AST1100 Lecture Notes

6 Electromagnetic radiation

1 The electromagnetic spectrum

To obtain information about the distant universe we have the following sources available:

1. **electromagnetic waves** at many different wavelengths.
2. **cosmic rays:** high energy elementary particles arriving from supernovae or black holes in our galaxy as well as from distant galaxies. The galactic magnetic field changes the direction of these particles making it impossible to determine the incoming direction and therefore the exact sources of the rays.
3. **neutrinos:** these extremely light elementary particles interact very rarely with other particles and can therefore arrive from huge distances without being scattered on the way. This property also makes neutrinos very difficult to detect and therefore a source of information with limited usefulness until better detection methods are discovered.
4. **gravitational waves:** spacetime distortions traveling through space as a wave. These are predicted by Einstein's general theory of relativity. Gravitational waves have still not been directly detected, but experiments are on their way.

Of these sources, electromagnetic waves is by far the most important. Practical problems limit the amount of information we can obtain from other sources with current technology. Since electromagnetic radiation is almost the only source which we use to get information about the distant universe, it is of high importance in astrophysics to know the processes which produce this kind of radiation. Here we will discuss some of the most important processes along with some discussion on how the radiation from these different

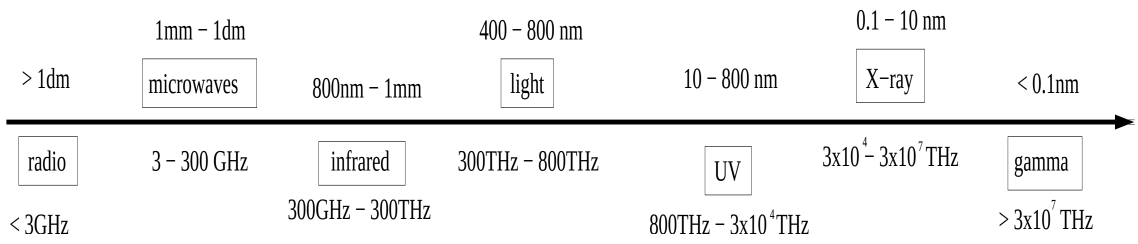

Figure 1: Intensity is the energy of radiation passing through area dA into a solid angle $d\Omega$ per time, per wavelength.

processes is used to obtain information about the universe. Some important types of radiation are

- **thermal radiation:** the thermal motion of atoms produces electromagnetic radiation at all wavelengths. For a *black body* (see later), the radiation emitted at a given frequency is distributed according to Planck's law of radiation.

- **synchrotron radiation:** radiation produced by energetic charged particles accelerated in a magnetic field. This process emits electromagnetic radiation at different wavelengths depending on the energies involved in the process. Our own galaxy emits synchrotron radiation as radio waves due to the acceleration of cosmic ray electrons in the magnetic field of the galaxy.

- **Bremsstrahlung:** radiation produced by the 'braking' of a charged particle, usually an electron, by another charged particle, typically a proton or atomic nucleus. Due to electromagnetic forces from ions, electrons are deflected, and hence accelerated, producing electromagnetic radiation at all wavelengths. The space between galaxies in the clusters of galaxies is called the *intergalactic medium (IGM)*. It contains a very hot plasma of electrons and ions emitting brehmsstralung mainly as X-rays. These X-rays constitute an important source of information about distant clusters of galaxies.

- **21cm radiation:** Neutral hydrogen emits radiation with wavelength 21 cm due to a so-called spin-flip: The quantum spin of the electron and proton may change direction such that the spin vectors go from having their orientation in the same direction to having their orientation in opposite directions. In this process, the total energy of the atom decreases and the energy difference between the two states is emitted as 21cm radiation. This is a so-called forbidden transition, meaning that it occurs very rarely. For a single atom one would on average need to wait about 10 millions years for the process to occur. However, in huge clouds of gas the number of hydrogen atoms is so large that the intensity of 21cm radiation can be quiet large even for such a rare process.

2 Solid angles

Before embarking on the properties of radiation, we will first introduce a new concept which will be widely used: *the solid angle.* The solid angle is a generalization of the concept of an *angle* from one to two dimensions. Looking at figure 2, we see that an angle measured in radians is simply a distance Δs taken along the rim of the unity circle

$$\theta = \Delta s.$$

To convince you about this, remember that the circumference of the unity circle, the full distance taken around the circle, is 2π. Now, the solid angle is measured in units of *steradians*, for short *sr*, and is a part of the *area* of the surface of the unit sphere as seen in figure 3. Thus,

$$\Omega = \Delta A.$$

The solid angle corresponding to the full unit sphere is then 4π sr which is the full area of the surface of the unit sphere. If we imagine a source of radiation in the center of the unit sphere, the solid angle can be used to describe the amount of radiation going in a certain direction as the energy transported per steradian. This is widely used in the study of radiative processes in stars.

3 Black body radiation

Thermal radiation is emitted from an object of temperature T because of the thermal motion of atoms at this temperature. Black body radiation is thermal radiation from a black body. A black body is defined as a body

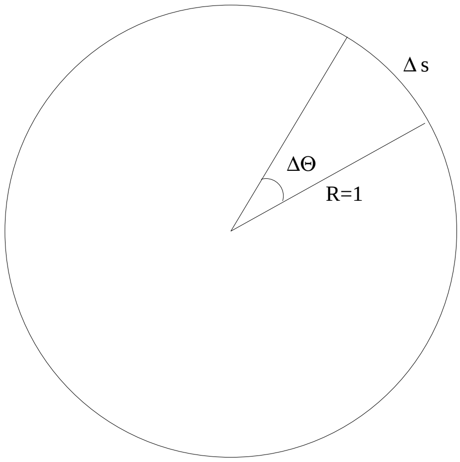

Figure 2: The angle measured in radians is defined as the length taken along the rim of the unit circle.

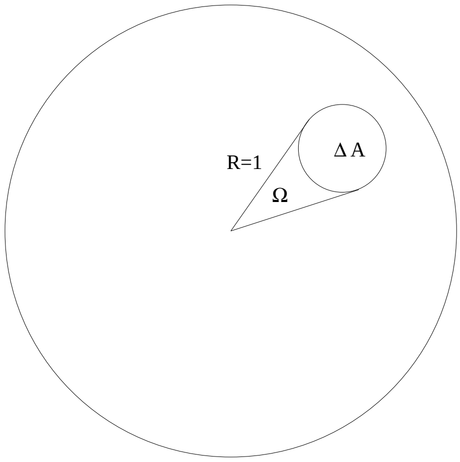

Figure 3: The solid angle measured in steradians is defined as the area taken on the surface of the unit sphere.

