AST1100 Lecture Notes

7 - 8 The special theory of relativity: Basic principles

1 Simultaneity

We all know that 'velocity' is a relative term. When you specify velocity you need to specify velocity **with respect to something**. If you sit in your car which is not moving (with respect to the ground) you say that your velocity is zero with respect to the ground. But with respect to the Sun you are moving at a speed of $30km/s$. From the point of view of an observer passing you in his car with a velocity of $100km/h$ with respect to the ground, your speed is $-100km/h$ (see figure 1). Even though you are not moving with respect to the ground, you are moving backwards at a speed of $100km/h$ with respect to the passing car. In the following we will use the expression 'frame of reference' to denote a system of observers having a common velocity. All observers in the same frame of reference have zero velocity with respect to each other. An observer always has velocity zero with respect to his own frame of reference. An observer on the ground measures the velocity of the passing car to be $100km/h$ with respect to his frame of reference. On the other hand, the driver of the car measures the velocity of the ground to be moving at $-100km/h$ with respect to his frame of reference. We will also use the term 'rest frame' to denote the frame of reference in which a given object has zero velocity. In our example we might say: In the rest frame of the passing car, the ground is moving backwards with $100km/h$.

You are observing a truck coming towards you at a speed of $v_{\text{truck}}^{\text{ground}} = -50km/h$ with respect to the ground (see figure 2, velocities are defined to be positive to the right in the figure). From your *frame of reference*, which is the same frame of reference as the ground, the speed of the truck is $|v_{\text{truck}}^{\text{ground}}| = 50km/h$ in the direction towards you. Now you start driving your car in the direction of the truck with a speed of $v_{\text{car}}^{\text{ground}} = +50km/h$

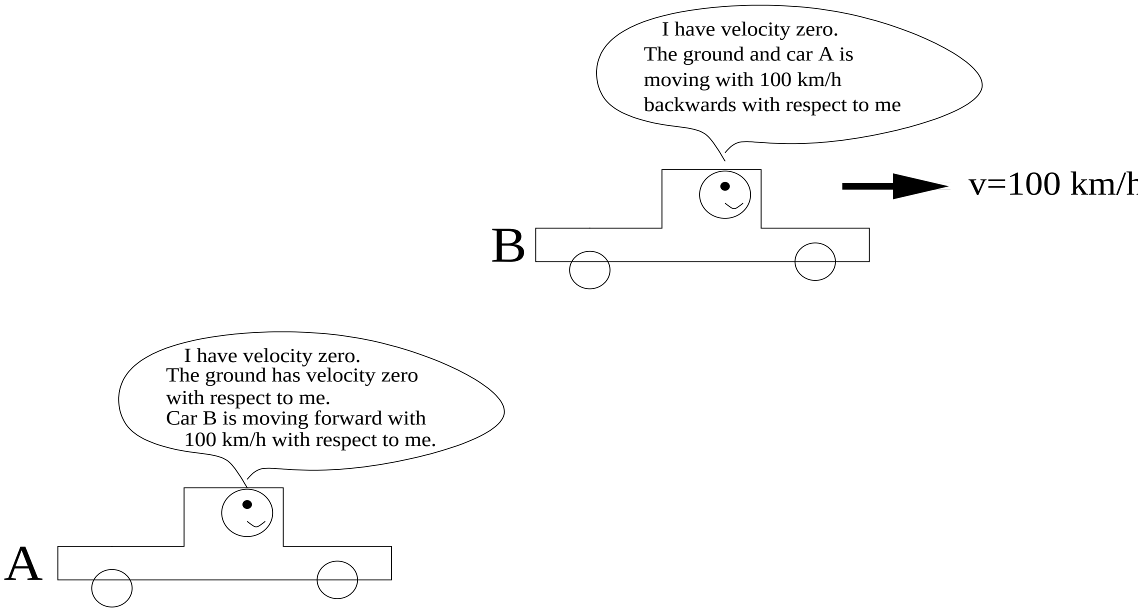

Figure 1: Velocities are relative

with respect to the ground (see again figure 2). From your frame of reference you observe the ground to be moving backwards with a velocity of $v_{\text{ground}}^{\text{car}} = -50km/h$. Again, from your frame of reference you now observe the velocity of the approaching truck to be $v_{\text{truck}}^{\text{car}} = v_{\text{truck}}^{\text{ground}} - v_{\text{car}}^{\text{ground}} = (-50km/h) - (50km/h) = -100km/h$ (whereas from the frame of reference of an observer on the ground, the truck still has $v_{\text{truck}}^{\text{ground}} = -50km/h$). Now you make a turn so that you drive in the opposite direction: Now your velocity is $-50km/h$ with respect to the ground, but now you are driving in the same direction as the truck. You are now moving in the same direction as the truck with exactly the same speed with respect to the ground. From your frame of reference (which is now the same frame of reference as the truck) the truck is not moving.

So far, so good. This was just stating some obvious facts from everyday life in a difficult way. Now, replace the truck with a beam of light (a photon) and the car with the Earth. The situation is depicted in figure 3. You observe the speed of light from a distant star at two instants: One at the

1st of January, another at the 1st of July. In January you are moving away from the photons approaching you from the distant star. In July you are moving towards the photons arriving from the star. If the speed of light with respect to the distant star is c, then in January you expect to measure the speed of the light beam from the star to be $c - v$ where $v = 30km/h$ is the speed of the Earth with respect to the same star (we assume that the star does not move with respect to the Sun, so this is also the orbital speed of the Earth). In July you expect to measure the speed of light from the star to be $c+v$, just as for the truck in the example above: The speed of the light beam seen from your frame of reference is supposed to be different depending on whether you move towards it or away from it.

In 1887 Michelson and Morley performed exactly this experiment which is now famous as the 'Michelson-Morley experiment'. The result however, was highly surprising: They measured exactly the same speed of light in both cases. The speed of light seemed to be the same independently of the frame of reference in which it is measured. This has some quite absurd consequences: Imagine that you see the truck driving at the speed of light (or very close to the speed of light, no material particle can ever travel at the speed of light). You are accelerating your car, trying to pass the truck. But no matter at which speed you drive, you see the truck moving with the speed of light with respect to your frame. Even when you reach half the speed of light, you still see the truck moving with velocity c. But how is this possible? An observer at rest with respect to the ground measures the truck moving with the speed of light as well, not with the velocity $c + c/2 = 3c/2$ as you would expect.

This was one of the first signs showing that something was wrong with classical physics. The fact that the speed of light seemed to be constant in all frames of reference led to several contradictions. We have already seen one example of such a contradiction. We will now look at another one which might shed some light on the underlying reason for these contradictions. In figure 4 we show the situation. Observer O is standing on the ground (at rest with respect to the ground), observer P is standing in the middle of a train of length L moving with velocity v with respect to the ground. Observer O sees two lightnings striking the front and the rear of the train simultaneously. We call the two events A and B (An event is a point in space and time, a point with a space and time coordinate): Event A is the lightning striking the front, event B is the lightning striking the rear. Events A and B are simulta-

