AST1100 Lecture Notes

5 The virial theorem

1 The virial theorem

We have seen that we can solve the equation of motion for the two-body problem analytically and thus obtain expressions describing the future motion of these two bodies. Adding just one body to this problem, the situation is considerably more difficult. There is no general analytic solution to the three-body problem. In astrophysics we are often interested in systems of millions or billions of bodies. For instance, a galaxy may have more than 2×10^{11} stars. To describe exactly the motion of stars in galaxies we would need to solve the 2×10^{11}-body problem. This is of course impossible, but we can still make some simple considerations about the general properties of such a system. We have already encountered one such general property, the fact that the center of mass maintains a constant velocity in the absence of external forces. A second law governing a large system is the *virial theorem* which we will deduce here. The virial theorem has a wide range of applications in astrophysics, from the formation of stars (in which case the bodies of the system are the atoms of the gas) to the formation of the largest structures in the universe, the clusters of galaxies. We will then apply the virial theorem to some of these problems in the coming lectures. Here we will show how to prove the theorem.

The virial theorem is a relation between the total kinetic energy and the total potential energy of a system in equilibrium. We will come back to the exact definition of the equilibrium state at the end of the proof.

We will consider a system of N particles (or bodies) with mass m_i, position vector $\vec{r}_i$, velocity vector $\vec{v}_i$ and momentum $\vec{p}_i = m_i\vec{v}_i$ (see figure 1). For this system, the total moment of inertia is given by (remember from your

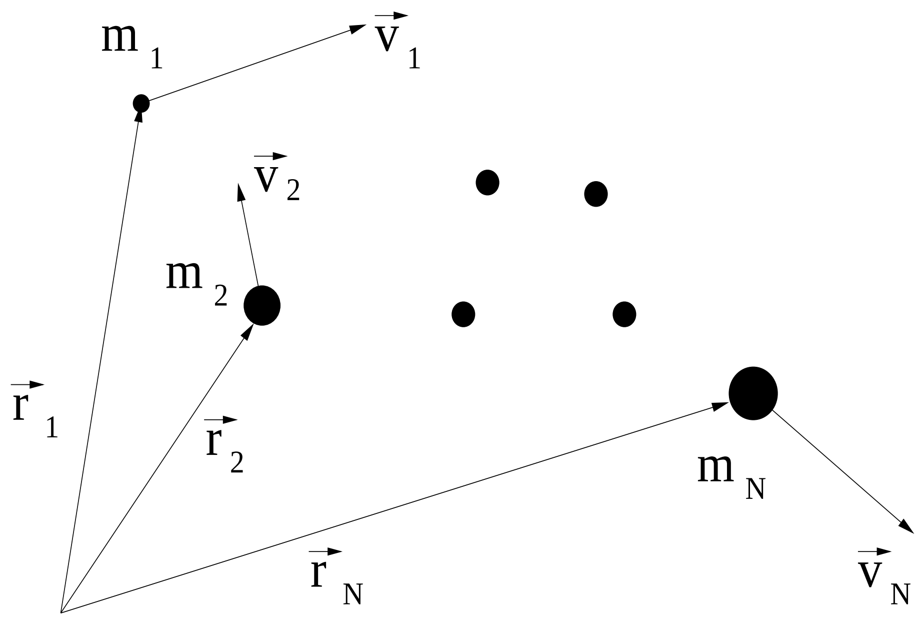

Figure 1: The N-body system.

mechanics classes?)

$$I = \sum_{i=1}^{N} m_i |\vec{r}_i|^2 = \sum_{i=1}^{N} m_i \vec{r}_i \cdot \vec{r}_i.$$

In mechanics one usually takes the moment of intertia with respect to a given axis, here we take the moment of intertia with respect to the origin. The time derivative of the moment of inertia is called the *virial*,

$$Q = \frac{1}{2}\frac{dI}{dt} = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i.$$

