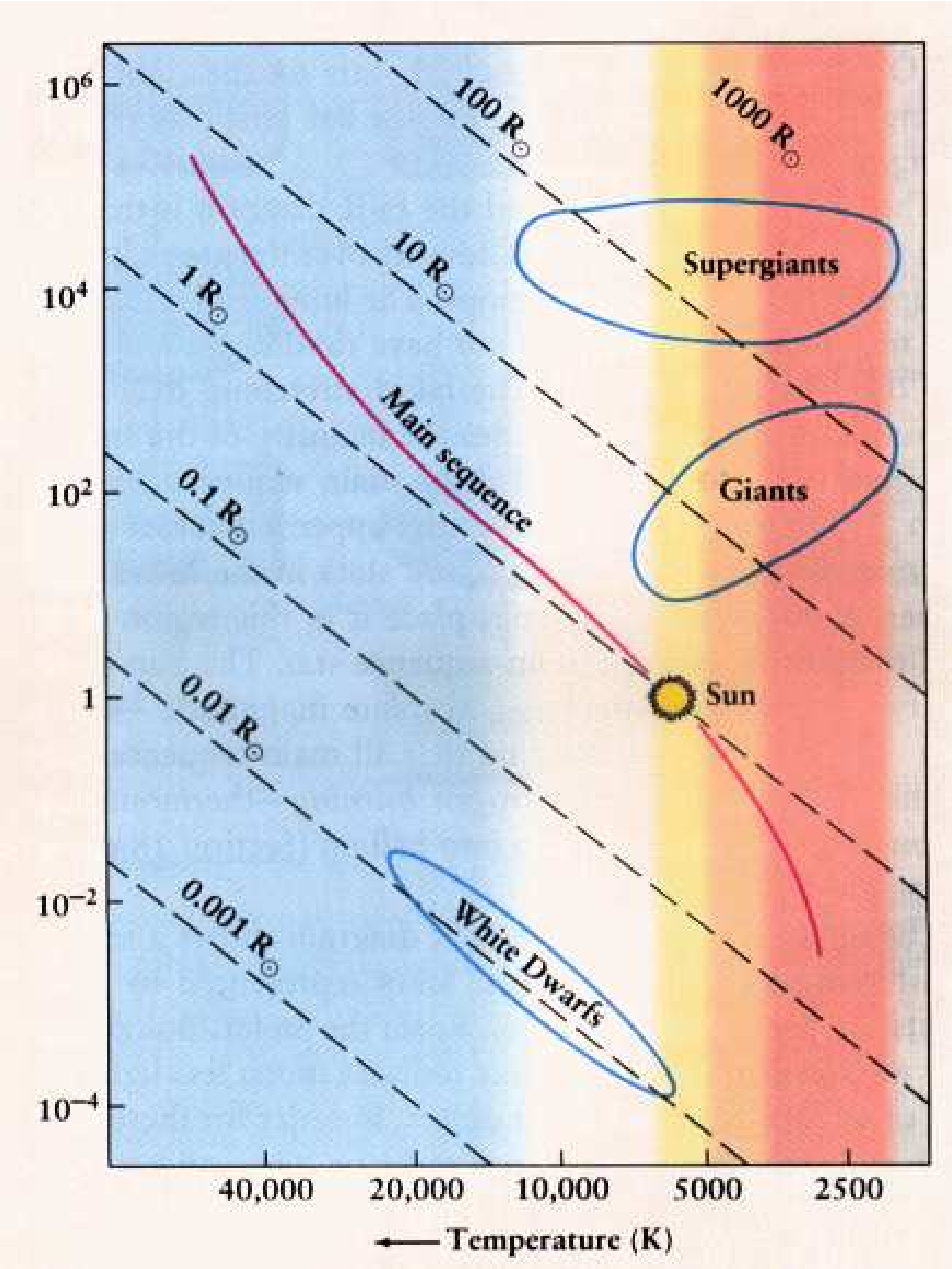

Figure 2: HR-diagram with constant radii lines plotted. From http://astro.wsu.edu/worthey/astro/html/im-Galaxy/

to about $10R_\odot$. Giant stars fall in the range between $10R_\odot$ to about $100R_\odot$ whereas super giants may have radii of several 100 solar radii. The masses of stars range from $0.08M_\odot$ for the least massive stars up to about $100M_\odot$ for the most massive stars. We will later discuss theoretical arguments explaining why there is a lower and an upper limit of star masses.