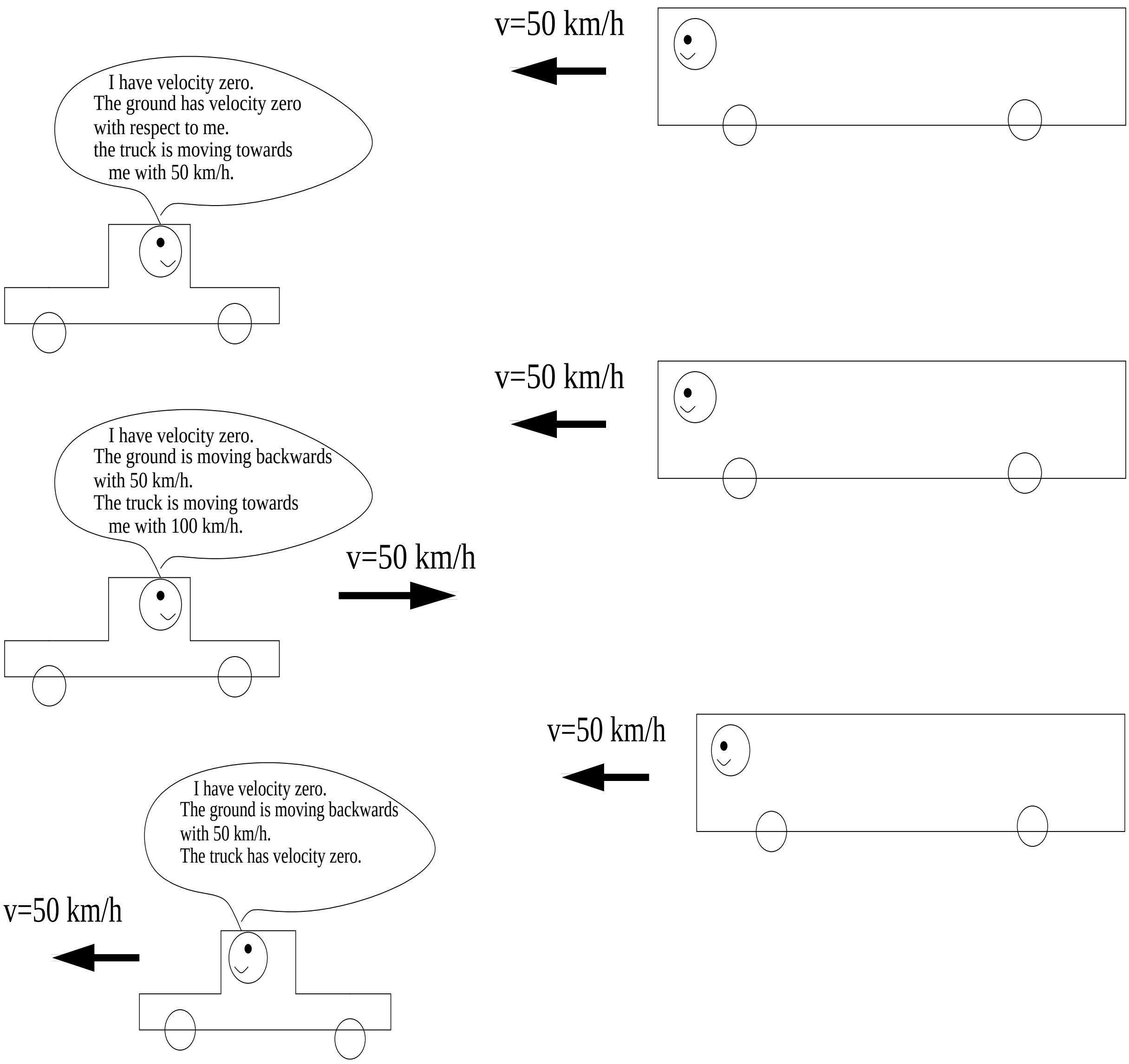

Figure 2: The velocity of the truck seen from the car depends on the velocity of the car

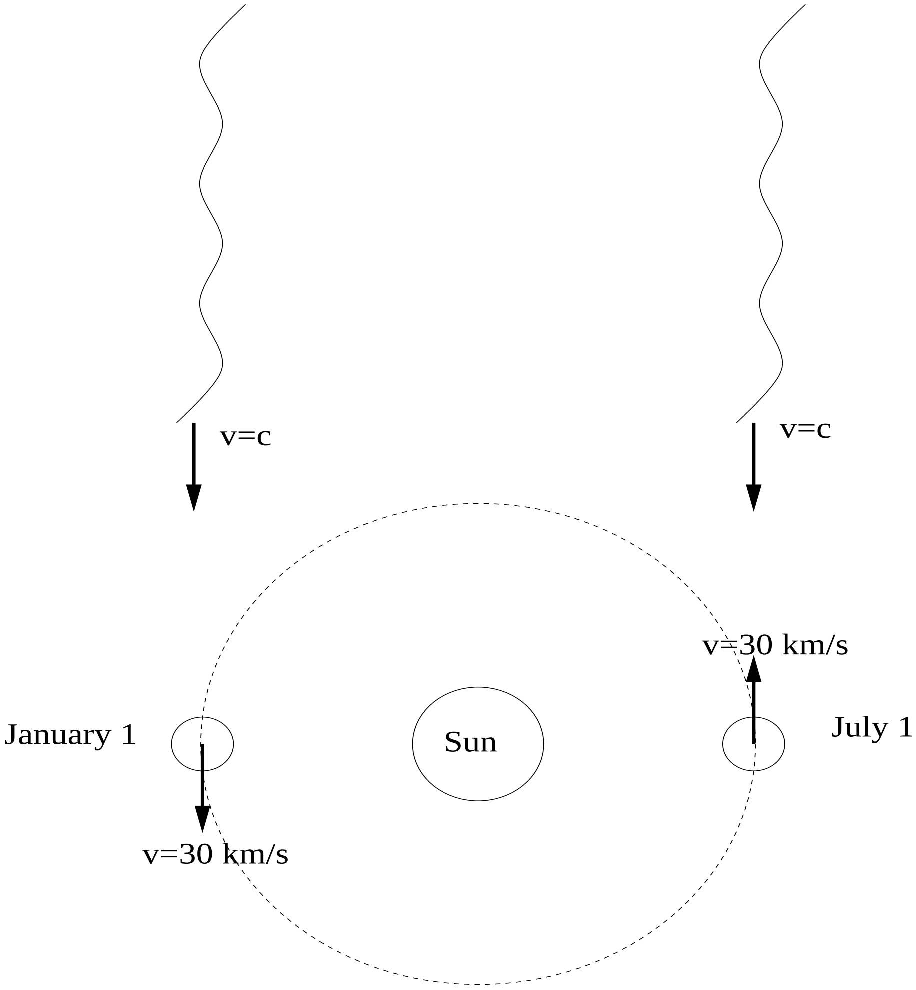

Figure 3: The velocity of the starlight is measured when the Earth has velocity 30 km/s towards and away from the light beam.

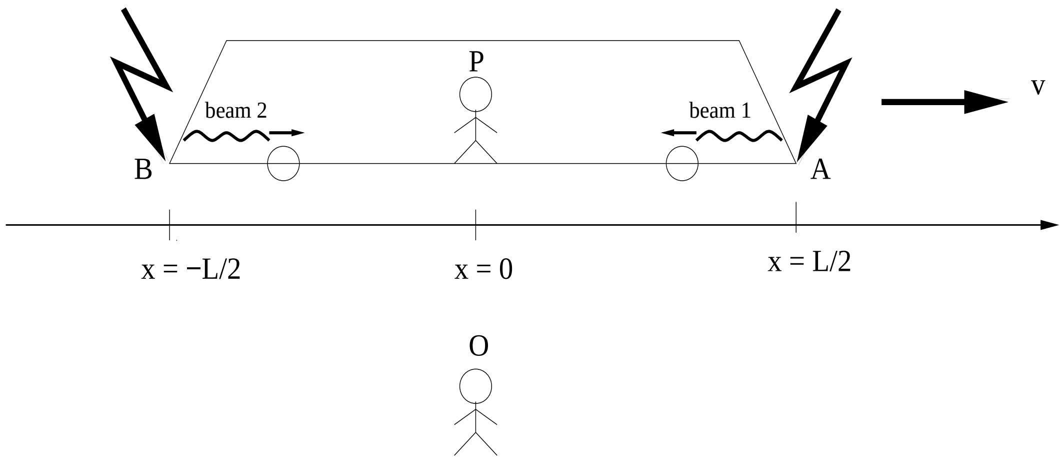

Figure 4: Event A: Lightning strikes the front part of the train. Event B: Lightning strikes the rear part of the train. These two events are observed by observer O on the ground and observer P in the train. The train has length L.

neous. The light from these two lightnings start traveling from the front and back end of the train towards observer P. The beam approaching observer P from the front is called beam 1 and the beam approaching from the rear is called beam 2. Both observers synchronize their clocks to $t = 0$ at the instant when the lightnings strike the train. Both observers have also defined their own coordinate systems x (observer on the ground) and x' (observer in the train) which is such that the position of observer P is at $x = x' = 0$ in both coordinate systems at the instant $t = 0$ when the lightenings strike. Thus the lightnings hit the train at the points $x = x' = L/2$ and $x = x' = -L/2$ as seen from both observers. We will now look how each of these observers experience these events:

From the point of view of observer O standing on the ground:

The frame of reference of observer O on the ground is often referred to as the *laboratory frame* . It is the frame of reference which we consider to be at rest. At what time $t = t_C$ does observer P see beam 1 (we call this event C) ? To answer this question, we need to have an expression for the x-coordinate of observer P and the x-coordinate of beam 1 at a given time t. Observer P moves with constant velocity v so his position at time t is $x_P = vt$. Beam 1 moves in the negative x-direction with the speed of light c starting from $x_1 = L/2$ at $t = 0$. The expression thus becomes $x_1 = L/2 - ct$. Observer P sees beam 1 when $x_1 = x_P$ at time t_C. Equating these two expressions, we find

$$t_C = \frac{L/2}{c+v}. \tag{1}$$

At what time $t = t_D$ does observer P see beam 2 (we call this event D) ? Following exactly the same line of thought as above, we find

$$t_D = \frac{L/2}{c-v}. \tag{2}$$

So according to observer O in the laboratory frame, $t_C < t_D$ and observer P should see the light beam from the lightning in front before the light from the back. This sounds reasonable: Observer P is moving towards beam 1 and away from beam 2 and should therefore see beam 1 first.

From the point of view of observer P standing in the train:

At what time $t = t_C$ does observer P see beam 1? We have just agreed on the fact that the speed of light is independent of the frame of reference. The result is that the speed of light is c also for the observer in the train. Seen from the frame of reference of observer P, observer P himself is at rest and the ground is moving backwards with speed v. Thus from this frame of reference, observer P is always standing at the origin $x'_P = 0$ (the coordinate system x' moves with observer P). The expression for x'_1 is the same as seen from observer O:$x'_1 = L/2 - ct$ (convince yourself that this is the case!). Again we need to set $x'_1 = x'_P$ giving

$$t_C = \frac{L/2}{c}$$

At what time $t = t_D$ does observer P see beam 2? Again we follow the same procedure and obtain

$$t_D = \frac{L/2}{c}$$

As calculated from the frame of reference of observer P, the two beams hit observer P at exactly the same time.