which absorbs all radiation it receives, no radiation is reflected or can pass through. Many objects in astrophysics are close to being a black body, a star is a typical example. For a black body, an expression for the intensity of the thermal radiation as a function of wavelength/frequency can be obtained analytically. A black body emits thermal radiation at all frequencies, but which frequency has the largest intensity depends on the temperature of the black body. To calculate the distribution of radiation per frequency quantum physics is needed. We will therefore not make the calculation here (you will come to this in physics courses later), but rather state the result

$$B(\nu) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)} - 1}.$$

This is called *Planck's law of radiation.* Here ν is the frequency, T is the temperature of the black body, h is Planck's constant and k is the Boltzmann constant. The quantity $B(\nu)$ is *intensity* defined such that

$$\Delta E = B(\nu)\cos\theta \Delta\nu \Delta A \Delta\Omega \Delta t \tag{1}$$

is the small energy passing through a small area ΔA into a small solid angle $\Delta\Omega$ (see figure 4) per small time interval Δt in the small frequency range $[\nu, \nu + \Delta\nu]$ measured in units of $W/m^2/sr/Hz$. Here the factor $\cos\theta$ comes from the fact that energy per solid angle per area is lower by a factor $\cos\theta$ for an observer making an angle θ with the normal to the area emitting radiation. Example: Imagine you have a light bulb which emits black body radiation at a certain temperature. You set up a wall between you and the light bulb and let light pass only through a small hole in the wall of area $\Delta A = 0.1mm^2$. Just around the hole you construct a unit sphere and put a detector at an angle $\theta = 30°$ with a line orthogonal to the wall. The detector occupies about 1/1000 of the unit sphere and thus absorbes light from $\Delta\Omega = 4\pi/1000sr$. Finally, the detector contains a material which only absorbes and measures radiation in the wavelenght range 600nm to 600.1nm, such that $\Delta\nu = 0.1nm$. The energy that the detector measures from the light during a period of $10^{-3}s$ is then:

$$\Delta E = B(600nm) \times \cos(30°) \times 0.1mm^2 \times (4\pi/1000)sr \times 0.1nm \times 10^{-3}s$$

In reality, the definition is made when we let all Δ be infinitesimally small, such that the definition reads

$$dE = B(\nu)\cos\theta d\nu dA d\Omega dt \tag{2}$$

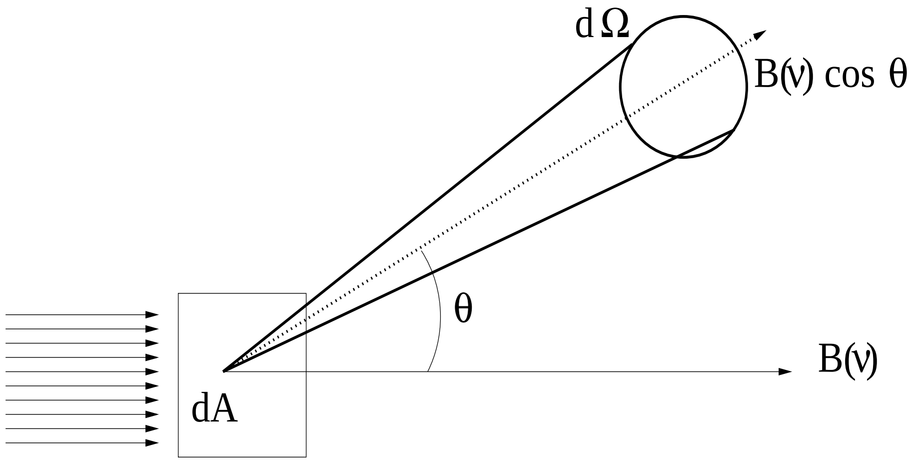

Figure 4: Intensity is the energy of radiation passing through area dA into a solid angle $d\Omega$ per time, per wavelength.

When we use differentials instead of finite differences Δ, we can use integrals to obtain the energy over large intervals in area, frequency, solid angle or time.

Note that in order to write Planck's law in terms of wavelength λ instead of frequency ν one can **not** simply replace $\nu = c/\lambda$. $B(\nu)$ is defined in terms of differentials, so we need to take these into account. When changing from frequncy to wavelength, the energy should be the same, we are just changing variables, so the energy should be constant. From equation 2, we thus see that $B(\nu)d\nu = B(\lambda)d\lambda$ for the energy to be constant. We can write

$$B(\nu)d\nu = -B(\nu)\frac{d\nu}{d\lambda}d\lambda \equiv B(\lambda)d\lambda,$$

where the minus sign comes from the fact that λ and ν increase in opposite directions, $\lambda + |\delta\lambda| \to \nu - |\delta\nu|$. We therefore obtain

$$B(\lambda) = -B(\nu)\frac{d\nu}{d\lambda} = -B(\nu)\left(-\frac{c}{\lambda^2}\right) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/(kT\lambda)} - 1}.$$

Figure (5) shows the intensity as a function of wavelength for black bodies with different temperature T. We see that the wavelength of maximum intensity is different for different temperatures. We can use the position of this peak to determine the temperature of a black body. We can find an

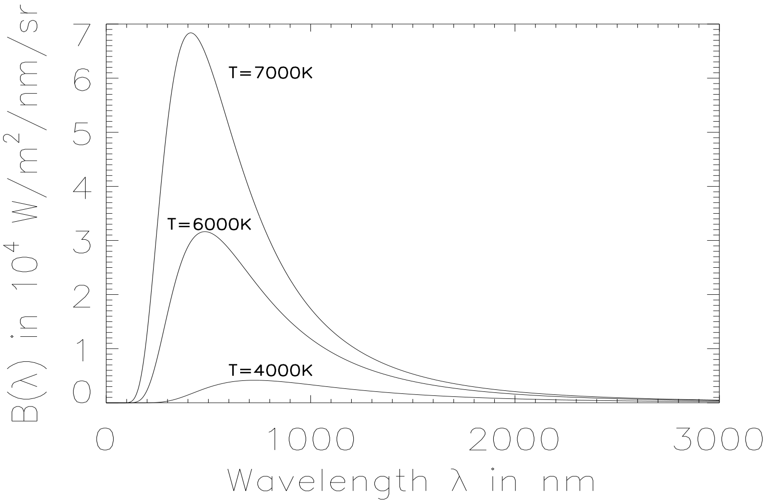

Figure 5: Planck's law for different black body temperatures.

analytical expression for the position of the peak by setting the derivative of Planck's law equal to zero,

$$\frac{dB(\lambda)}{d\lambda} = 0$$

In the exercises you will show that the result gives

$$T\lambda_{\max} = 2.9 \times 10^{-3} mK.$$

This is called *Wien's displacement law.*

Another way to obtain the temperature of a black body is by taking the area under the Planck curve, i.e. by integrating Planck's law over all wavelengths. This area is also different for different temperatures T. Integrating this over all solid angles $d\Omega$ and frequencies $d\nu$, we obtain an expression for the *flux*, energy per time per area,

$$F = \frac{dE}{dAdt}.$$

The integral can be written as (here we are just integrating equation (2) over

$d\nu$ and $d\Omega$)

$$F = \int_0^\infty d\nu \int d\Omega B(\nu)\cos\theta.$$

Using that $d\Omega = d\phi \sin\theta d\theta = d\phi(d\cos\theta)$ and substituting $u = h\nu/kT$, we get

$$\begin{aligned} F &= \int_0^{2\pi} d\phi \int_0^1 d\cos\theta \cos\theta \int d\nu \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)}-1} \\ &= \frac{2k^4T^4\pi}{h^3c^2}\int \frac{u^3 du}{e^u - 1} \\ &= \frac{2\pi k^4T^4}{h^3c^2}\underbrace{\zeta(4)}_{\pi^4/90}\underbrace{\Gamma(4)}_{3!} \\ &= \underbrace{\frac{2\pi^5k^4}{15h^3c^2}}_{\equiv\sigma}T^4. \end{aligned}$$

Here the solution of the u-integral can be found in tables of integrals expressed in terms of ζ, the Riemann zeta-function and Γ, the gamma-function, both of which can be found in tables of mathematical functions. The final result is called the Stefan-Boltzmann law,

$$F = \sigma T^4,$$

the flux emitted from a black body is proportional to the temperature to the fourth power.