We will now start to look at the evolution of stars, from birth to death. Stars start out as huge clouds of gas contracting due to their own gravity. Thus a star starts out on the far right side of the HR-diagram, with a very low temperature. Then, as it contracts, the radius decreases and the temperature increases. It moves leftwards and finally after nuclear reactions have begun, the star settles on the main sequence. Where it settles on the main sequence depends on the mass of the star. As we will show later, the larger the mass, the higher the luminosity and the higher the surface temperature. So the more massive stars settles on the left side of the main sequence whereas the less massive stars settles on the right side of the main sequence. Stars spend the largest part of their lives on the main sequence. During the time on the main sequence they move little in the HR-diagram. Towards the end of their lives, when the hydrogen in the core has been exhausted, the stars increase their radii several times becoming giants or supergiants. The surface temperature goes down, but due to the enormous increase in radius the luminosity increases. After a short time as a giant, the star dies: Low mass stars die silently, blowing off the outer layers and leaving behind a small white dwarf star in the lower part of the HR-diagram. The more massive stars die violently in a supernova explosion leaving behind a so-called neutron star or a black hole. We will now discuss the physics behind each of these steps in turn. Beginning here with star birth: a gas of cloud contracting.

2 The Jeans criterion

A star forms from a cloud of gas, a so-called *molecular cloud*, undergoing gravitational collapse. These molecular clouds consist mainly of atomic and molecular hydrogen, but also contain dust and even more complex organic molecules. The question is whether a cloud will start collapsing or not. In the lectures on the virial theorem we saw that the condition for stability is $2K + U = 0$. If the kinetic energy is larger compared to the potential energy, the system does not stabilize, the gas pressure is larger than the gravitational forces and the cloud expands. On the other hand, if the potential energy is dominating, the cloud is gravitationally bound and undergoes collapse. For a cloud to collapse we thus have the condition (why?),

$$2K < |U|.$$

In the lectures on the virial theorem, we found an expression for the potential energy of the cloud:

$$U \propto \frac{3GM^2}{5R},$$

where M is the mass of the cloud and R is the radius. From thermodynamics, we learn that the kinetic energy of a gas is given by

$$K = \frac{3}{2}NkT,$$

where N is the number of particles in the gas, k is the Boltzmann constant and T is the temperature. We can write N as

$$N = \frac{M}{\mu m_H}, \tag{1}$$

where $\overline{m} = \mu m_H$ is the mean mass per gas particle. The *mean molecular weight*

$$\mu = \frac{\overline{m}}{m_H},$$

is simply the mean mass per particle measured in units of the hydrogen mass m_H (check now that expression 1 for N makes sense for you! This is important!). So the condition $2K < |U|$ becomes simply

$$\frac{3MkT}{\mu m_H} < \frac{3GM^2}{5R}.$$

We can write this as a criterion on the mass

$$M > \frac{5kT}{G\mu m_H}R.$$

This minimum mass is called the *Jeans mass* M_J which we can write in terms of the mean density of the cloud as

$$M_J = \left(\frac{5kT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2},$$

where we used $\rho = M/(4/3\pi R^3)$ assuming constant density throughout the cloud. Thus, clouds with a larger mass than the Jeans mass $M > M_J$ will have $2K < |U|$ and therefore start a gravitational collapse. We can also write this in terms of a criterion on the radius of the cloud. Using again the expression for the density we have the *Jeans length* (check again that you can deduce this expression from the expression above)

$$R_J = \left(\frac{15kT}{4\pi G\mu m_H \rho}\right)^{1/2}.$$

A cloud with a larger radius than the Jeans length $R > R_J$ will undergo gravitational collapse. The Jeans criterion for the collapse of a cloud is a good approximation in the absence of rotation, turbulence and magnetic fields. In reality however, all these factors do contribute and far more complicated considerations are needed in order to calculate the exact criterion.