To deduce the virial theorem we need to take the time derivative of the virial

$$\frac{dQ}{dt} = \sum_{i=1}^{N} \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i + \sum_{i=1}^{N} \vec{p}_i \cdot \vec{v}_i,$$

where Newton's second law gives

$$d\vec{p}_i/dt = \vec{F}_i$$

$\vec{F}_i$ being the sum of all forces acting on particle i. We may write this as

$$\frac{dQ}{dt} = \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i + \sum_{i=1}^{N} m_i v_i^2,$$

where the last term may be expressed in terms of the total kinetic energy of the system $K = \sum_i 1/2 m_i v_i^2$

$$\frac{dQ}{dt} = \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i + 2K. \tag{1}$$

We will now try to simplify the first term on the right hand side. If no external forces work on the system and the only force which acts on a given particle is the gravitational force from all the other particles, we can write

$$\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i = \sum_{i=1}^{N} \sum_{j \neq i} \vec{f}_{ij} \cdot \vec{r}_i,$$

where $\vec{f}_{ij}$ is the gravitational force on particle i from particle j. The last sum is a sum over all particles j except particle $j = i$. The double sum thus expresses a sum over all possible combinations of two particles i and j, except the combination where $i = j$. We may view this as an $N \times N$ matrix where we sum over all elements ij in the matrix, except the diagonal elements ii. We divide this sum into two parts separated by the diagonal (see figure 2),

$$\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i = \underbrace{\sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot \vec{r}_i}_{\equiv A} + \underbrace{\sum_{i=1}^{N} \sum_{j>i} \vec{f}_{ij} \cdot \vec{r}_i}_{\equiv B}$$

We now rewrite the sum B as

$$B = \sum_{i=1}^{N} \sum_{j>i} \vec{f}_{ij} \cdot \vec{r}_i = \sum_{j=1}^{N} \sum_{i<j} \vec{f}_{ij} \cdot \vec{r}_i,$$

where the sums have been interchanged (you can easily convince yourself that this is the same sum by looking at the matrix in figure 2). We can also interchange the name of the indices i and j (this is just renaming the indices, nothing else)

$$B = \sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ji} \cdot \vec{r}_j.$$

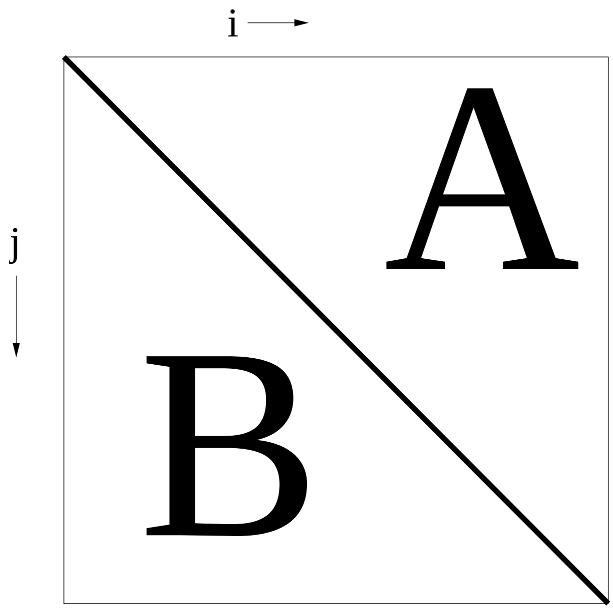

Figure 2: The matrix visualizing the summation

From Newton's third law, we have $\vec{f}_{ij} = -\vec{f}_{ji}$,

$$B = -\sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot \vec{r}_j.$$

Totally, we have,

$$\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i = A + B = \sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot \vec{r}_i - \sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot \vec{r}_j = \sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot (\vec{r}_i - \vec{r}_j). \quad (2)$$

Did you follow all steps so far? Here, the force $\vec{f}_{ij}$ is nothing else than the well known gravitational force,

$$\vec{f}_{ij} = G \frac{m_i m_j}{r_{ij}^3} (\vec{r}_j - \vec{r}_i),$$

where $r_{ij} = |\vec{r}_j - \vec{r}_i|$. Note that the force points in the direction of particle j. Inserting this into equation (2) gives

$$\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i = -\sum_{i=1}^{N} \sum_{j<i} G \frac{m_i m_j}{r_{ij}^3} r_{ij}^2 = \sum_{i=1}^{N} \sum_{j<i} U_{ij},$$