So not only are the exact times t_C and t_D different as calculated from the two frames of reference, but there is also an even stronger contradiction: Observer P should be hit by the two beams simultaneously as calculated from the frame of reference of observer P himself, but as calculated from the laboratory frame, beam 1 hits observer P before beam 2. What does really happen? Do the beams hit observer P simultaneously or not? Well, let's ask observer P himself:

So observer P, two lightnings stroke your train simultaneously at the front and rear end. Did you see these two lightnings simultaneously or did you see one flash before the other?
Observer P: Sorry? I think you are not well informed. The two lightnings did not happen simultaneously. There was one lightning which stroke the front part and then shortly afterwards there was another one striking the rear. So clearly I saw the flash in the front first.
Observer O: No, no, listen, the lightnings did strike the train simultaneously, there was no doubt about that. But you were moving in the direction of beam 1 and therefore it appeared to you that the front was hit by the lightning first.
Observer P: So you didn't watch very carefully I see. It is impossible that the two lightnings stroke at the same time. Look, I was standing exactly in the middle of the train. The speed of light is always the same, no matter from which direction it arrives. Beam 1 and beam 2 had to travel exactly the same distance $L/2$ with exactly the same speed c. If the beams were emitted simultaneously I MUST have seen the two flashes at the same time. But I didn't....beam 1 arrived before beam2, and so event A must have happened before event B

So beam 1 did indeed hit observer P before beam 2. And indeed, observer P has got a point: From observer P the two lightnings could not have occurred at the same time. Asking observer O one more time he says that he is absolutely certain that the two lightnings stroke simultaneously. Who is right?

We have arrived at one of the main conclusions that Einstein reached when he was discovering the theory of relativity: **Simultaneity is relative**. If two events happen at the same time or not depends on who you ask. It depends on your frame of reference. In the example above, the two lightnings were simultaneous for the observer at rest on the ground, but not for the observer moving with velocity v. This has nothing to do with the movement of the light beams, it is simply time itself which is different as seen from two different frames of reference. Simultaneity is a relative term in exactly the same way as velocity is: When you say that two events are simultaneous you need to specify that they are simultaneous with respect to some frame of reference.

In order to arrive at the conclusion of the relativity of simultaneity, Einstein excluded an alternative: Couldn't it be that the laws of physics are different in different frames of reference? If the laws of physics in the train were different from those in the laboratory frame, then simultaneity could still be absolute. The problem then is that we need to ask the question 'Physics is different in frames which move with respect to *which* frame of reference?'. In order to ask this question, velocity would need to be absolute. If velocity is relative, then we can just exchange the roles: The observer in the train is at rest and the observer on the ground is moving. Then we would need to change the laws of physics for the observer on the ground. This would lead to contradictions. In order to arrive at the theory of relativity, Einstein postulated the *Principle of Relativity* . The principle of relativity states that all laws of physics, both the mathematical form of these laws as well as the physical constants, are the same in all *free float frames*. In the lectures on general relativity we will come back to a more precise definition of the free float frame. For the moment we will take a free float frame to be a frame which is not accelerated, i.e. a frame in which we do not experience fictive forces. You can deduce the laws of physics in one free float frame and apply these in any other free float frame. Imagine two space ships, one is moving with the velocity $v = 1/2c$ with respect to the other. If you close all windows in these spaceship there is no way, by performing experiments inside these spaceships, that you can tell which is which. All free float frames are equivalent, there is no way to tell which one is at rest and which one is moving. Each observer in a free float frame can define himself to be at rest.

2 Invariance of the spacetime interval

We have seen that two events which are simultaneous in one frame of reference are not simultaneous in another frame. We may conclude that time itself is relative. In the same way as we needed two coordinate systems x and x' to specify the position in space relative to two different frames, we need two time coordinates t and t' to specify the time of an event as seen from two different frames. We are used to think of time as a quantity which has the same value for all observers but we now realize that each frame of reference has its own measure of time. Clocks are not running at the same pace in all frames of reference. Observers which are moving with respect to each other will measure different time intervals between the same events. Time is not absolute and for this reason simultaneity is not absolute.

Look at figure 5. It shows two points A and B and two coordinate systems (x, y) and (x', y') rotated with respect to each other. The two points A and B are situated at a distance $\Delta x_{AB} = L$ and at the same y-coordinate $\Delta y_{AB} = 0$ in the (x, y) system. In the rotated (x', y') system however, there is a non-zero difference in the y-coordinate, $\Delta y_{AB} \neq 0$. Now, replace y with t. Do you see the analogy with the example of the train above?

If we replace y with t and y' with t', then the two points A and B are the events A and B in *spacetime*. Our diagram is now a *spacetime diagram* showing the position of events in space x and time t, rather than a coordinate system showing the position of a point in space (x, y). Consider the two coordinate systems (x, t) and (x', t') as measurements in two different frames of reference, the lab frame and the frame of observer P. We see that in the (x, t) system, the two events are simultaneous $\Delta t_{AB} = 0$ whereas in the (x', t') system, the events take place at two different points in time.

We are now entering deep into the heart of the special theory of relativity: We need to consider time as the *fourth dimension*. And moreover, we need to treat this fourth dimension similar (but not identical) to the three spatial dimensions. That is, we need to talk about distances in space and distances in time. But, you might object, we measure distances in space in meters and time intervals in seconds. Can they really be similar? Yes they can, and you will soon get rid of the bad habit of measuring space and time in different units. From now on you will either measure space AND time in meters, or time AND space in seconds. By the time you have finished this course you will, without thinking about it, ask the lecturer how many meters the exam lasts or complain to your friends about how small your room in the dormitory is, giving them the size in square seconds.

How do you convert from meters to seconds and vice versa? The conver-

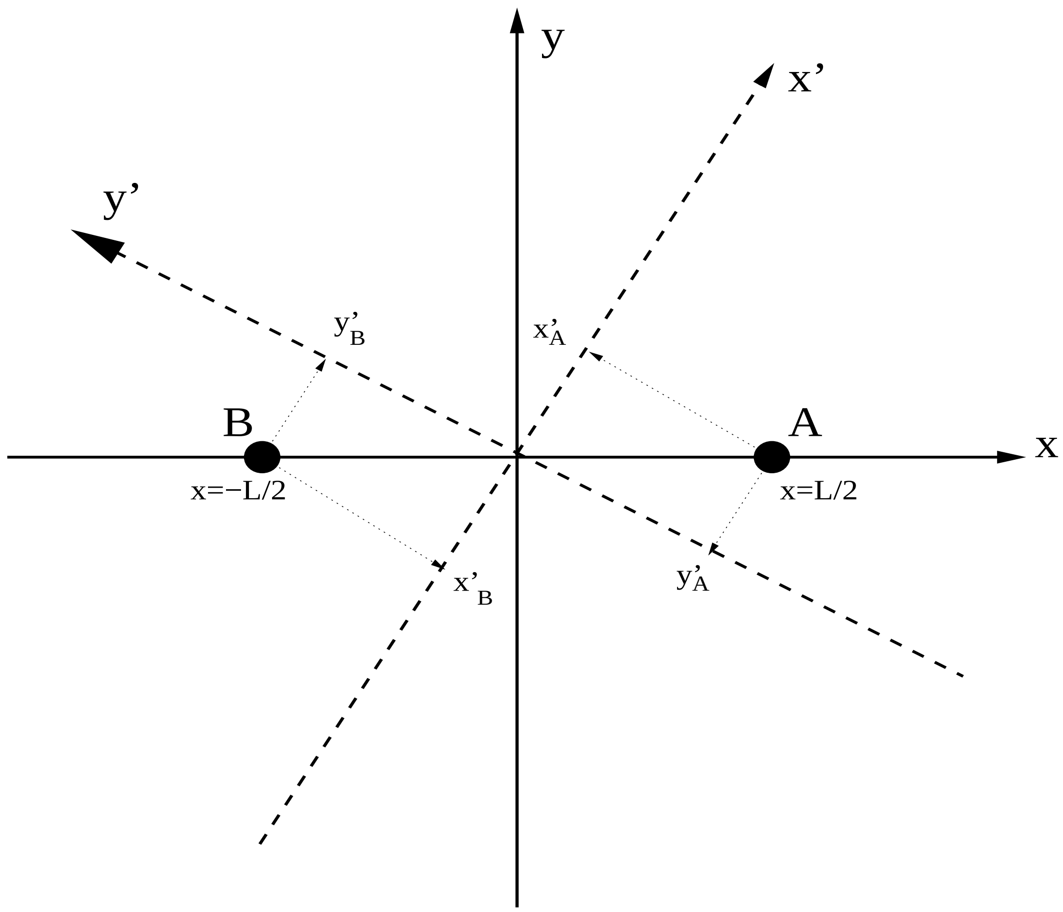

Figure 5: The position of two points A and B measured in two different coordinate systems rotated with respect to each other

sion factor is given by the universal factor c, the speed of light. If you have a time interval measured in seconds, multiply it by c and you have the time interval in meters. If you have a distance in space measured in meters, divide it by c and you obtain the distance measured in seconds:

$$x = ct, \quad t = x/c.$$

From now on we will drop the factor c and suppose that distances in space and time are measured in the same units. When you put numbers in your equations you need to take care that you always add quantities with the same units, if you need to add two quantities with different units, the conversion factor is always a power of c.