The collapsing cloud will initially be in free fall, a period when the photons generated by the converted potential energy are radiated away without heating the cloud (the density of the cloud is so low that the photons can easily escape without colliding with the atoms/molecules in the gas). The initial temperature of the cloud of about $T = 10K - 100K$ will not increase. After about one million years, the density of the cloud has increased and the photons cannot easily escape. They start heating the cloud and potential energy is now radiated away as thermal radiation. In the lectures on the virial theorem we made an approximate calculation of the time it would take the Sun to collapse to its present size assuming a constant luminosity. We found a collapse time of about 10 million years. Proper calculations show that this process would take about 40 million years for a star similar to the Sun. The contracting star is called a *protostar*.

When the core of the collapsing protostar has reached sufficiently high temperatures, thermonuclear fusion begins in the center. The luminosity starts to get dominated by the energy produced by nuclear fusion rather than converted potential energy from the gravitational collapse. The protostar keeps contracting until hydrostatic equilibrium is reached and the star has entered the main sequence.

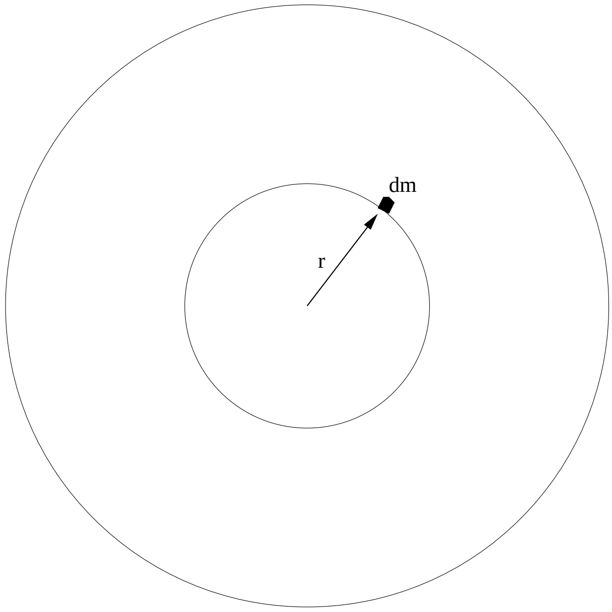

Figure 3: The mass element dm inside a main sequence star is not moving.

3 Settling on the main sequence: Hydrostatic equilibrium

In figure 3 we show a mass element with mass dm inside a star at a radius r from the center. We know that gravity pulls this element towards the center. But a main sequence star does not change its radius with time, so there must be a force working in the opposite direction keeping this mass element stable at radius r. This force is the pressure. In a main sequence star, the pressure forces must exactly equal the force of gravity, otherwise the star would change its radius. This fact, called *hydrostatic equilibrium*, gives us an invaluable source of information about a star's interior. We can't observe the interior of a star directly, but the equation of hydrostatic equilibrium together with other thermodynamic relations combined with observations of the star's surface allow detailed computer modeling of the interior of stars. Here we will deduce this important equation.

In figure 4 we have zoomed in on the mass element dm. Because of the symmetry of the problem (the fact that gravitation only works radially), we

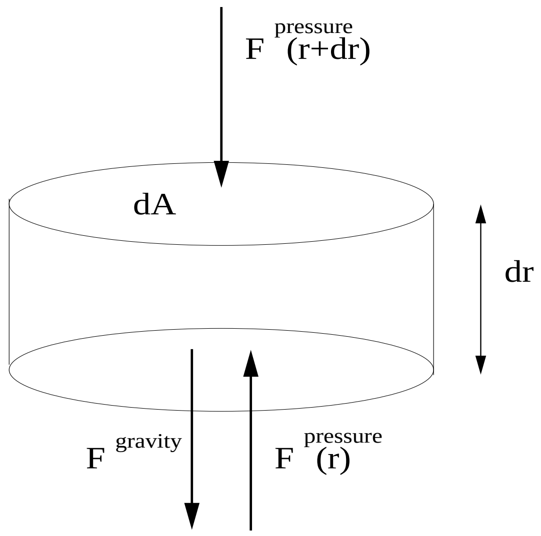

Figure 4: The mass element dm inside a main sequence star is not moving: The forces add to zero.