where U_{ij} is the gravitational potential energy between particle i and j. This sum is the total potential energy of the system (do you see this?), the sum of the potential between all possible pairs of particles (note that one pair of particle should be counted only once, this is why there is a $j < i$ in the latter sum). Thus, we have obtained an expressions for the two terms in equation (1) expressing the time derivative of the virial

$$\frac{dQ}{dt} = U + 2K.$$

Finally we will use the equilibrium condition. We will take the mean value of this expression over a long period of time,

$$\langle \frac{dQ}{dt} \rangle = \langle U \rangle + 2\langle K \rangle,$$

where

$$\langle \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt.$$

For the term on the left hand side, we find

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \frac{dQ}{dt} dt = \lim_{\tau \to \infty} \frac{Q(\tau) - Q(0)}{\tau} \equiv 0,$$

for a system in equilibrium. The last equality here is the definition of the equilibrium state in which the system needs to be for the virial theorem to hold: the mean value of the time derivative of the virial must go to zero. In order for this to be fulfilled, the quantities $Q(\tau)$ and $Q(0)$ need to have finite values. If, for instance, the system is bound and the particles go in regular orbits, the virial Q will oscillate regularly between two finite values. In this case, the last expression above will go to zero as $\tau \to \infty$. If Q had not been limited, which could happen for a system which is not bound, then Q could attain large values with time and it would not be clear that this expression would approach zero as $\tau \to \infty$. Using the above equation and the equilibrium condition we see that a bound system in equilibrium obeys

$$\langle K \rangle = -\frac{1}{2} \langle U \rangle.$$

This is the virial theorem. In order to obtain $\langle K \rangle$ and $\langle U \rangle$ we need to take the average of the kinetic and potential energy over a long time period. In the case of the solar system, this is easy: The orbits are periodic so it suffices to take the average over the longest orbital period.

Averaging a system over a long time period may be equal to averaging the system over the ensemble. This is the *ergodic hypothesis*. Mathematically it can be written as

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} dt \to \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} .$$

If a bound system has a huge number of particles ($N \to \infty$), it is equivalent to seeing the system over a long period of time ($\tau \to \infty$). Thus, we can apply the virial theorem to a galaxy by taking the mean of the kinetic and potential energy of all stars in the galaxy in a given instant. According to the ergodic hypothesis, it is not necessary in this case to take the mean of the kinetic and potential energy over a very long period of time. Since the time scales for changes for such huge systems is very long, it is much easier to simply take the average over all stars. The ergodic theorem thus says that we can replace the mean value from being a time average to be an average over all bodies in the system.

2 Applying the virial theorem to a collapsing cloud of gas

To show the power of the virial theorem we will apply it to a system with very many particles and show how properties of this complex system may be calculated. In the exercises you will find two more examples of applications of the virial theorem to problems of a very different nature. The example presented in this section is also an appetiser for the lectures on stellar evolution coming later.

Before the advent of the theory of relativity, the source of the energy that powers stars was sought. One suggestion was that the stellar energy was gravitational energy that is being radiated away as the cloud of gas retracts. A star starts out as a huge cloud of gas which starts collapsing due to its own force of gravity. Gas falls towards the center of the cloud and releases gravitational energy in the form of electromagnetic radiation as it falls. As long as the cloud keeps collapsing, energy is radiated away and could possibly explain the energy production in stars. To check if this is a plausible explanation, we will need to calculate the total energy, kinetic plus potential, that the star could possibly radiate away during its collapse and compare this with the energy output from the Sun. To calculate the total energy of such a cloud, we need to invoke the virial theorem. A collapsing cloud of gas is a bound many-body system and the virial theorem should apply.

We will assume that the cloud is spherically symmetric with radius R and mass M. We need to calculate the total energy, kinetic plus potential, of such a cloud. Thanks to the virial theorem, it suffices to calculate only the potential energy. The total energy is given by

$$E = K + U = -\frac{1}{2}U + U = \frac{1}{2}U,$$

where K is kinetic energy and U is potential energy. Using the virial theorem $K = -U/2$, we replace K by U and obtain an expression for the total energy given only in terms of the total potential energy. I have skipped time average here since this is a system with very many particles and we can use the ergodic hypothesis and simply sum over all particles.