We see that we have two ways of measuring the temperature of a star, by looking for the wavelength were the intensity is maximal, or by measuring the energy per area integrated over all wavelengths. If a star had been a black body, these two temperatures would have agreed. However, a star is not a perfect black body. A star has different temperatures at different depths in the star's atmosphere. At different wavelengths we receive radiation from different depths and the final radiation is a combination of Planck radiation at several temperatures. Since the intensity as a function of wavelength is not a perfect Planck curve at a fixed temperature T, the two ways of measuring the temperature will also disagree,

- From Wien's displacement law, we get the *color temperature*, $T = \text{constant}/\lambda_{\text{max}}$.

- From Stefan-Boltzmann's law we get the *effective temperature*, $T = (F/\sigma)^{1/4}$.

The first temperature is called the color temperature since it shows for which wavelength the radiation has it's maximal intensity and hence which color the star appears to have. The second temperature is based on the total energy emitted.

We have so far introduced two measures for the energy of electromagnetic radiation:

- **intensity:** energy received per frequency, per area, per solid angle and per time:
$$I(\nu) = \frac{dE}{cos\theta d\nu dA d\Omega dt}$$
- **flux:** (or total flux) total energy received per area and per time:
$$F = \frac{dE}{dAdt}$$

You will now soon meet the following expressions:

- **flux per frequency:** total energy received per area, per time and per frequency. The total flux above is just the flux per frequency integrated over all frequencies.
$$F(\nu) = \frac{dE}{dAdtd\nu}$$
- **luminosity:** total energy received per unit of time (thus integrated over the whole area which the radiation reaches):
$$L = \frac{dE}{dt}$$
- **luminosity per frequency:** total energy received per frequency per time. The total luminsity above is just the luminosity per frequency integrated over all frequencies.
$$L(\nu) = \frac{dE}{dtd\nu}$$

You will soon see more uses of all these expressions in practise, but it is already now a good idea to memorize the meaning of intensity, flux and luminosity.

4 Spectral lines

When looking at the spectra of stars you will discover that they have thin dark lines at some specific wavelengths. Something has obscured the radiation at these wavelengths. When the radiation leaves the stellar surface it passes through the stellar atmosphere which contains several atoms/ions absorbing the radiation at specific wavelengths corresponding to energy gaps in the atoms. According to Bohr's model of the atom, the electrons in the atom may only take certain energy levels $E_0, E_1, E_2, ...$ The electron cannot have an energy between these levels. This means that when a photon with energy $E = h\nu$ hits an atom, the electron can only absorb the energy of the photon if the energy $h\nu$ corresponds exactly to the difference between two energy levels $\Delta E = E_i - E_j$. Only in this case is the photon absorbed and the electron is excited to a higher energy level in the atom. Photons which do not have the correct energy will pass the atom without being absorbed. For this reason, only radiation at frequency ν with photon energy $E = h\nu$ corresponding to the difference in the energy level of the atoms in the stellar atmosphere will be absorbed. We will thus have dark lines in the spectra at the wavelengths corresponding to the energy gaps in the atoms in the stellar atmosphere (see figure 6). By studying the position of these dark lines, the *absorption lines*, in the spectra we get information about which elements are present in the stellar atmosphere.

The opposite effect also takes place. In the hotter parts of the stellar atmospheres, electrons are excited to higher energy levels due to collisions with other atoms. An electron can only stay in an excited energy level for a limited amount of time after which it spontaneously returns to the lowest energy level, emitting the energy difference as a photon. In these cases we will see bright lines, *emission lines*, in the stellar spectra at the wavelength corresponding to the energy difference, $h\nu = \Delta E$ (see figure 7).

The exact energy levels in the atoms and thus the wavelengths of the absorption and emission lines can be calculated using quantum physics, or they can be measured in the laboratory. However, the actual wavelength where the spectral line is found in a stellar spectra may differ from the predicted value. One reason for this could be the Doppler effect. If the star has a non-zero radial velocity with respect to the Earth, all wavelengths and

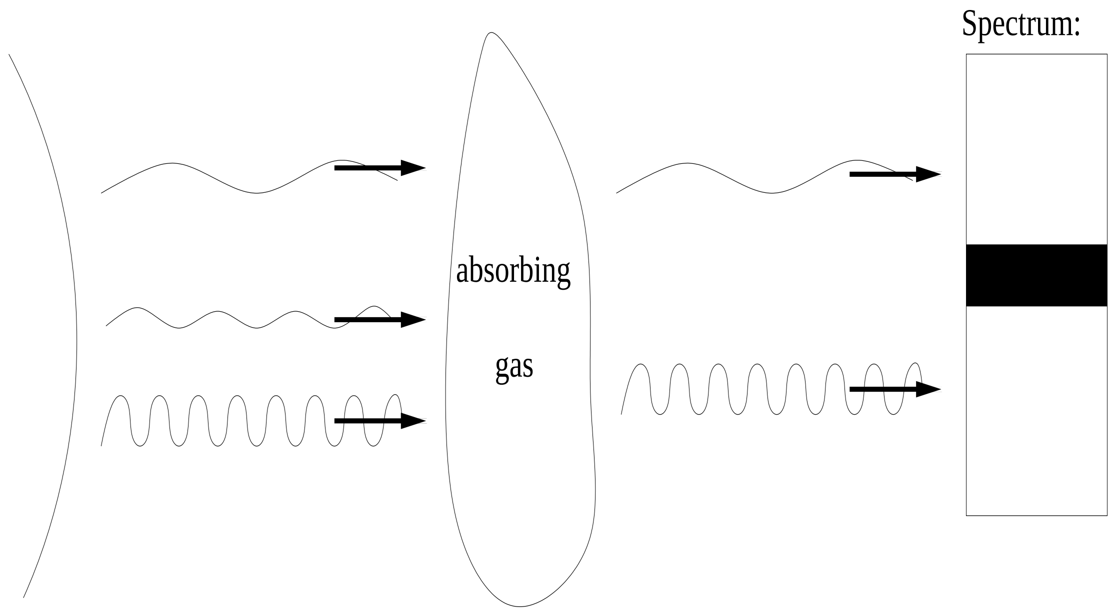

Figure 6: Formation of absorption lines.