can assume spherical symmetry, i.e. that density, pressure and temperature are all only a function only of the distance r from the center. We show the forces of pressure pushing on the mass element from above and below, as well as the force of gravity. Assuming that the element is infinitesimally small, there are no gravitational forces pushing on the sides and the pressure forces on the sides will be equal since the distance r from the center is the same on both sides. The forces on the sides must therefore sum up to zero. We will now look at a possible radial movement of the mass element. Newton's second law on the mass element gives

$$dm\frac{d^2r}{dt^2} = -F^{\text{grav}} - F^{\text{pressure}}(r+dr) + F^{\text{pressure}}(r),$$

where all forces are defined to be positive. The minus sign on the two first forces show that they push towards the center in negative r direction. The area of the upper and lower sides of the element is dA. Pressure is defined as force per area, so

$$P = \frac{F^{\text{pressure}}}{dA},$$

giving

$$dm\frac{d^2r}{dt^2} = -G\frac{M(r)dm}{r^2} - P(r+dr)dA + P(r)dA,$$

(check that you understand where each term comes from here) where $M(r)$ is the total mass inside radius r:

$$M(r) = \int_0^r dr' 4\pi (r')^2 \rho(r') \tag{2}$$

The infinitesimal difference in pressure between r and $r+dr$ is $dP = P(r+dr) - P(r)$. We have

$$\frac{dm}{dA}\frac{d^2r}{dt^2} = -\frac{dm}{dA}\frac{GM(r)}{r^2} - dP$$

We write the mass of the element as the density $\rho(r)$ at radius r times the volume $dAdr$ of the mass element $dm = \rho dAdr$. Dividing by dr on both sides give

$$\rho\frac{d^2r}{dt^2} = -G\frac{\rho M(r)}{r^2} - \frac{dP}{dr}.$$

(did you understand all parts of the deduction?) For a main sequence star, the radius is not changing so the mass element cannot have any acceleration in r direction giving $d^2r/dt^2 = 0$. This gives the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho(r)g(r),$$

where $g(r)$ is the local gravitational acceleration

$$g(r) = G\frac{M(r)}{r^2}.$$

The equation of hydrostatic equilibrium tells us how the pressure $P(r)$ must change as a function of radius in order for the star to be stable. In the following we will study what kind of pressure we might experience inside a star and which effect it has.

From thermodynamics we learn that the gas pressure in an *ideal gas* can be written as

$$P = \frac{\rho kT}{\mu m_H}.$$

An ideal gas is a gas where the atoms or molecules of which the gas consists do not interact with each other. This is not the case in real gases but often a good approximation. In the stellar interior, there is a high density of photons traveling in all possible directions. The photons behave like the atoms or molecules in a gas. So we may consider the collection of photons as a *photon gas*. This photon gas also has a pressure in the same way as a normal gas has. Thermodynamics tells as that the pressure of a photon gas is given by

$$P = \frac{1}{3}aT^4,$$

where $a = 7.56 \times 10^{-16}$ J/m^{3}K^4 is the *radiation constant*.

4 Problems

Problem 1 (10 - 20 min.) Look at the HR-diagram in figure 1. Assume that you observe a main sequence star with spectral class G0. The apparent magnitude of the star is $m = 1$.

1. Roughly what luminosity and absolute magnitude would you expect the star to have? (use the diagram)
2. Using this result, can you give a rough approximation of the distance?
3. Looking again at the HR-diagram. Roughly what is the minimum and maximum absolute magnitude you would expect the star to have?
4. What is the range of distances the star could have?

This method for measuring distances is called *spectroscopic parallax* (although it has nothing to do with normal parallax). I have not included this method in the lectures on distance measurements. From the answer to the last question you will understand why it is not a very exact method.

Problem 2 (30 - 45 min.) A Giant Molecular Cloud (GMC) has typically a temperature of $T = 10K$ and a density of about $\rho = 3 \times 10^{-17}$kg/m^3. A GMC has been observed at a distance of $r = 200$pc. It's angular extension on the sky is $3.5'$. Assume the cloud to be spherical with uniform density.