We see that if we are able to calculate the total potential energy of the cloud, we would also obtain the total mechanical energy (kinetic+potential). To obtain the total potential energy, we will start by considering the potential du of a tiny particle of mass dm inside the cloud at a distance r from the center. We have learned (see the lectures on dark matter) that the gravitational forces from a spherical shell of matter add to zero inside this shell. Thus we need only to consider the gravitational attraction on the mass dm from the sphere of matter inside the position of the mass. This is a sphere of radius r with mass $M(r)$. Being a sphere, Newton's law of gravitation applies as if it were a point mass located at the center with mass $M(r)$. Thus the potential energy between the particle dm and the rest of the cloud (the part inside the particle) is

$$du = -G\frac{M(r)dm}{r}.$$

We integrate this equation over all masses dm in the shell of thickness dr at distance r from the center. We assume that the mass density in the shell is given by $\rho(r)$. We then obtain the potential energy dU between the shell and the spherical mass $M(r)$ inside the shell.

$$dU = -G\frac{M(r)4\pi r^2\rho(r)dr}{r}.$$

To obtain the total potential energy U, we need to integrate this expression over all radii r out to the edge of the cloud at $r = R$,

$$U = -4\pi G\int_0^R M(r)\rho(r)rdr.$$

We would generally need to know the density $\rho(r)$ in order to obtain $M(r)$ and to integrate this equation. The scope here is to obtain an approximate expression giving us an idea about the mass and radius dependence of the energy and to obtain an order of magnitude estimate. For this purpose, we assume that the density is constant with a value equal to the mean density of the cloud,

$$\rho = \frac{M}{4/3\pi R^3}.$$

This gives $M(r) = 4/3\pi r^3\rho$ and we can integrate the equation

$$\begin{aligned} U &= -4\pi G\left(\frac{M}{4/3\pi R^3}\right)^2 4/3\pi\int_0^R r^4 dr \\ &= -\frac{3GM^2}{5R} \end{aligned}$$

From the virial theorem, the total energy is then (check!)

$$E = \frac{1}{2}U = -\frac{3GM^2}{10R}.$$

This is the total energy of a cloud of gas with mass M and radius R. The energy that the Sun has radiated away during its lifetime can be written as

$$E_{\text{radiated}} = E(\text{big}R) - E(R_\odot),$$

where 'big R' refers to the radius of the cloud when it started collapsing and $R_\odot$ is the current radius of the Sun. The total energy of the cloud goes as

$\propto 1/R$, so for the initial cloud this quantity can be approximated to zero. Thus we are left with

$$E_{\text{radiated}} = \frac{3GM_{\odot}^2}{10R_{\odot}},$$

where $M_{\odot}$ is the mass of the Sun. Inserting numbers for the mass and radius of the Sun we obtain $E_{\text{radiated}} \approx 1.1 \times 10^{41} J$. Assuming that the Sun has been radiating with the same luminosity $L_{\odot}$ (dE/dt) during its full lifetime, we can calculate the age of the Sun,

$$\Delta t = \frac{E_{\text{radiated}}}{L_{\odot}} \approx 10^7 \text{years}.$$

If gravitational collapse was indeed the source of solar energy, the Sun couldn't have lived longer than about 10 millions years. Several geological findings have shown that the Earth and therefore also the Sun has existed for about 500 times as long. Thus using the virial theorem we have shown (using some assumptions) that gravitational collapse cannot satisfactory explain the generation of energy in the Sun.

3 Problems

Problem 1 (10 min. - 20 min.)

In a way we can look at the virial theorem as a generalization of Kepler's third law to a many-body system. Show that for the two-body problem, the virial theorem is identical to Kepler's third law in the Newtonian form (as deduced in the exercises in lecture notes 1-2). Assume circular orbits. Start with the virial theorem, insert expressions for the energies and show Kepler's third law. (you won't get more help here...).

Problem 2 (2 - 2.5 hours)

Fritz Zwicky was the first to note that there is some missing matter in the universe. In 1933, several years before the discovery of the flat rotation curves in the galaxies, he used the virial theorem to calculate the mass of galaxies in the Coma Cluster. A cluster of galaxies is a cluster of a few hundred galaxies orbiting a common center of mass. The Coma Cluster is one of our neigbouring clusters of galaxies. He found that the mass of the Coma Cluster calculated using the virial theorem was much larger than the mass expected from the visible luminous matter. In this problem we will try to follow his example and estimate the mass of galaxies in a cluster of galaxies. We will consider a simulated cluster of about 100 galaxies. We will assume that the cluster consists of these 100 brightest galaxies and assume that the remaining galaxies are too small to affect our calculations significantly.