Measuring time in meters might seem strange, but physically you can think about it this way: Since the conversion factor is the speed of light, a time interval measured in meters is simply the distance that light travels in the given time interval. If the time interval between two events is 2 meters, it means that the time interval between these events equals the time it takes for light to travel 2 meters. We might say that the time interval between these events is 2 meters of light travel time. Similarly for measuring distances in seconds: If the spatial distance between two events is 10 seconds, it means that the distance equals the distance that light travels in 10 seconds. The distance is 10 light seconds. Actually you are already accustomed to measure distances in time units: You say that a star is 4 light years away, meaning that the distance equals the distance that light travels in four years. Note also one more effect of measuring space and time in the same units: Velocities will be dimensionless. Velocity is simply distance divided by time, if both are measured in meters, velocity becomes dimensionless. We can write this as $v_{\text{dimensionless}} = dx/(cdt) = v/c$ (to convert dt to units of length we need to multiply it by c, thus cdt). If the velocity $v = dx/dt = c$ is just the speed of light, we get $v_{\text{dimensionless}} = 1$. From now on we will just write v for $v_{\text{dimensionless}}$. Note that some books use β to denote dimensionless velocity, here we will use v since we will always use dimensionless velocities when working with the theory of relativity. The absolute value of velocity v is now a factor in the range $v = [0, 1]$ being the velocity relative to the velocity of light.

This was the first step in order to understand the foundations of special relativity. Here comes the second: Let us, for a moment, return to the spatial coordinate systems (x, y) and (x', y') in figure 5. Clearly the coordinates of the points A and B are different in the two coordinate systems. But there is one thing which is identical in all coordinate systems: The distance between points A and B. If we call this distance Δs_{AB} we can write this distance in the two coordinate systems as

$$\begin{aligned}(\Delta s_{AB})^2 &= (\Delta x_{AB})^2 + (\Delta y_{AB})^2 \\ (\Delta s'_{AB})^2 &= (\Delta x'_{AB})^2 + (\Delta y'_{AB})^2\end{aligned}$$

(check that you understand why!). The distance between A and B has to be equal in the two coordinate systems, so

$$(\Delta s_{AB})^2 = (\Delta s'_{AB})^2.$$

Is this also the case in spacetime? Can we measure intervals between events in spacetime? This is now, at least in theory, possible since we measure space and time separations in the same units. In a spatial (x, y, z) system we know the geometrical relation,

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2,$$

from Euclidean geometry: The square of the distance between two points (called the *line element*) is simply the sum of the squares of the coordinate distances between these two points. But do the rules of Euclidean geometry apply to spacetime? No, not entirely. The geometry of spacetime is called *Lorentz geometry*. The distance between two events (line element) in Lorentz spacetime Δs^2, is given by

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2).$$

Note the minus sign. This minus sign is the only thing which distinguishes space from time. The square of the spacetime distance between two events equals the square of the time separation between these events **minus** the square of the spatial separations between the events. And in the same way as the distance between two points in space is the same in all coordinate systems, the distance in spacetime, *the spacetime interval* is the same in all frames of reference. We say that the spacetime interval is *invariant*. A quantity is invariant if it has the same value in all frames of reference. We already know another invariant quantity: the speed of light.

So, that was it. We're done. Now you know what the special theory of relativity is all about. Congratulations! You now see that we may write the special theory of relativity in two sentences: Measuring space and time intervals in the same units, you can calculate the spacetime interval between two events using the formula for the line element in Lorentz geometry. This spacetime interval between two events is invariant, it has the same value as measured from all frames of reference. We will now see what this means in practice. But before you continue, take a walk, go for a coffee or simply take half an hour in fresh air. Your brain will need time to get accustomed to this new concept.

3 An example

A train is moving along the x-axis of the laboratory frame. The coordinate system of the laboratory frame is (x, y) and of the train, (x', y'). In the train a light signal is emitted directly upwards along the y-axis (event A). Three meters above, it is reflected in a mirror (event B) and finally returns to the point where it was emitted (event C). In the train frame it takes the light beam 3 meters of time to reach the mirror and 3 meters of time to return to the point where it was emitted. The total up-down trip (event A to event C) took 6 meters of time in the frame of the train (light travels with a speed of $v = 1$, one meter per meter of light travel time). From event A to event C, the train had moved 8 meters along the x-axis in the laboratory frame. Because of the movement of the train, the light beam moved in a pattern as shown in figure 6 seen from the lab frame.

1. *Use the figure to find the total distance d traveled by the light beam in the laboratory frame.* Dividing the triangle into two smaller triangles (see the figure), we find from one triangle that the distance traveled from the emission of the light beam to the mirror is $d/2 = \sqrt{(4m)^2 + (3m)^2} = 5m$ and similarly for the return path. Thus, the total distance traveled by the light beam from event A to event C is $d = 10m$.

2. *What was the total time it took for the light beam from event A to event C in the laboratory frame?* We have just seen that in the laboratory frame, the light beam traveled 10 meters from event A to event C. Since light travels at the speed of one meter per meter of time, it took 10

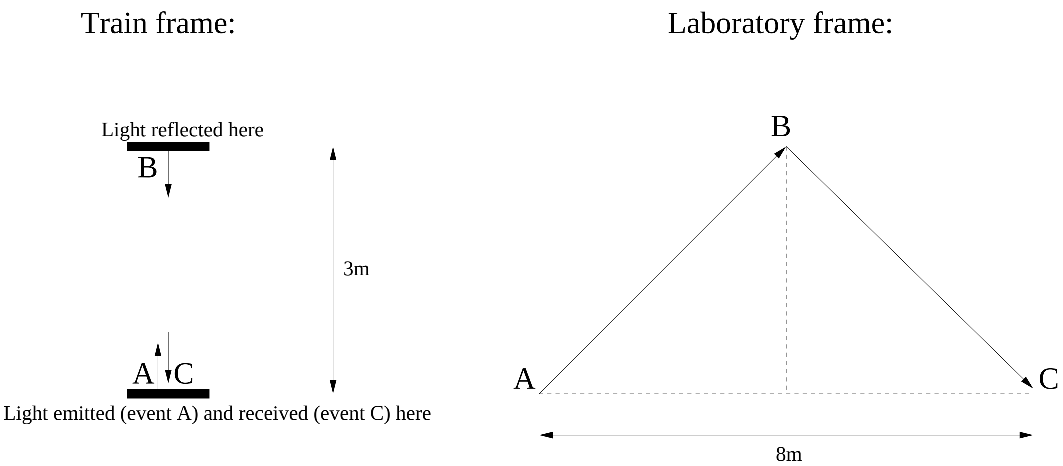

Figure 6: The light emitted (event A) upwards in the train is reflected (event B) and received (event C) at the same place (in the train frame) as it was emitted.

meters of time from event A to event C. In the frame of the train, it took only 6 meters of time.

3. *What is the speed of the train?* The train moved 8 meters in 10 meters of time, so the speed is $v = 8/10 = 4/5$, 4/5 the speed of light.

4. *What is the spacetime interval $\Delta s'$ between event A and event C with respect to the train frame?* In the train frame, event A and event C happened at the same point, so $\Delta x' = 0$. It took 6 meters of time from event A to event C, so $\Delta t' = 6m$. The spacetime interval is thus $\Delta s' = \sqrt{(6m)^2 - 0} = 6m$.

5. *What is the spacetime interval Δs between event A and event C with respect to the laboratory frame?* In the laboratory frame, the distance between the events were $\Delta x = 8m$ and the time interval was $\Delta t = 10m$. The spacetime interval is thus $\Delta s = \sqrt{(10m)^2 - (8m)^2} = 6m$, exactly the same as $\Delta s'$ in the train frame.

6. *Was there an easier way to answer the previous question?* Oh...uhm, yes, you're right, the spacetime interval is the same in all frames of reference so I should immediately had answered $\Delta s = \Delta s' = 6m$ without

any calculation....much easier!

Indeed much easier....remember that this will be very useful when calculating distances and intervals with respect to frames moving close to the speed of light.

4 Observer O and P revisited

Armed with the knowledge of the invariance of the spacetime interval we now return to observer O and P in order to sort out exactly what happened for each of the observers. We know that with respect to the laboratory frame, the two lightnings stroke simultaneously (events A and B were simultaneous) at points $x = \pm L/2$ at the time $t = 0$ when observer P was at the origin $x_P = 0$. But at what time did the two lightnings strike with respect to observer P in the train? We have learned that with respect to the frame of reference following the train, the events A and B were **not** simultaneous. But in the reference frame of observer P, at what time t'_A and t'_B did the two lightnings strike? The two observers exchange a signal at $t = 0$ such that their clocks are both synchronized to $t = t' = 0$ at the instant when observer P is at the origin in both coordinate systems $x_P = x'_P = 0$. Did event A and B happen before or after $t' = 0$ on observer P's wristwatch ? (it is common to talk about *wristwatches* when referring to the time measured in the rest frame of a moving object, i.e. the time measured by observers moving with the object. This wristwatch time is also called *proper time*).