1. What is the actual radius of the cloud?

2. What is the mass of the cloud?

3. Is the mass larger than the Jeans mass? Is the cloud about to collapse and form a protostar?

4. A supernova explodes in the vicinity of the star emitting a pressure wave which passes through the cloud. If an external pressure is pushing the cloud together, could this possibly lead to a decrease in the minimum mass required for collapse (give arguments in terms of K and U)? Argue why a decrease in minimum mass is more probable than an increase (hint: does K really increase for all particles when you compress the cloud?)

5. Could the supernova thus have contributed to the collapse of a cloud which has a mass less than the Jeans mass?

6. The galaxy has a fairly uniform distribution of hydrogen in the galactic disc. If a pressure wave is moving around the center of the disc in a spiral like shape, would this explain why we observe galaxies as spirals and not as a disc?

Problem 3 (2 - 3 hours) We will now assume a very simple model of the Sun in order to show how one can use the equation of hydrostatic equilibrium to understand stellar interiors and the nuclear reactions taking place in the stellar cores. We will assume that the density of the Sun $\rho = \rho_0$ is uniform throughout.

1. Find an expression for the total mass $M(r)$ inside a radius r.

2. We will now assume that the only pressure in the Sun is the gas pressure from an ideal gas. We ignore the radiation pressure. Insert this expression for $M(r)$ into the equation of hydrostatic equilibrium and show that it can be written as
$$\frac{dT}{dr} = -\frac{4\pi}{3} G \rho_0 r \frac{\mu m_H}{k}$$

3. Integrate this equation from the core at $r = 0$ to the surface of the Sun at $r = R$ and show that the temperature T_c in the core of the Sun can be written
$$T_C = T(R) + \frac{2\pi}{3} G R^2 \rho_0 \frac{\mu m_H}{k}.$$

4. Assume that the Sun consists entirely of protons with a mass of 1.67×10^{-27} kg. Use the solar mass of 2×10^{30} kg, the solar radius of 700 000 km and the surface temperature of the Sun $T = 5780K$ to obtain the density ρ_0 and thereby the core temperature T_C. (By doing this calculation properly taking into account variations of the density with distance from the core, one obtains a core temperature of about 15 million Kelvin)

5. In the coming lectures, we will learn that hydrogen can fuse to Helium by two different processes, the pp-chain and the CNO-cycle. The pp chain is more efficient at temperatures below 20 million Kelvin whereas the CNO-cycle starts dominating at temperatures above 20 million Kelvin. Use your result for the core temperature of the Sun to decide which of these processes produces most of the energy in the Sun.

6. Write ρ_0 in terms of the mass M and the radius R of the Sun. We have seen that the surface temperature of the Sun is much smaller than the core temperature and might therefore be neglected. Show that the core temperature of a star depends on the mass and radius as

$$T_C \propto \frac{M}{R}$$

7. In later lectures we will discuss in detail the evolution of a star. We will learn that when the Hydrogen in the core of a star has been exhausted, the nuclear fusion processes cease. In this case the pressure forces cannot sustain the force of gravity and the radius of the core starts shrinking. It will continue shrinking until some other force can oppose the force of gravity. If Helium, an element which is now found in large abundances in the core, starts to fuse to heavier elements this would create a photon pressure high enough to sustain gravity. A temperature of at least 100 million degrees Kelvin is needed in order for this fusion process to start. By how much does the core radius of the Sun need to shrink in order for Helium fusion to start?

8. In the last case, the radiation pressure is giving the dominant contribution to the forces of pressure. Show that in this case, the temperature

of the core can be written as

$$T_C = \left(T(R)^4 + \frac{2\pi G}{a}\rho_0^2 R^2\right)^{1/4},$$

again assuming a constant density.

Problem 4 (2 - 3 hours) We will now assume a slightly more realistic model of the Sun. Assume that the density of the Sun as a function of distance r from the core can be written as

$$\rho(r) = \frac{\rho_C}{1 + (r/R)^2},$$

where ρ_C is the density in the core of the Sun and R is the radius at which the density has fallen by a factor 1/2 (check this by inserting $r = R$ in the expression). In this exercise we will use our knowledge about the minimum temperature which is needed to obtain nuclear reactions in order to calculate the density in the solar core.