1. Looking in the telescope we see that the cluster is spherical, the galaxies are evenly distributed inside a spherical volume. The distance to the cluster is 85 Mpc. You observe the radius of the cluster to be $32'$. What is the radius of the cluster in Mpc?

2. All galaxies in the cluster appear to be very similar to the Milky Way, both in the number of stars and the type of stars. The galaxies look so similar to each other that we can assume that all the galaxies have the same mass m. We know that the Milky Way has about 2×10^{11} stars. Assuming that the mean mass of a star equals the mass of the Sun, what is the estimated total luminous mass m of these galaxies?

3. Use the virial theorem to show that the mass m of a galaxy in the cluster can be written as

$$m = \frac{\sum_{i=1}^{N} v_i^2}{G \sum_{i=1}^{N} \sum_{j>i} 1/r_{ij}},$$

   where r_{ij} is the distance between galaxy i and galaxy j.

4. You will find a file with data for each of the galaxies here:

   http://folk.uio.no/frodekh/AST1100/lecture5/galaxies.txt

   The first column in the file is the observed angular distance (in arcminutes) from the center of the cluster along an x-axis. The second column in the file is the observed angular distance (in arcminutes) from the center of the cluster along an y-axis. (the x-y coordinate system is chosen with an arbitrary orientation on the plane of observation (which is perpendicular to the line of sight)). The third column is the measured distance to the galaxy (from Earth) in Mpc. The fourth column is the position of the spectral line at $21.2cm$ for the given galaxy in units of m.

(a) Using these data, what is the radial velocity of the cluster with respect to us ? Remember that the velocity of a galaxy can be written as

$$v(gal) = v(cluster) + v(rel),$$

where $v(gal)$ is the total velocity of the galaxy with respect to us, $v(cluster)$ is the velocity of the cluster (of the center of mass of the cluster) with respect to us and $v(rel)$ is the relative velocity of a galaxy with respect to the center of mass of the cluster. The relative velocities with respect to the center of mass are random, so for a large number of galaxies the mean

$$\frac{1}{N}\sum_{\mathrm{i=1}}^{N} v_i(rel) \to 0$$

goes to zero.

(b) Make a plot showing how this cluster appears in the telescope: draw the x-y axes (using arcminutes as units on the axes) and make a dot at the position for each galaxy. Remember that in Python you can plot for instance a circle at each data point by using plot(x,y,'o').

(c) Use these data and the expression above for the mass of a galaxy from the virial theorem to obtain a minimum estimate of the total mass of a galaxy in the cluster. How does it compare to the estimate you obtained for luminous matter above ? **hint 1:** To make the double sum in Python you can construct two FOR-loops, one over the index i and one over the index j. Inside the two FOR-loops, you add the expression inside the sum for indices i and j to the final result. **hint 2:** To find the distance between two galaxies i and j, it is convenient to find the x, y and z coordinates of each galaxy in meters.

(d) Your measured velocities are based on the Doppler effect and are therefore radial velocities. Because the inclinations of the velocities with the line of sight is not 90°, your estimate is a minimum estimate of the mass. We will now use the fact that you have many galaxies and that you know that the orientation is random to get a more exact estimate. As a first step you will need to find the mean of $\sin^2 i$ (where i is inclination) taken over many galaxies with random orientations: What is the expected mean value taken over many galaxies of the expression $\sin^2 i$? We assume that the inclination is random (with a uniform distribution). Remember that the mean value of a function $f(x)$ is defined statistically by

$$\langle f(x) \rangle = \frac{\int dx f(x) P(x)}{\int dx P(x)},$$

where $P(x)$ is the statistical distribution, i.e. the probability of having a value x. In this case, the distribution is uniform, meaning that there is an equal probability for getting any value of the inclination i. We may thus set $P(x) = 1$. The integration in this general expression is done over all possible values of x.

(e) Can you use this to obtain a more accurate estimate of the mass ?