We know that an event is characterized by a position x and a time t in each of the frames of reference. Let's collect what we know about the position and time of event A, B and the event when observer P passes $x = x' = 0$ which we call event P:

event P:

$$\begin{aligned} x = 0 \quad t = 0 \\ x' = 0 \quad t' = 0 \end{aligned}$$

event A:

$$\begin{aligned} x = L/2 \quad t = 0 \\ x' = L_0/2 \quad t' = t'_A \end{aligned}$$

event B:

$$x = -L/2 \quad t = 0$$
$$x' = -L_0/2 \quad t' = t'_B$$

Note that the length of the train is L_0 for observer P and L for observer O. We have already seen that observers in different frames of reference only agree on the length of the spacetime interval, NOT on lengths in space or intervals in time separately. For this reason, we do expect L and L_0 to be different. Look also at figure 5, the distance Δx_{AB} between the points A and B differ between the two coordinate systems, in the system (x, y) it is $\Delta x_{AB} = L$, but in the system (x', y') it is $\Delta x'_{AB} = x'_B - x'_A \equiv L'$. The length of the train in the rest frame of the train, L_0, is called *the proper lenght.* We will later come back to why it is given a particular name.

We want to find at which time t'_A and t'_B observed from the wristwatch of observer P, did events A and B happen? Did they happen before or after event P? For observer O all these events were simultaneous at $t = 0$, the moment in which the two observers exchanged a signal to synchronize their clocks. For observer P, could these events possibly had happened before they happened for observer O? Or did they happen later than for observer O?

In order to solve such problems, we need to take advantage of the fact that we know that the spacetime interval between events is invariant. Let's start with the spacetime interval between events A and B.

Spacetime interval AB: From each of the frames of reference it can be written as

$$\begin{aligned} \Delta s^2_{AB} &= \Delta t^2_{AB} - \Delta x^2_{AB}, \\ \Delta(s'_{AB})^2 &= (\Delta t'_{AB})^2 - (\Delta x'_{AB})^2. \end{aligned}$$

(note that the y and z coordinates are always 0, so $\Delta y = \Delta y' = 0$ and $\Delta z = \Delta z' = 0$). In order to calculate the spacetime interval, we need the space and time intervals Δx^2_{AB}, Δt^2_{AB}, $(\Delta x'_{AB})^2$ and $(\Delta t'_{AB})^2$ separately. In both frames, the spatial distance between the two events equals the length of the train in the given frame of reference. So $\Delta x_{AB} = L$ and $\Delta x'_{AB} = L_0$. For observer O the events were simultaneous $\Delta t_{AB} = 0$, whereas for observer P the events happened with a time difference $\Delta t'_{AB} = t'_A - t'_B$. Setting the two expressions for the spacetime interval equal we obtain,

$$L^2 = L_0^2 - (t'_A - t'_B)^2. \tag{3}$$

(check that you obtain this as well!). We have arrived at one equation connecting observables in one frame with observables in the other. We need more equations to solve for t'_A and t'_B. Let's study the spacetime interval between events A and P.

Spacetime interval AP: From each of the frames of reference it can be written as

$$\begin{aligned}\Delta s_{AP}^2 &= \Delta t_{AP}^2 - \Delta x_{AP}^2\\ \Delta(s'_{AP})^2 &= (\Delta t'_{AP})^2 - (\Delta x'_{AP})^2\end{aligned}$$

In order to calculate the spacetime interval, we need the space and time intervals Δx_{AP}^2, Δt_{AP}^2, $(\Delta x'_{AP})^2$ and $(\Delta t'_{AP})^2$ separately. In both frames, the spatial distance between the two events equals half the length of the train in the given frame of reference. So $\Delta x_{AP} = L/2$ and $\Delta x'_{AP} = L_0/2$. For observer O the events were simultaneous $\Delta t_{AP} = 0$, whereas for observer P the events happened with a time difference $\Delta t'_{AP} = t'_A - 0 = t'_A$. Setting the two expressions for the spacetime interval equal we obtain,

$$(L/2)^2 = (L_0/2)^2 - (t'_A)^2. \tag{4}$$

Note that we have three unknowns, t'_A, t'_B and L. We need one more equation and therefore one more spacetime interval. The spacetime interval between B and P does not give any additional information, so we need to find one more event in order to find one more spacetime interval. We will use event C, the event that beam 1 hits observer P.

Spacetime interval CP: Again, we need

$$\begin{aligned}\Delta s_{CP}^2 &= \Delta t_{CP}^2 - \Delta x_{CP}^2,\\ \Delta(s'_{CP})^2 &= (\Delta t'_{CP})^2 - (\Delta x'_{CP})^2.\end{aligned}$$

In the first section we calculated the time t_C when beam 1 hit observer P in the frame of observer O. The results obtained in the laboratory frame were correct since the events A and B really were simultaneous in this frame. As we have seen, the results we got for observer P were wrong since we assumed that events A and B were simultaneous in the frame of observer P as well. Now we know that this was not the case. We have $\Delta t_{CP} = t_C - 0 = t_C = L/2/(v+1)$ (from equation 1, note that since we measure time and space in the same units $c = 1$). As event C happens at the position of observer P, we can find the position of event C by taking the position of observer P at time t_C giving $\Delta x_{CP} = v\Delta t_{CP} = vL/2/(v+1)$. In the frame of observer P, event C clearly happened at the same point as event P so $\Delta x'_{CP} = 0$. The time of event C was just the time t'_A of event A plus the time $L/2$ it took for the light to travel the distance $L/2$ giving $\Delta t'_{CP} = t'_A + L_0/2$. Equating the line elements we have

$$\frac{L^2/4}{(v+1)^2}(v^2-1) = -(t'_A + L_0/2)^2 \tag{5}$$

Now we have three equations for the three unknowns. We eliminate L from equation (5) using equation (4). This gives a second order equation in t'_A with two solutions, $t'_A = -L_0/2$ or $t'_A = -vL_0/2$.

The first solution is unphysical: The time for event C is in this case $t'_C = t'_A + L_0/2 = 0$ so observer P sees the lightning at $t' = 0$. Remember that at $t = t' = 0$ observer O and observer P are synchronizing their clocks, so at this moment, and only this moment, their watch show the same time. This means that observer P sees flash A at the same moment as observer O sees the lightning. Thus at $t = t' = 0$, observer O would see the lightning hit the front of the train, but at the same time he would see it hit observer P.

Disregarding the unphysical solution we are left with

$$t'_A = -v\frac{L_0}{2}.$$

Thus event A happened for observers in the train before it happened for observers on the ground. Now we can insert this solution for t'_A in equation 4 and obtain L,

$$L = L_0\sqrt{1-v^2} \equiv L_0/\gamma, \tag{6}$$

with $\gamma \equiv 1/\sqrt{1-v^2}$. So the length of the train is smaller in the frame of observer O than in the rest frame of the train. We will discuss this result in detail later, first let's find t'_B. Substituting for t'_A and L in equation (3) we find

$$t'_B = v\frac{L_0}{2} = -t'_A.$$

So event B happened later for observers in the train than for observers on the ground. To summarize: Event A and B happened simultaneously at $t = t' = 0$ for observers on the ground. For observers in the train event A had already happened when they synchronize the clocks at $t = 0$, but event B happens later for the observers in the train. Note also that the time t'_A and t'_B are symmetric about $t' = 0$. If you look back at figure 5 we see that the analogy with two coordinate systems rotated with respect to each other is quite good: If we replace y by t we see that for the events which were simultaneous $\Delta y_{AB} = 0$ in the (x, y) frame, event A happens before $y = 0$ and event B happens after $y = 0$ in the rotated system (x', y'). But we need to be careful not taking the analogy too far: The geometry of the two cases are different. The spatial (x, y) diagram has Euclidean geometry whereas the spacetime diagram (x, t) has Lorentz geometry. We have seen that this simply means that distances are measured differently in the two cases (one has a plus sign the other has a minus sign in the line element).

We have seen that for observer P event A happens before event P when they synchronize their clocks. But does he also see the lightning before event P? As discussed above, this would be unphysical, so this is a good consistency check:

$$t'_C = t'_A + \frac{L_0}{2} = -v\frac{L_0}{2} + L_0/2 = L_0/2(1 - v),$$

which is always positive for $v < 1$. Thus observer P sees the flash after event P. When does observer P see the second flash (event D) measured on the wristwatch of observer P? Again we have $t'_D = t'_B + L_0/2$ giving

$$t'_D = L_0/2(1 + v),$$

so the time interval between event C and D measured on the wristwatch of a passenger on the train is

$$\Delta t' = t'_D - t'_C = vL_0$$

How long is this time interval as measured on the wristwatch of observer O? We already have t_C and t_D from equations (1) and (2). Using these we get the time interval measured from the ground,

$$\Delta t = vL_0/\sqrt{1 - v^2}$$

So we can relate a time interval in the rest frame of the train with a time interval on the ground as

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2}} = \gamma \Delta t'. \tag{7}$$

Note that I have skipped index CD here since this result is much more general: It applies to any two events taking place at the position of observer P. This is easy to see. Look at figure 7. We define an observer O which is at rest in the laboratory frame using coordinates (x, t) and an observer P moving with velocity v with respect to observer O. In the frame of reference of observer P we use coordinates (x', t').