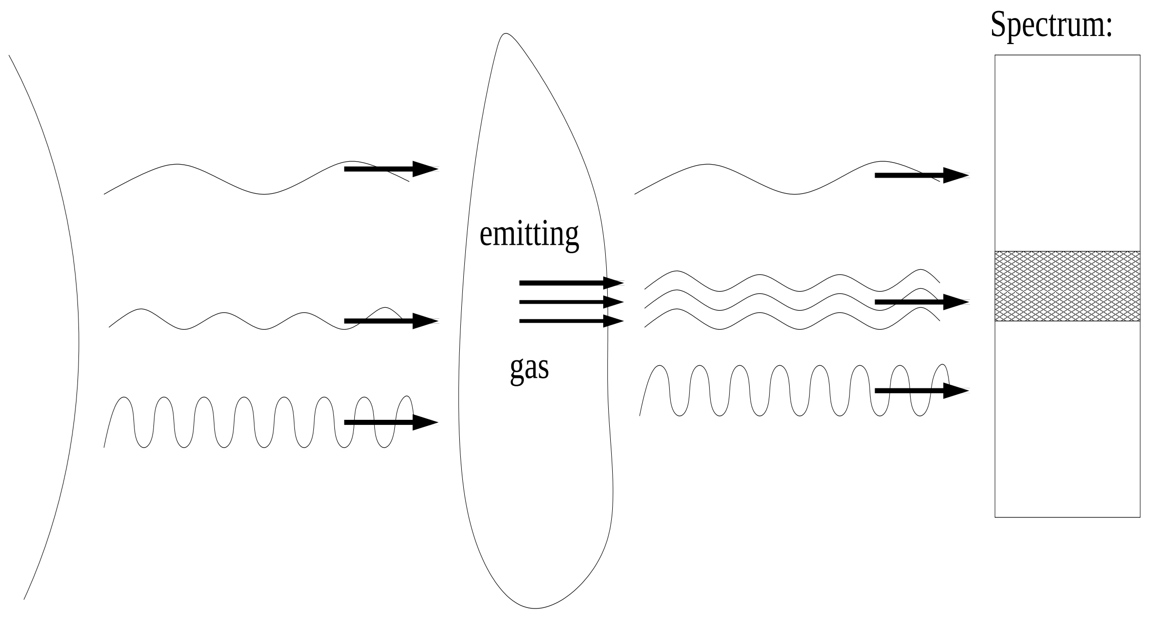

Figure 7: Formation of emission lines.

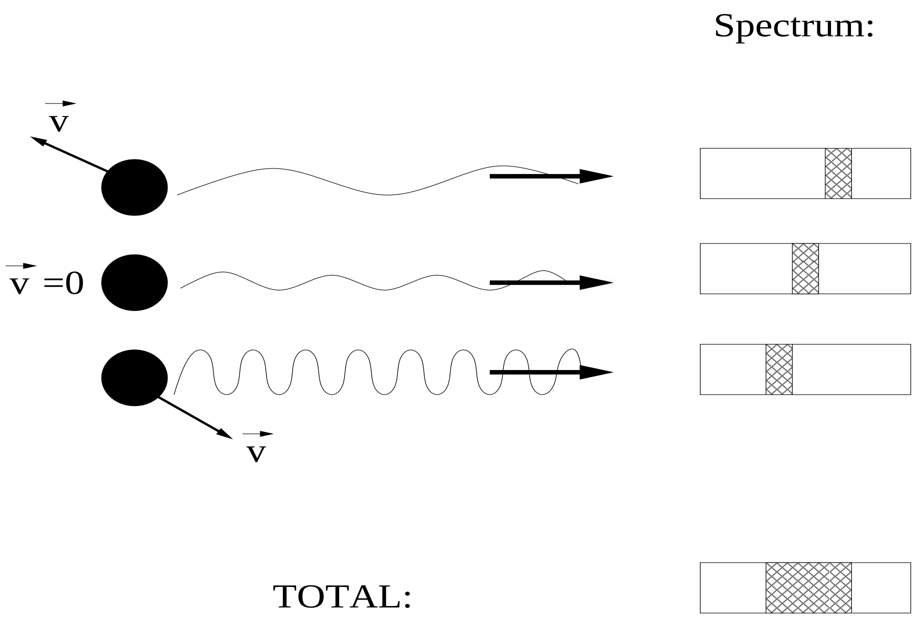

Figure 8: Broadening of spectral lines due to thermal motion.

hence also the position of the spectral lines will move according to

$$\frac{\Delta\lambda}{\lambda_0} = \frac{v_r}{c},$$

where v_r is the radial component of the velocity. By taking the difference $\Delta\lambda$ between the observed wavelength (λ) and predicted wavelength (λ_0) of the spectral line, one can measure the velocity of a star or any other astrophysical object as we discussed in the lecture on extrasolar planets.

Note that even if the star has zero-velocity with respect to Earth, we will still measure a Doppler effect: The atoms in a gas are always moving in random directions with different velocities. This thermal motion of the atoms will induce a Doppler effect and hence a shift of the spectral line. Since the atoms have a large number of different speeds and directions, they will also induce a large number of different Doppler shifts $\Delta\lambda$ with the result that a given spectral line is not seen as a narrow line exactly at $\lambda = \lambda_0$, but as a sum of several spectral lines with different Doppler shifts $\Delta\lambda$. The total effect of all these spectral lines is one single broad line centered at $\lambda = \lambda_0$

(see figure 8). The width of the spectral line will depend on the temperature of the gas, the higher the temperature, the higher the dispersion in velocities and thus in shifts $\Delta\lambda$ of wavelengths.

We can estimate the width of a line by using some elementary thermodynamics. From the above discussion, we see that we will need information about the velocity of the atoms in the gas. For an ideal gas at temperature T (measured in Kelvin K), the number density of atoms (number of atoms per volume) in a given velocity range $[v, v + dv]$ is given by the Maxwell-Boltzmann distribution function,

$$n(v)dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}} 4\pi v^2 dv.$$

Here the mass of the atoms in the gas is given by m and n is the total number density of atoms per unit volume. The meaning of the function is thus the following: if you insert the mass of the atoms in the gas m, the temperature T of the gas and the number density of the gas, n, you can now find out how many atoms in this gas which have a velocity in the range $[v, v + \Delta v]$. Say you need to find out how many atoms have the velocity in the range between 2 and 2.01 km/s. Then you insert $v = 2$ km/s to obtain $n(v)$, and use $\Delta v = 0.01 km/s$ and use $n(v)\Delta v$ to find the total number of atoms with the given velocity per volume.

In figure 9 we see two such distributions (what is plotted is $n(v)/n$), both for hydrogen gas (the mass m has been set equal to the mass of the hydrogen atom), solid line for temperature $T = 6000K$ which is the temperature of the solar surface and dashed line for $T = 373K$ (which equals $100°C$). We can thus use this distribution to find the percentage of molecules in a gas which has a certain velocity. Now, before you read on, go back and make sure that you understand well the meaning of the function $n(v)$.

We see that the peak of this distribution, i.e. the velocity that the largest number of atoms have, depends on the temperature of the gas,