1. We will now find an expression for the total mass $M(r)$ inside a radius r using this density profile. In order to perform the integral in equation (2) we make the substitution $x = r/R$ and integrate over x instead of r. Show that $M(r)$ can be written

$$M(r) = 4\pi\rho_C R^3 \int_0^{r/R} dx \frac{x^2}{1 + x^2}$$

2. In order to perform such integrals, the Mathematica package is very useful. Not everybody has access to Mathematica, but a free web interface exists for performing integrals. Go to
http://integrals.wolfram.com/index.jsp
Type
$x^2/(1 + x^2)$,
and you get a nice and easy answer. Using this result, together with the assumption of pure ideal gas pressure, show that the equation of hydrostatic equilibrium can now be written

$$\frac{d}{dr}(\rho(r)T(r)) = -\frac{\mu m_H}{k} 4\pi G \rho_C^2 R^3 \frac{r/R - \arctan(r/R)}{r^2} \frac{1}{1 + (r/R)^2}.$$

3. We now need to integrate this equation from radius 0 to an arbitrary radius r. Again the substitution $x = r/R$ is useful. Show that the equation of hydrostatic equilibrium now reads

$$\rho(r)T(r) - \rho_C T_C = -\frac{\mu m_H}{k} 4\pi G \rho_C^2 R^2 \int_0^{r/R} dx \left(\frac{1}{x(1+x^2)} - \frac{\arctan(x)}{x^2(1+x^2)} \right)$$

4. To solve this integral you need to type the following in the 'Integrator'
$1/(x(1+x^2))$
and
$ArcTan[x]/(x^2(1+x^2))$
Using these results, show that the core temperature T_c can be written

$$T_C = T(r)/(1+x^2) + \frac{\mu m_H}{k} 4\pi G \rho_C R^2 \left(\frac{1}{2}(\arctan x)^2 + \frac{\arctan(x)}{x} - 1 \right)$$

5. We will now try to obtain values for the central density ρ_C. In order to obtain that, we wish to get rid of x and r from the equation. When $x \to \infty$, that is, when going far from the center, show that the equation reduces to

$$T_C = \frac{\mu m_H}{k} 4\pi G \rho_C R^2 (\frac{\pi^2}{8} - 1)$$

6. Before continuing, we need to find a number for R, the distance from the center where the density has fallen by 1/2. Assume that considerations based on hydrodynamics and thermodynamics tell us that the core of the Sun extends out to about $0.2R_\odot$ and that the density has fallen to 10 percent of the central density at this radius. Using this information, show that

$$R = \frac{0.2R_\odot}{\sqrt{9}} \approx 0.067R_\odot.$$

7. We know that a minimum core temperature of about 15 million degrees is needed in order for thermonuclear fusion to be an efficient source of energy production. What is the minimum density in the center of the Sun? Assume the gas in the Sun to consist entirely of protons. Express the result in units of the mean density $\rho_0 = 1400\text{kg/m}^3$ of the Sun. (More accurate calculations show that the core density of the Sun is about 100 times the mean density)

In the last two exercises we have used some very simplified models together with some rough assumptions and observed quantities to obtain knowledge about the density and temperature in the interior of the Sun. These exercises were made to show you the power of the equation of hydrostatic equilibrium: By combing this equation with the knowledge we have about the Sun from observations of its surface together with knowledge about nuclear physics, we are able to deduce several facts about the solar interior. In higher courses in astrophysics, you will also learn that there are more equations than the equation of hydrostatic equilibrium which must be satisfied in the solar interior. Most of these equations come from thermodynamics and fluid dynamics. In the real case, we thus have a set of equations for $T(r)$ and $\rho(r)$ enabling us to do *stellar model building*, without using too many assumptions we can obtain the density and temperature of stars at different distances from the center. These models have been used to obtain the understanding we have today of how stars evolve. Nevertheless many questions are still open and poorly understood. Particularly towards the end of a star's life, the density distribution and nuclear reactions in the stellar interior become very complicated and the equations become difficult to solve. But solving these equations is important in order to understand the details of supernova explosions.