We now look at two ticks on the wristwatch of observer P. Observer P himself measures (on his wrist watch) the time between two ticks to be $\Delta t'$ whereas observer O measures the time intervals between these two ticks on P's watch to be Δt (measured on observer O's wrist watch). In the coordinate system of observer P, the wristwatch does not move, hence the space interval between the two events (the two ticks) is $\Delta x' = 0$. For observer O, observer P and hence his wristwatch is moving with velocity v. So observer O measures a space interval of $\Delta x = v\Delta t$ between the two events. The spacetime interval in these two cases becomes

$$\begin{aligned}(\Delta s)^2 &= \Delta t^2 - \Delta x^2 = \Delta t^2 - (v\Delta t)^2 = (\Delta t)^2(1 - v^2) \\ (\Delta s')^2 &= (\Delta t')^2.\end{aligned}$$

Spacetime intervals between events are always equal from all frames of reference so we can equate these two intervals and we obtain equation (7).

Going back to the example with the train: If the train moves at the speed $v = 4c/5$ then we have $\Delta t = 5/3\Delta t' \approx 1.7\Delta t'$. When observer O on the ground watches the wristwatch of observer P, he notes that it takes 1.7 hours on his own wristwatch before one hours has passed on the wristwatch of observer P. If observer P in the train is jumping up and down every second on his own wristwatch, it takes 1.7 seconds for each jump as seen from the ground. For observers on the ground it looks like everything is in slow-motion inside the train.

How does it look for the observers in the train? Remember that velocity is relative. Being inside the train, we define ourselves as being at rest. From this frame of reference it is the ground which is moving at the speed $-v$. Everything has been exchanged: Since we now define the train to be at rest, the coordinate system (x, t) is now for the train whereas the coordinate system (x', t') is for the ground which is moving at velocity $-v$ (see figure 7). Note the minus sign: The motion of the ground with respect to the train is in the opposite direction than the motion of the train with respect to the ground.

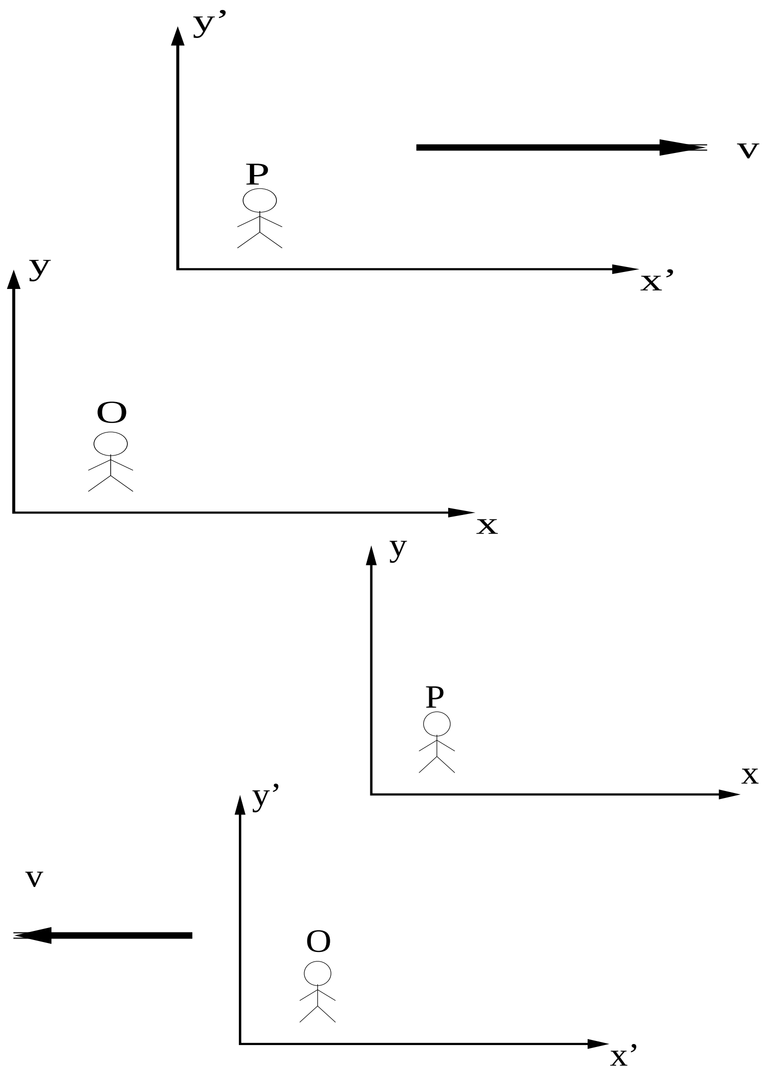

Figure 7: Two reference frames: (x, y) coordinates are used for the system defined to be at rest and (x', y') coordinates are used for the system defined to be moving. In the upper figure, observer O is in the laboratory frame with observer P in the frame moving with velocity v. In the lower figure, the two systems have exchanged roles and $v \to -v$. All equations derived in the above system will be valid for the system below by exchanging $v \to -v$.

We can now follow exactly the same calculations as above for two events happening at the position of observer O instead of observer P. For instance we watch two ticks on the clock of observer O. Then we find again formula (7) but with the meaning of Δt and $\Delta t'$ interchanges. Assuming again a speed of $v = -4c/5$ (note again the minus sign), observer P sees that it takes 1.7 hours on his wristwatch for one hour to pass on the wristwatch of observer O. It is the opposite result with respect to the above situation. While observers on the ground observe everything in the train in 'slow-motion', the observers on the train observe everything on the ground in 'slow-motion'. This is a consequence of the principle of relativity: There is no way to tell whether it is the train which is moving or the ground which is moving. We can define who is it rest and who is moving, the equations of motion that we obtain will then refer to one observer at rest and one observer in motion. When we change the definition, the roles of the observers in the equation will necessarily also change. Thus, if we define the ground to be at rest and the train to be moving and we deduce that observers on the ground will see the persons in the train in 'slow-motion', the opposite must also be true: If we define the train to be at rest and the ground to be moving, then the observers on the train will observe the observers on the ground in 'slow-motion'. Confused? Welcome to special relativity!

Consider two observers, both with their own wristwatch, one at rest in the laboratory frame (observer O) another moving with velocity v with respect to the laboratory frame (observer P). Going back to equation (7) we now know that if $\Delta t'$ is the interval between two ticks on the wristwatch of observer P, then Δt is the time interval between the same two ticks of observer P's watch measured on observer O's wristwatch. Using equation 7 we see that **the shortest time interval between two ticks is always the time measured directly in the rest frame of the wristwatch producing the ticks.** Any other observer moving with respect to observer P will measure a longer time interval for the ticks on observer P's wristwatch. This is of course also valid for observer O: The shortest time interval between two ticks on observer O's wristwatch is the time that observer O himself measures. The wristwatch time is called the *proper time* and is denoted τ.

Note that the proper time between two events (two ticks on a wristwatch) also equals the spacetime interval between these events. This is easy to see: consider again the ticks on observer P's wristwatch. In the rest frame of observer P, the wristwatch is not moving and hence the spatial distance between the two events (ticks) is $\Delta x = 0$. The time interval between these two events is just the proper time $\Delta\tau$. Consequently we have for the spacetime interval $\Delta s^2 = (\Delta t')^2 - (\Delta x')^2 = \Delta\tau^2 - 0 = \Delta\tau^2$.

Now, let's return to another result, the length of the train L as measured by observer O on the ground. Again, the result in equation 6 can be shown in a similar manner to be more general. The length L_0 can be the length of any object in the rest frame of this object. We see from equation 6 that **any observer which is not at rest with respect to the object will observe the length L which is always smaller than the length L_0**. The length of an object measured in the rest frame of the object is called the *proper length* of the object. An observer in any other reference frame will measure a smaller length of the object. The proper length L_0 is the longest possible length of the object. This also means that an observer in the moving train will measure the shorter length L for another identical train being at rest with respect to the ground (being measured to have length L_0 by observers on the ground).