$$\frac{dn(v)}{dv} = 0 \rightarrow \frac{d}{dv}(e^{-mv^2/(2kT)}v^2) = 0.$$

Taking the derivative and setting it to zero gives the following relation

$$v_{\max}^2 = \frac{2kT}{m},$$

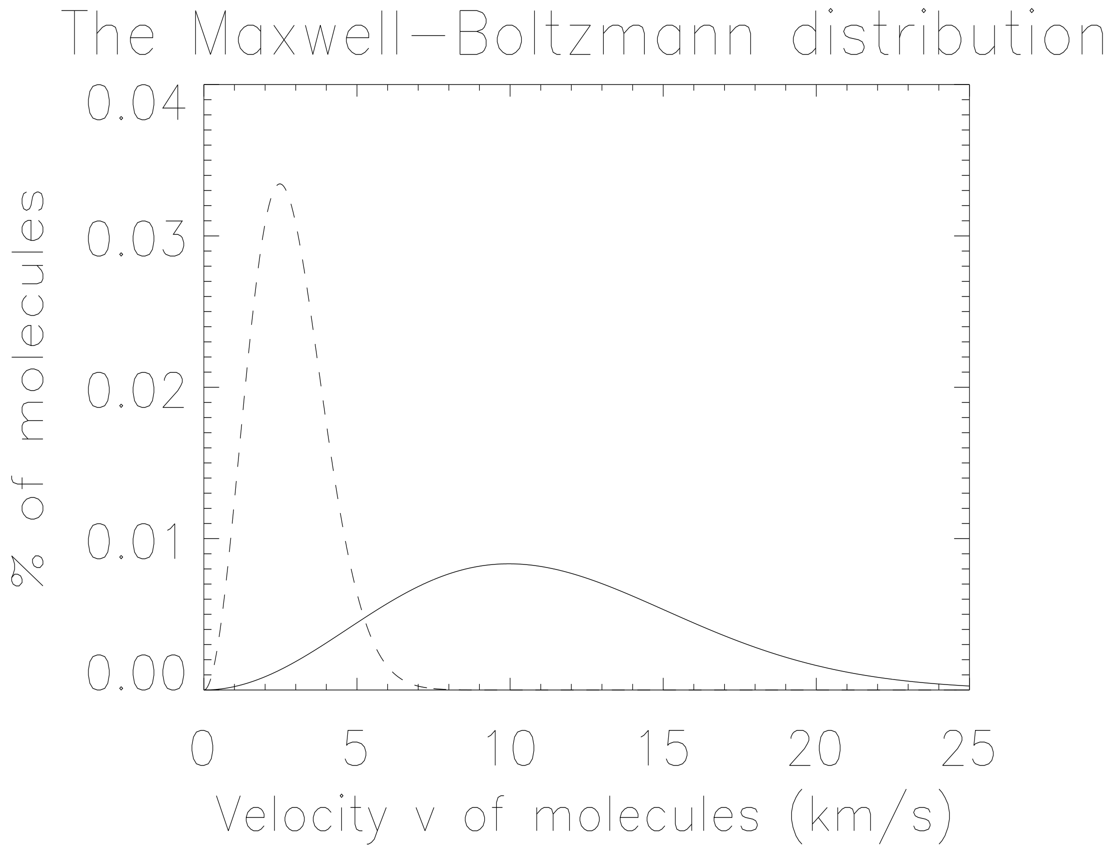

Figure 9: The Maxwell-Boltzmann distribution for hydrogen gas, showing the percentage of molecules in the gas having a certain thermal velocity at temperature $T = 6000K$ (solid line) and $T = 373K$ (dashed line)

i.e. the most probable velocity for an atom in the gas is given by $v_{\rm max}$ (NOTE: 'max' does **not** mean highest velocity, but highest **probability**). Most of the atoms will have a velocity close to this velocity (see again figure 9).

The Maxwell-Boltzmann distribution only tells you the absolute value v of the velocity. When measuring the Doppler effect, only the radial (along the line of sight) component v_r has any effect. The atoms in a gas have random directions and therefore atoms with absolute velocity v will have radial velocities scattered uniformly in the interval $v_r = [-v, v]$ (why this interval? do you see it?). Since the most probable absolute velocity is $v_{\rm max}$ the most probable radial velocity will be all velocities in the interval $v_r = [-v_{\rm max}, v_{\rm max}]$ (you see that for instance $v_r = 0$ is in this interval, do you understand why $v_r = 0$ is at as common as $v_r = v_{\rm max}$?) . The atoms with absolute velocity $v_{\rm max}$ will thus give Doppler shifts uniformly distributed between $\Delta\lambda/\lambda_0 = -v_{\rm max}/c$ and $\Delta\lambda/\lambda_0 = v_{\rm max}/c$. Few atoms have a much higher velocity than $v_{\rm max}$ and therefore the spectral line starts to weaken (less absorption/emission) after $|\Delta\lambda|/\lambda_0 = v_{\rm max}/c$. We will thus see a spectral line with the width given roughly by

$$2\Delta\lambda = \frac{2\lambda_0}{c}v_{\rm max} = \frac{2\lambda_0}{c}\sqrt{\frac{2kT}{m}},$$

using the expression for $v_{\rm max}$ above. Do you see how this comes about? Try to imagine how the spectral line will look like, thinking how atoms at different velocities (above and below the most probable velocity) will contribute to v_r and thereby to the the spectral line. Try to make a rough plot of how $F(\lambda)$ for a spectral line should look like. Do not proceed until you have made a suggestion for a plot for $F(\lambda)$.

Of course, there are atoms at speeds other than $v_{\rm max}$ contributing to the spectral line as well. The resulting spectral line is thus not seen as a sudden drop/rise in the flux at $\lambda_0 - \Delta\lambda$ and a sudden rise/drop again at $\lambda_0 + \Delta\lambda$. Contributions from atoms at all different speeds make the spectral line appear like a Gaussian function with strongest absorption/emission at $\lambda = \lambda_0$. We say that the *line profile* is Gaussian. More accurate thermodynamic calculations show that we can approximate an absorption line with the Gaussian function

$$F(\lambda) = F_{\rm cont}(\lambda) + (F_{\rm min} - F_{\rm cont}(\lambda))e^{-(\lambda-\lambda_0)^2/(2\sigma^2)}, \tag{3}$$

where $F_{\rm cont}(\lambda)$ is the *continuum flux*, the flux $F(\lambda)$ which we would have if the absorption line had been absent. The width of the line is defined by σ.

For a Gaussian curve one can write σ in terms of the Full Width at Half the Maximum (FWHM, see figure 10) as $\sigma = \Delta\lambda_{\mathrm{FWHM}}/\sqrt{8\ln 2}$ where

$$\Delta\lambda_{\mathrm{FWHM}} = \frac{2\lambda_0}{c}\sqrt{\frac{2kT\ln 2}{m}},$$

We see that this exact line width differs from our approximate calculations above only by $\sqrt{\ln 2}$. With this expression we also have a tool for measuring the temperature of the elements in the stellar atmosphere.

5 Stellar magnitudes

The Greek astronomer Hipparchus (about 150 BC) made a catalogue of about 850 stars and divided them into 6 *magnitude* classes, depending on their brightness: the brightest stars were classified as magnitude 1 stars, and the stars which could barely be seen were classified as magnitude 6. Little did Hipparchus know about the fact that more than 2000 years later his system would still be used, and not only that, it would be used by all astronomers in the (now much bigger) world. Whereas Hipparchus classified the stars by eye, a more scientific method is used today. The eye reacts to differences in the logarithm of the brightness. For this reason, the magnitude classification is logarithmic. For a difference in magnitude of 5 between two stars, the ratio of the fluxes (energy received per area per time $F = dE/dt/dA$) of these stars is defined to be exactly 100.