5 The Lorentz transformations

Given the spacetime position (x, t) for an event in the laboratory frame, what are the corresponding coordinates (x', t') in a frame moving with velocity v along the x-axis with respect to the laboratory frame? So far we have found expressions to convert time intervals and distances from one frame to the other, but not coordinates. The transformation of spacetime coordinates from one frame to the other is called the Lorentz transformation. In the exercises you will deduce the expressions for the Lorentz transformations. Here we state the results. We start by the equations converting coordinates (x', y', z', t') in the frame moving along the x-axis to coordinates (x, y, z, t) in the laboratory frame,

$$\begin{aligned}
t &= v\gamma x' + \gamma t', \\
x &= \gamma x' + v\gamma t', \\
y &= y', \\
z &= z'.
\end{aligned}$$

To find the inverse transformation, we have seen that we can exchange the roles of the observer at rest and the observer in motion by exchanging the coordinates and let $v \to -v$ (see figure 7),

$$\begin{aligned} t' &= -v\gamma x + \gamma t, \\ x' &= \gamma x - v\gamma t, \\ y' &= y, \\ z' &= z. \end{aligned}$$

Here

$$\gamma = \frac{1}{\sqrt{1-v^2}}.$$

6 List of expressions you should know by now

laboratory frame → page 6
principle of relativity → page 9
free float frame → page 9
space time diagram → page 10
line element → page 13
Lorentz geometry → page 13
space time interval → page 13
invariance → page 13
proper time → page 16
proper length → page 17

7 Problems

Problem 1 (10 min. -15 min.)

We have seen the effect of Lorentz contraction, namely that a stick of proper length L_0 (measured in the rest frame of the stick) moving at a speed v along the x-axis in the laboratory frame, is measured to have a shorter length $L = L_0/\gamma$ in the laboratory frame. But what happens to the size of the stick in y and z directions measured from the laboratory frame? Do we correspondingly measure the stick to become thinner? We will now investigate this:

To check this possibility, imagine two identical cylinders A and B which are hollow such that if one cylinder becomes smaller (smaller radius) than the

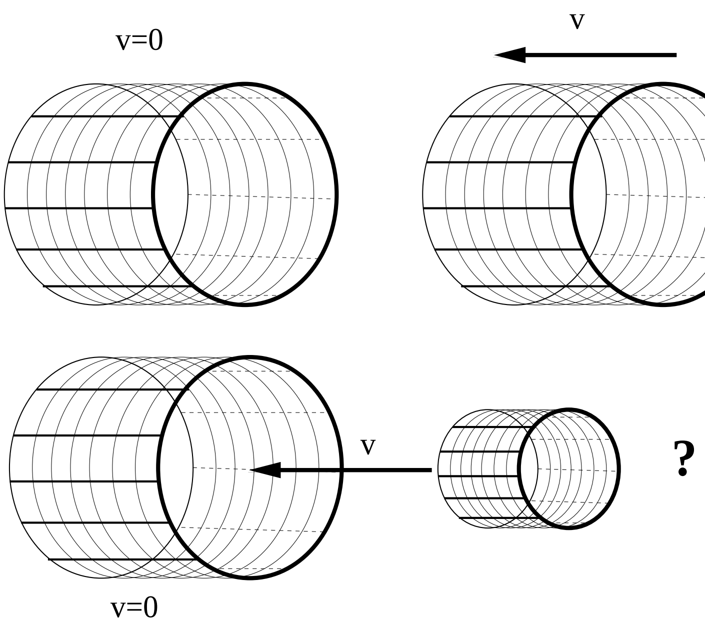

Figure 8: Does a moving cylinder become thinner as well as contracted seen from the laboratory frame? In problem 1 we study this more closely.

other, it might pass inside the larger cylinder (see figure 8). The axis of both cylinders are aligned with the x-axis at $y = z = 0$. Thus, the axis of both cylinders are exactly along the x-axis. Cylinder A is at rest in the laboratory frame, cylinder B is moving with velocity v along the x-axis, approaching cylinder A.

1. We know that the length of cylinder B as measured from the laboratory frame shrinks. Assume that the same effect takes place in the y and z directions such that the radius of cylinder B gets smaller measured in the laboratory frame. What happens when the two cylinders meet?

2. Now, look at exactly the same situation but from the point of view of an observer sitting on cylinder B. What happens when the two cylinders meet?

3. Can you give a good arguments to explain why $y = y'$ and $z = z'$ in the Lorentz transformations? (Note: this transformation is for movements along the x-axis. If there are movements along the y and z axes as well, the Lorentz transformation will look different and much more complicated. This is outside the scope of this course.)

Problem 2 (10 min. - 20 min.) A proton and an electron separated by a distance L_0 are at rest in a train.

1. What is the electric field E' from the proton at the location of the electron? (as measured in the rest frame of the train)

2. The train moves with velocity v with respect to the laboratory frame. Show that the electric field E as measured in the laboratory frame can be written as $E = E'/(1 - v^2)$.

3. Based on this result, can you now use the principle of relativity to find general qualitative arguments showing that the electric feld must be a relative quantity depending on the frame of reference in which it is measured?

Problem 3 (20 min. - 1 hour) When high energy cosmic ray protons collide with atoms in the upper atmosphere, so-called muon particles are produced. These muon particles have a mean life time of about $2\mu s$ ($2 \times 10^{-6} s$) after which they decay into other types of particles. They are typically produced about 15km above the surface of the Earth. We will now study a cosmic ray muon approaching the surface with the velocity of $0.999c$.

1. How long time does it take for a muon to arrive at the surface of the Earth as measured from the Earth frame?

2. Ignore relativistic effects: Do you expect many muons to survive to the surface of the Earth before decaying? (compare with the mean life time)

3. From relativity, we know that from the rest frame of the muon, the time it takes to reach the surface of the Earth is different. We will now use invariance of the spacetime interval to find the time it takes in the frame of the muon to reach the surface of the Earth.

   (a) Find the space and time distances Δx and Δt in the Earth frame and use these to obtain the spacetime interval Δs. Give all the answers in seconds.

   (b) What is $\Delta x'$, the spatial distance traveled by the muon in the muon rest frame?

   (c) Use invariance of the line element to obtain the travel time $\Delta t'$ in the muon rest frame. Will we detect muons at the surface of the Earth?

4. The diameter of the galaxy is about 100 000 light years, thus even with the speed of light it would take 100 000 years to pass the galaxy. How long time does it take to transverse the galaxy in the reference frame of a cosmic ray particle traveling at the speed of $v = 0.999999999999c$? (Use again invariance of the spacetime interval). Does this give some hope for future long distance space travel?

Problem 4 (1 - 2 hours)

You have devised a clock which works the following way: It consists of two mirrors a distance L_0 apart. A light ray is emitted along the positive x-axis at one of the ends and then reflected back and forth between the two mirrors. Each time it hits one of the mirrors it gives a 'tick'. See figure 9.

1. How long does it take between each tick in the reference frame of the clock?

2. Now we observe the clock from a passing train. The clock is at rest in the laboratory frame with coordinates (x, t) and we observe it from the

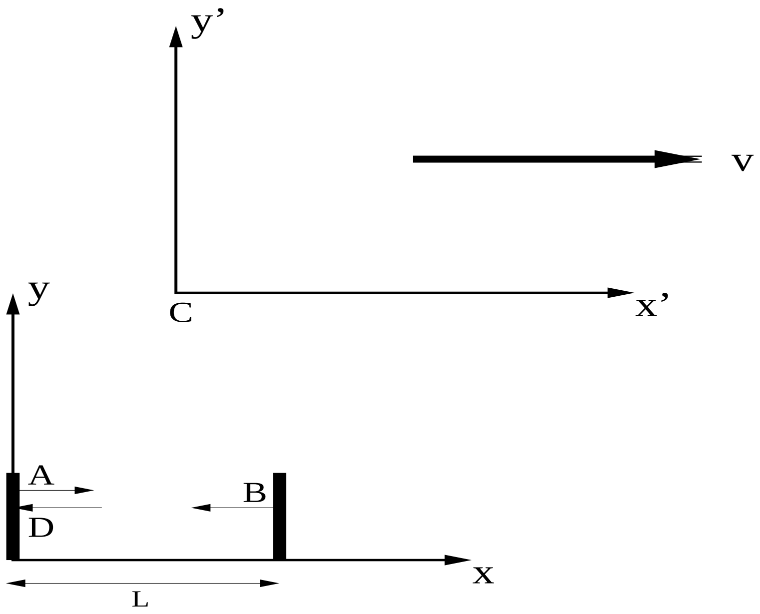

Figure 9: The situation in problem 4: A light beam is emitted when $x = x' = 0$ and $t = t' = 0$ (event A). Then the beam is reflected in the right mirror (event B) and reflected again in the left mirror (event D). This picture is taken from the laboratory frame at event B $t = t_B$ (the position of event A and D are just marked, they are not happening at this moment). Event C happens at the same time as event B in the laboratory frame. The position of event C in the laboratory frame is the position $x = x_C$ of the origin of the train frame.

train moving with velocity v along the positive x-axis of the laboratory frame. We use coordinates (x', t') for the train frame (see figure 9). Event A is the emission of light at the left mirror. This is the reference event occurring at $x = x' = 0$ and $t = t' = 0$. Event B is when the light ray hits the opposite mirror. We also introduce event C which takes place at the position of the middle point of the train (where $x' = 0$) at the same time as event B seen from the laboratory frame. We want to find out how long time $\Delta t'_{AB}$ it took for the light beam to reach the right mirror in the train frame. Write a list of events A, B and C and write the position (x, t) and (x', t') in the two frames for all three events. The only unknowns here are x'_B, t'_B and t'_C. All the other coordinates should be expressed in terms of the known quantities, L_0 and v.