The flux we receive from a star depends on the distance to the star. We define the *luminosity* L of a star to be the total energy emitted by the whole star per unit time (dE/dt). This energy is radiated equally in all directions. If we put a spherical shell around the star at distance r, the energy received per unit area on this shell would equal the total energy L divided by the surface area of the shell,

$$F = \frac{L}{4\pi r^2}.$$

Thus, the larger distance r, the larger the surface area of the shell $4\pi r^2$ and the smaller the energy received per unit area (flux F). If we have two stars with observed fluxes F_1 and F_2 and magnitudes m_1 and m_2, we have learned that if $F_1 = F_2$ then $m_1 = m_2$ (agree?). We have also learned that if

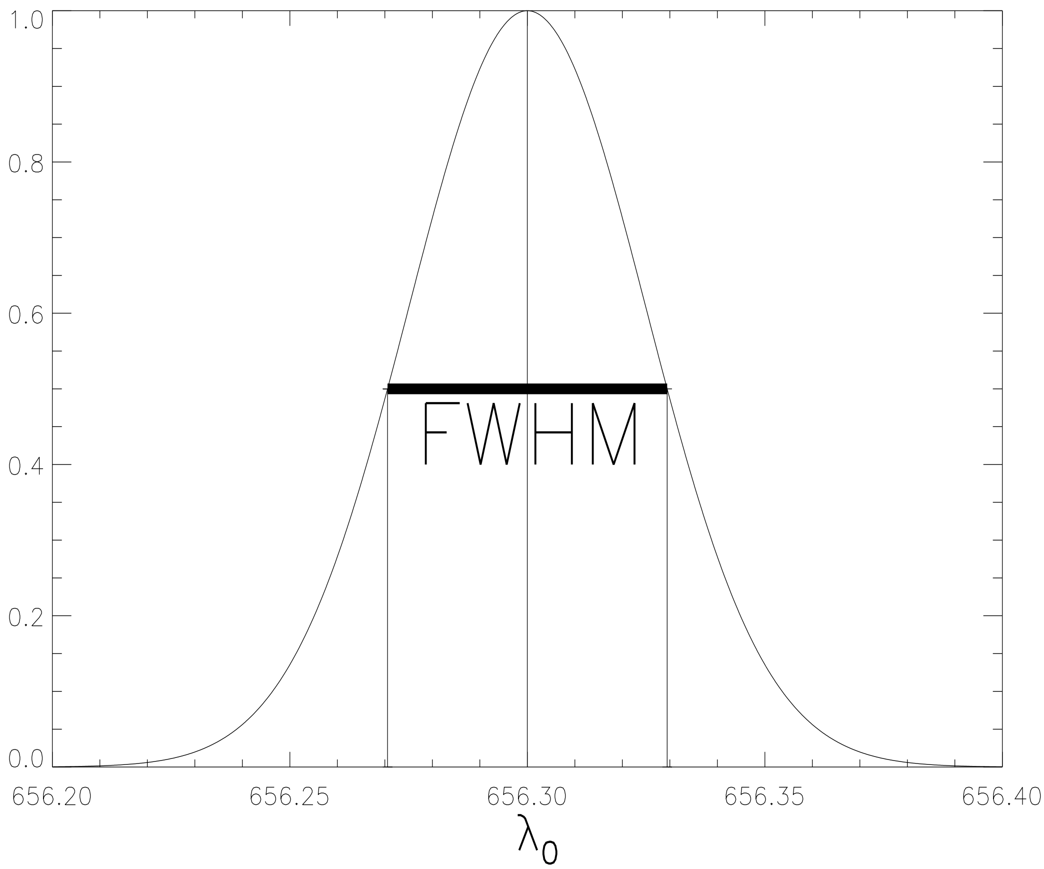

Figure 10: A Gaussian profile: The horizontal line shows the Full Width at Half Maximum (FWHM) which is where the curve has fallen to half of its maximum value. This is an emission line, an absorption line would look equal, just upside down.

$F_1 = 100F_2$ then $m_2 - m_1 = 5$ (remember that in Hipparchus' system $m = 1$ stars were the brightest and $m = 6$ stars were the faintest).

The magnitude scale is logarithmic, thus we obtain the following general relation between magnitude and flux

$$\frac{F_1}{F_2} = 100^{(m_2 - m_1)/5},$$

or

$$m_1 - m_2 = -2.5 \log_{10} \left(\frac{F_1}{F_2} \right).$$

(Check that you can go from the previous equation to this one!) Given the difference in flux between two stars, we can now find the difference in magnitude.

We have so far discussed the *apparent magnitude* m of a star which depends on the distance r to the star. If you change the distance to the star, the flux and hence the magnitude changes. We can also define *absolute magnitude* M which only depends on the total luminosity L of the star. The absolute magnitude M does not depend on the distance. It is defined as the star's apparent magnitude if we had moved the star to a distance of exactly 10 pc. We can find the relation between apparent and absolute magnitude of a star,

$$\frac{F_r}{F_{r=10pc}} = \frac{L/(4\pi r^2)}{L/(4\pi(10pc)^2)} = \left(\frac{10pc}{r} \right)^2 = 100^{(M-m)/5},$$

giving

$$m - M = 5 \log_{10} \left(\frac{r}{10pc} \right).$$

With this new more precise definition, stars can have magnitudes lower than 1. The brightest star in the sky, Sirius, has apparent magnitude -1.47 (note that the logarithmic dependence actually gives the brightest stars negative apparent magnitude). The planet Venus at maximum brightness has apparent magnitude -4.7 and the Sun has magnitude -26.7. The faintest object in the sky visible with the Hubble Space Telescope has apparent magnitude of about 30, about 100^5 times fainter than the faintest star visible with the naked eye. Originally the zero point of the magnitude scale was defined to be the star Vega. This has now been slightly changed with a more technical definition (outside the scope of this course).

NOTE: In order to define the magnitude we use the flux which we receive on Earth, the *received flux*. In some situations you will also need the *emitted flux*, the flux measured on the surface of the star emitting the radiation. It is important to keep these apart as they are calculated in a different manner (what is the difference?).

6 Problems

Problem 1 (20-30 min.)) At very large ($h\nu >> kT$) and very small ($h\nu << kT$) frequencies, Planck's law can be written in a simpler form. The first limit is called the Wien limit and the second limit is called the Rayleigh-Jeans limit or simply the Rayleigh-Jeans law.

1. Show that Planck's law can be written as
$$B(\nu) = \frac{2h\nu^3}{c^2} e^{-h\nu/(kT)}$$
in the Wien limit.

2. Show that Planck's law can be written as
$$B(\nu) = \frac{2kT}{c^2}\nu^2$$
in the Rayleigh-Jeans limit. What kind of astronomer do you think uses Rayleigh-Jeans' law regularly ?

Problem 2 (60 min. - 90 min.) Now we will deduce Wien's displacement law by finding the peak in $B(\lambda)$.

1. Use the expression in the text for $B(\lambda)$ and take the derivative with respect to λ. After taking the derivative, replace λ everywhere with x given by
$$x = \frac{hc}{kT\lambda}.$$

2. To find the peak in $B(\lambda)$, we need to set the derivative equal to zero. Show that this gives us the following equation
$$\frac{xe^x}{e^x - 1} = 5.$$

3. We now want to solve this equation numerically. We see that all we need to do is to find a value for x such that the expression on the left hand side equals 5. The easiest way to do this is to try a lot of different values for x in the expression on the left hand side. When the expression on the left hand side has got a value very close to 5, we have found x.

   (a) The solution to x will be in the range $x = [1, 10]$. Define an array x in Python with 1000 elements going from 1.0 as the lowest value to 10.0 as the highest value. Make a plot of the expression on the left hand side as a function of the array x. Can you see by eye at which value for x the curve crosses 5 ? Then you have already solved the equation.

   (b) To make it slightly more exact, we try to find which x gives us the closest possible value to 5. We define the difference Δ between our expression and the value 5 which we want for this expression

   $$\Delta = (\frac{xe^x}{e^x - 1} - 5)^2,$$

   where we have taken the square to get the absolute value. Define an array in Python which contains the value of Δ for all the values of x. Plot Δ as a function of x. By eye, for which value of x do you find the minimum ?

   (c) Use Python to find the exact value of x (from the 1000 values defined above) which gives the minimum Δ.

   (d) Now use the definition of x to obtain the constant in Wien's displacement law. Do you get a value close to the value given in the text ?

Problem 3 (optional 1 hour - 90 min.) Here we will assume that the Sun is a perfect black body. You will need some radii and distances. By searching i.e. Wikipedia for 'Sun' or 'Saturn' you will find these data.

1. The surface temperature of the Sun is about $T = 5778K$. At which wavelength λ does the Sun radiate most of its energy?

2. Plot $B(\lambda)$ for the Sun. What kind of electromagnetic radiation dominates?

3. What is the total energy emitted per time per surface area (flux) from the solar surface?

4. Repeat the previous exercise, but now by numerical integration. Do the integration with the box method (rectangular method) and Simpson's method. Compare the results. Compare your answer to values that you find in internet. If there is a difference, why do you think that is?

5. Use this flux to find the luminosity L (total energy emitted per time) of the Sun ? (here you need the solar radius)

6. What is the flux (energy per time per surface area) that we receive from the Sun? (here you need the Sun-Earth distance) (see figure 11).

7. Spacecrafts are often dependent on solar energy. What is the flux received from the Sun by the spacecraft Cassini-Huygens orbiting Saturn ? (here you need the distance Sun-Saturn)

8. Assume that the efficiency of solar panels is 12%, i.e. that the electric energy that solar panels can produce is 12% of the energy that they receive. How many square meters of solar panel does the Cassini-Huygens spacecraft need in order to keep a $40W$ light bulb glowing ?

Problem 4 (optional 30 min. - 60 min.) We will now study a simple climate model. You will here need the results from question 1-6 in the previous problem.

1. We assume that the Earth's atmosphere is transparent for all wavelengths. How much energy per second arrives at the surface of Earth? The flux that you calculated in the previous exercises is the flux received by an area located at the earth's surface with orientation perpendicular to the distance-vector between the Sun and the Earth. **Hint** - Since the Earth is a sphere, the flux is not at all constant over the surface. However, we do not need to calculate the density for each square meter (fortunately). We can just look at the size of the effective absorption area (shadow area) which is shown in figure 12. Since the distance between the Sun and the Earth is so large, we can assume that the rays arriving at earth are travveling in the same direction (parallell). The radius r of the shaddow area is then equal to Earth's radius. The rest should be straight forward.

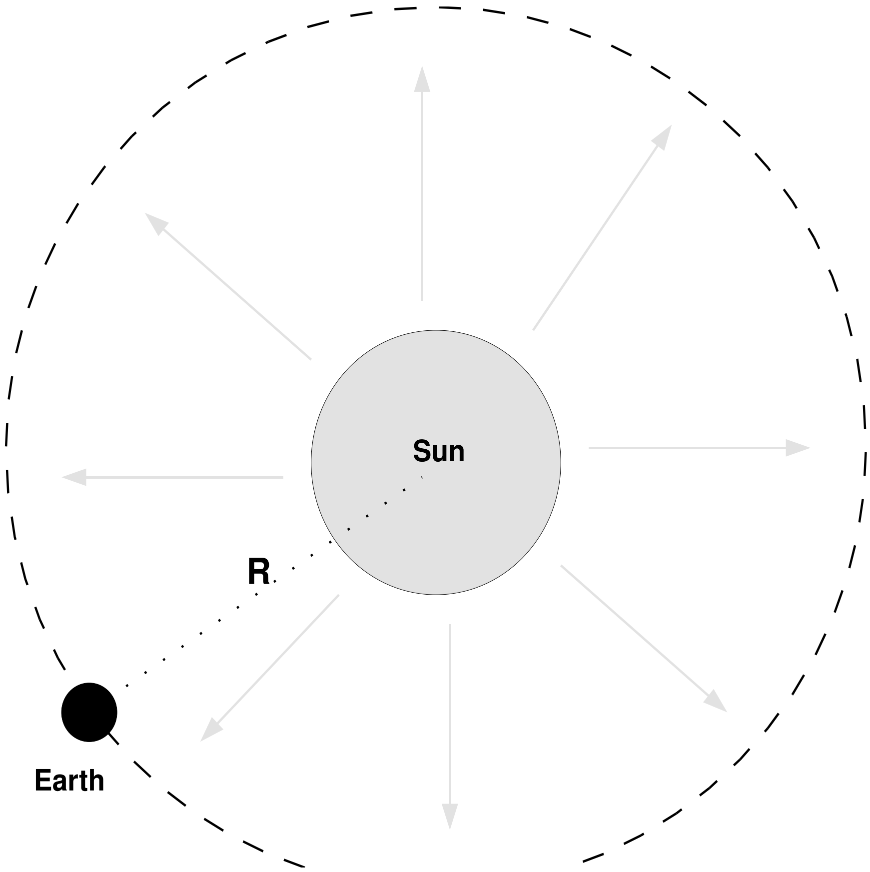

Figure 11: Radiation from the Sun. The flux is constant on a spherical surface with center at the Sun's center of mass.

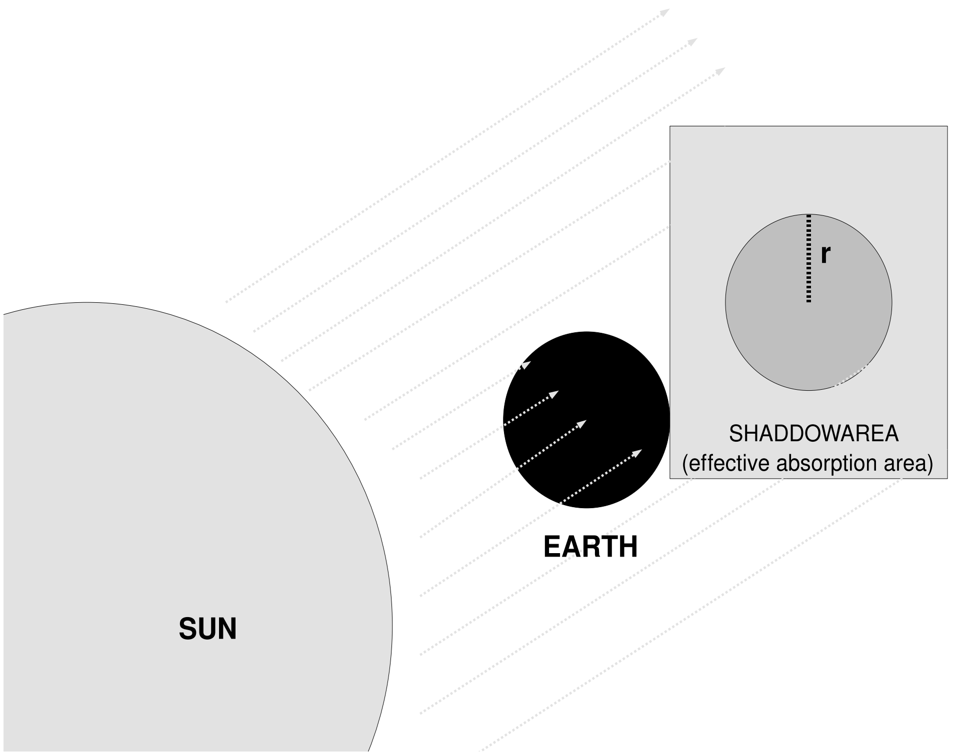

Figure 12: Shaddowarea (effective absorption area)

2. You are now going to estimate Earth's temperatur by using a simple climate model that just takes into account the radiation from the Sun and the Earth. We still assume that the atmosphere is transparent for all wavelengths. The model says that the Earth is a blackbody with a constant temperature. This means that it absorbs all incoming radiation and emits the same amount (in energy/sec) in *all* directions. Calculate the Earth's temperature by using the simple climate model. You will need the result from the previous question and Stefan Boltzmann's law. Compare your result to the empirical average value $\mathrm{T}_{av} \approx$ 290K? Why do you think there is a difference?