3. Write the spacetime intervals Δs_{AB} and $\Delta s'_{AB}$ between events A and B in the two frames. Show that invariance of the interval gives $x'_B = t'_B$. Could you have guessed this using physical arguments without any calculations?

4. Write the spacetime intervals Δs_{AC} and $\Delta s'_{AC}$ between events A and C in the two frames. Show that invariance of the interval gives $t'_C = L_0/\gamma$.

5. Write the spacetime intervals Δs_{BC} and $\Delta s'_{BC}$ between events B and C in the two frames. Show that invariance of the interval gives $t'_B = L_0\gamma(1 - v)$.

6. Now define event D which is when the light ray returns to the first mirror at $x = 0$. Use invariance of the spacetime interval for appropriate events to find at what time t'_D event D happened in the train frame.

7. In the frame of the train, how long time did it take from the light was emitted to the first 'tick'? And how long time did it take from the first tick to the second tick? Compare this to the results in the lab frame. Is this a useful clock in the frame of reference of the train?

Problem 5 (30 min. - 1 hour) Quasars are one of the most powerful sources of energy in the universe. They are smaller than galaxies, but emit about 100 times as much energy as a normal galaxy. The engine in a quasar is believed to be a black hole. Jets of plasma are ejected into space from areas close to the black hole.

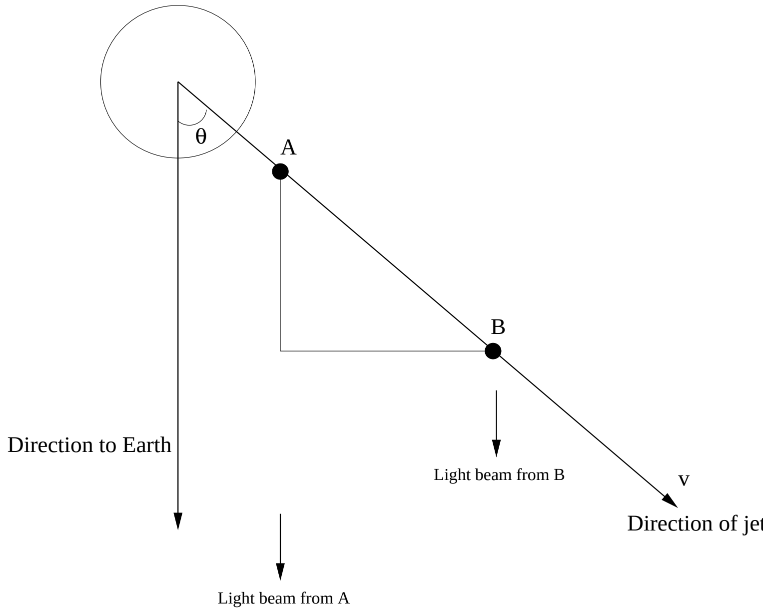

Figure 10: The quasar ejecting matter at an angle θ with the line of sight. The speed of the ejected matter is v. We define two events A and B which are the emission of photons from the ejected matter at the points A and B. At event A, the ejected matter passes point A and emits photons towards Earth. Three years later, the ejected matter passes point B and again emits photons.

1. In a Quasar called 3C273 at a distance of 2.6×10^9 light years from Earth, such a jet was observed during a period of three years. During this period it was found to have moved an angular distance of 2×10^{-3} arc seconds transversally on the sky. Show that the physical speed of the jet was $v = 8.4c$, more than eight times the speed of light.

2. We will now look at the physics of this process in order to understand what is going on. In figure 10 you can see the jet and two events A and B which are the events that photons were emitted as the jet moved through space. The photons emitted in event B were observed three years later than the photons emitted in point A. Here v is the real physical speed of the jet and θ is the angle between the direction of the jet and the line of sight. Show that the time interval $\Delta t_{\text{observed}}$ between the reception of photons (observations) from these two events is

$$\Delta t_{\text{observed}} = \Delta t(1 - v\cos\theta),$$

where Δt is the real time interval (in the Earth frame) between these two events. **hint:** No theory of relativity is needed in this calculation, all quantities you need are taken in the same frame of reference.

3. Show that the apparent transversal speed of the jet can be written as

$$v_{\text{observed}} = \frac{v\sin\theta}{1 - v\cos\theta}.$$

4. Assume that $\theta = 45°$. For which range of real speeds v do we observe an apparent speed v_{observed} which is larger than the speed of light?

5. The theory of relativity says that no signal can travel faster than the speed of light. Is this principle violated?

The effect we have seen here, an apparent speed of an object which exceeds the speed of light, is called *superluminal motion.*

Problem 6 (30 min. - 2 hours) In this exercise we will deduce the Lorentz transformations. We start by noting that the transformation equations must be linear in x and t. This is because the inverse transformation needs to have the same form as the original transformation by the principle of relativity: We can exchange the definition of who is at rest and who is moving only if the transformation is linear such that if $x \propto x'$ then $x' \propto x$. For instance if we had a coordinate transformation $x \propto (x')^2$, the inverse transformation would read $x' \propto \sqrt{x}$. These two equations would be completely different and the principle of relativity would be violated: The two observers would have completely different equations for transforming from one system to the other. Thus we can write the Lorentz transformations on the form

$$\begin{aligned}
t &= f(v)x' + g(v)t', && (8)\\
x &= h(v)x' + k(v)t', && (9)\\
y &= y',\\
z &= z',
\end{aligned}$$

where $f(v)$, $g(v)$, $h(v)$ and $k(v)$ are unknown functions of v. Note that the motion is along the x-axis, so no transformation is needed for the other two spatial dimensions. And again, by the principle of relativity, the inverse transformation must be obtained by exchanging the roles of the observers $(x, y) \leftrightarrow (x', y')$ and the velocity $v \to -v$ (see again figure 7),

$$\begin{aligned}
t' &= f(-v)x + g(-v)t, && (10)\\
x' &= h(-v)x + k(-v)t, && (11)\\
y' &= y,\\
z' &= z.
\end{aligned}$$

We need to solve for our unknown functions of v, namely $f(v)$, $g(v)$, $h(v)$ and $k(v)$.

1. Consider two events A and B. Event A happens at $x = x' = 0$ at $t = t' = 0$. Event B happens at (x, t) in the laboratory frame and at the origin $x' = 0$ at time t' in the moving frame (which moves with velocity v with respect to the laboratory frame). Write the time intervals Δt_{AB} and $\Delta t'_{AB}$ in terms of the coordinates x, t, x', t'. Then use equation (7) to find a relation between t and t'. You see that this relation already resembles one of equations (8)-(11) with one term missing. Look at at your coordinates and compare with the equations (8)-(11) and you will realize that the missing term vanishes. Show that

$$g(v) = \gamma.$$

2. We will still study the same two events. At what position x in the laboratory frame does event B happen? Express the answer in terms of t and v. Then use the previous result to elimintate t and write this in terms of t' and v. This gives you a relation between x and t'. You would need either an x' or t to obtain one of the relations above (equations 8-11), but show that one of these vanishes. Then show that

$$k(v) = v\gamma.$$

3. We will now study two different events A and B. Event A is again $x = x' = 0$ and $t = t' = 0$. But event B now happens at the position $x' = L_0$ in the moving frame and $x = L$ in the laboratory frame. In the laboratory frame, the two events happen at the same time. Use equation 6 to obtain a relation between x and x'. Look again at the Lorentz transformation equations (equations 8-11): Your expression needs either a t or a t' but one of these vanishes. You can thus conclude that

$$h(-v) = \gamma = h(v)$$

4. Now we are only missing f(v) in order to have deduced the full Lorentz transformations. Consider two other events A and B: Event A is again for $x = x' = 0$ at $t = t' = 0$ and event B is at position (x, t) in the laboratory frame and (x', t') in the moving frame. Use equations (8)-(9) to show that the spacetime interval between A and B for the two frames can be written

$$\begin{aligned} \Delta s^2 &= (f(v)x' + \gamma t')^2 - (\gamma x' + v\gamma t')^2 \\ (\Delta s')^2 &= (t')^2 - (x')^2 \end{aligned}$$

Show that invariance of the spacetime interval gives

$$f(v) = \gamma v.$$

The Lorentz transformations have been deduced.

Problem 7 (20 min. - 1 hour)

We will now return to the clock in problem 4 and solve this using the Lorentz transformations instead of the spacetime interval. We want to find at what time t'_B does the light hit the right mirror and at what time t'_D it has returned to the left mirror. Using the Lorentz transformations we will only need events A, B and D.

1. Again, write up the coordinates (x, t) and (x', t') for these three events. The following are unknown: x'_B, t'_B, x'_D and t'_D.

2. Use the Lorentz transformations to find t'_B and t'_D. You do not need to find x'_B and x'_D.

3. Use the Lorentz transformations to find the time (in the train frame) of the next two ticks of the clock. Are the intervals consistent with the first two ticks?