Problem 5 (20 min. - 30 min.)
Here you will deduce a general expression for the flux per wavelength, $F(\lambda) = dE/dA/dt/d\lambda$, that we receive from a star with radius R at a distance r with surface temperature T. Assume that the star is a perfect black body. You can solve this problem in two steps,

1. Find the luminosity per wavelength $L(\lambda) = dE/d\lambda/dt$, i.e. the energy per time per wavelength, emitted from the star. The intensity $B(\lambda)$ is defined as $dE/dA/d\Omega/d\lambda/dt/cos\theta$. You need to integrate over solid angle and area to obtain the expression for the luminosity. **hint:** Look at the derivation of Stefan-Boltzmann's law in the text.

2. Find the flux $F(\lambda)$ using $L(\lambda)$

3. Does the expression for the flux peak at the same wavelength as for Planck's law? Can we simply use the maximum wavelength from flux measurements to obtain $\lambda_{\max}$ to be used in Wien's displacement law?

Problem 6 (4 hours - 4.5 hours) I have produced a set of simulated spectra for a star. You will find the spectra in 10 files at

$$http://folk.uio.no/frodekh/AST1100/lecture6/spectrum\_day*.txt$$

These files show spectra taken of the same star at 10 different moments. The filename indicates the time of observation given in days from the first observation taken at $t = 0$. The first column of the file is the wavelength of observation in nm, the second column is the flux relative to the continuum flux around the spectral absorption line $H\alpha$ at $\lambda_0 = 656.3nm$. Due to the

Doppler effect, the exact position of the spectral line is different from λ_0. You will also see that this difference changes in time. As we have seen before, real life observations are noisy. It is not so easy to see exactly at which wavelength the center of the spectral line is located.

1. Plot each of the spectra as a function of wavelength. Can you see the absorption line ?

2. Make a bye-eye estimate of the position of the center of the spectral line for each observation. Use the Doppler formula to convert this into relative velocity of the star with respect to Earth for each of the 10 observations (neglect the fact that the velocity of the Earth changes with time).

3. Now we will make a more exact estimate of the spectral line position using a least squares fit. As discussed in the text, we can model the spectral line as a Gaussian function (see equation(3)),

$$F^{\text{model}}(\lambda) = F_{\text{max}} + (F_{\text{min}} - F_{\text{max}})e^{-(\lambda-\lambda_{\text{center}})^2/(2\sigma^2)}.$$

   When $\lambda = \lambda_{\text{center}}$, the model gives $F^{\text{model}}(\lambda) = F_{\text{min}}$. When λ is far from λ_{center} the model becomes $F^{\text{model}}(\lambda) = F_{\text{max}}$ as expected (check!). Thus the flux in this wavelength range if there hadn't been any spectral line would equal F_{max}. The flux at the wavelength for which the absorption is maximal is F_{min}. The spectra are normalized to the continuum radiation meaning that $F_{\text{max}} = 1$. We are left with three unknown parameters, F_{min}, σ and λ_{center}. The first parameter gives the flux at the center of the spectral line, the second parameter is a measure of the width of the line and the third parameter gives the central wavelength of the spectral line. In order to estimate the speed of the star with the Doppler effect, all we need is λ_{center}. But in order to get the best estimate of this parameter, we need to find the best fitting model to the spectral line, so we need to estimate all parameters in order to find the one that interests us. Again we will estimate the parameters using the method of least squares. We wish to minimize

$$\Delta(F_{\text{min}}, \sigma, \lambda_{\text{center}}) = \sum_{\lambda}(F^{\text{obs}}(\lambda) - F^{\text{model}}(\lambda, F_{\text{min}}, \sigma, \lambda_{\text{center}}))^2,$$

   where $F^{\text{obs}}(\lambda)$ is the observed flux from the file and the sum is performed over all wavelengths available.

(a) For each spectrum, plot the spectrum as a function of wavelength and identify the range of possible values for each of the three parameters we are estimating. Define three arrays $fmin$, $sigma$ and $lambdacenter$ in Python which contain the range of values for each of F_{min}, σ and $\lambda_{\mathrm{center}}$ where you think you will find the true values. Do not include more values of the parameters than necessary, but make sure that the true value of the parameter must be within the range of values that you select. Do not use more than 50 values for each parameter, preferably less. **hint:** The FOR loop over λ might be easier if you use indices instead of actual values for λ. That is, the FOR loop runs over index i in the array, and then you find the lambda value which corresponds to this index to use in the expression for Δ.

(b) Define a 3-dimensional array *delta* where you calculate Δ for all the combinations of parameters which you found reasonable.

(c) Find for which combination of the parameters F_{min}, σ and $\lambda_{\mathrm{center}}$ that Δ is minimal. These are your best estimates.

(d) Repeat this procedure for all 10 spectra and obtain 10 values for the Doppler velocity v_r.

4. Make an array of the 10 values you have obtained for the velocities and plot it as a function of time.

5. Assume that the change of velocity with time indicates the presence of a planet around the star (is there something in your observations which indicates this ?). The mass of the star was found to be 0.8 solar masses. Find the minimum mass of this planet (find v_r and the period 'by eye' looking at the velocity curve). Is this really a planet? **hint:** Remember that you need to subtract the peculiar velocity (velocity of the center of mass of the system), found by taking the mean of the velocity.

Problem 7 (10 min. - 15 min.)

In the text you find the apparent magnitudes of Sirius, Vega and the Sun. Look up the distances to these objects (again, wikipedia is a useful source of information) and calculate the absolute magnitude. Which of these three stars is actually the brightest?

Problem 8 (15 min. - 20 min.)

1. Use the flux calculated in Problem 3.6 to check that the apparent magnitude of the Sun used in the text is correct. In order to calibrate the magnitude you also need to know that the star Vega has been defined to have zero apparent magnitude (actually with newer definitions it has magnitude 0.03) and that the absolute magnitude of Vega is 0.58. You also need to know the luminosity of Vega: Look it up in Wikipedia.

2. The faintest objects observed by the Hubble Space Telescope (HST) have magnitude 30. Assume that this is the limit for HST. How far away can a star with the same luminosity as the Sun be for HST to see it? (here you will need the luminosity of the Sun calculated in problem 3.5)

3. Assume that the luminosity of a galaxy equals the luminosity of 2×10^{11} Suns. How far away can we see a galaxy using HST?