AST1100 Lecture Notes

9 - 10 The special theory of relativity: Four vectors and relativistic dynamics

1 Worldlines

In the spacetime diagram in figure 1 we see the path of a particle (or any object) through spacetime. We see the different positions (x, t) in space and time that the particle has passed through. Such a path showing the points in spacetime that an object passed is called a *worldline*. We will now study two events A and B (on the worldline of a particle) which are separated by a small spacetime interval Δs. These events could be the particle emitting two flashes of light or the particle passing through two specific points in space. The corresponding space and time intervals between these two events in the laboratory frame are called Δt and Δx. From the figure you see that $\Delta t > \Delta x$. You can see that this also holds for every small spacetime interval along the path. This has to be this way: The speed of the particle at a given instant is $v = \Delta x/\Delta t$. If $\Delta x = \Delta t$ then $v = 1$ and the particle travels at the speed of light. That $\Delta t > \Delta x$ simply means that the particle travels at a speed $v < c$ which it must. The worldline of a photon would thus be a line at 45° with the coordinate axes. The worldline of any material particle will therefore always make less than 45° with the time axis.

Events which are separated by spacetime distances such that $\Delta t > \Delta x$ are called *timelike events*. Timelike events may be causally connected since a particle with velocity $v < c$ would have the possibility to travel from one of the events to the other event. There is a possibility that the second event could have been caused by the first event since it is possible for a signal to travel between the events. Timelike events have positive line elements,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 > 0.$$

Events for which $\Delta t = \Delta x$ are called *lightlike events*. Only a particle

Figure 1: The worldline, the trajectory of a particle in a spacetime diagram. Two events A and B along the path of the particle have been marked.

traveling at the speed of light ($v = \Delta x/\Delta t = 1$) could travel from the first event to the second. Lightlike events have zero spacetime interval,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 = 0.$$

Note one consequence of this: Remember that the proper time interval $\Delta\tau^2$ equals the spacetime interval Δs^2. Thus, photons always have $\Delta\tau = 0$, the wristwatch attached to a photon would not change. Photons and other particles traveling at the speed of light do not feel the effect of time.

Events for which $\Delta x > \Delta t$ are called *spacelike events*. For these events, the spatial component of the distance is larger than the time component. No worldline could ever connect two spacelike events as it would require a particle to travel faster than light. Thus, spacelike events are not causally connected. The first event could not have caused the second. The spacetime interval for spacelike events is negative,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 < 0.$$

In figure 2 we see two events A and B and three different worldlines between these events. These events could be a car passing position x_A and

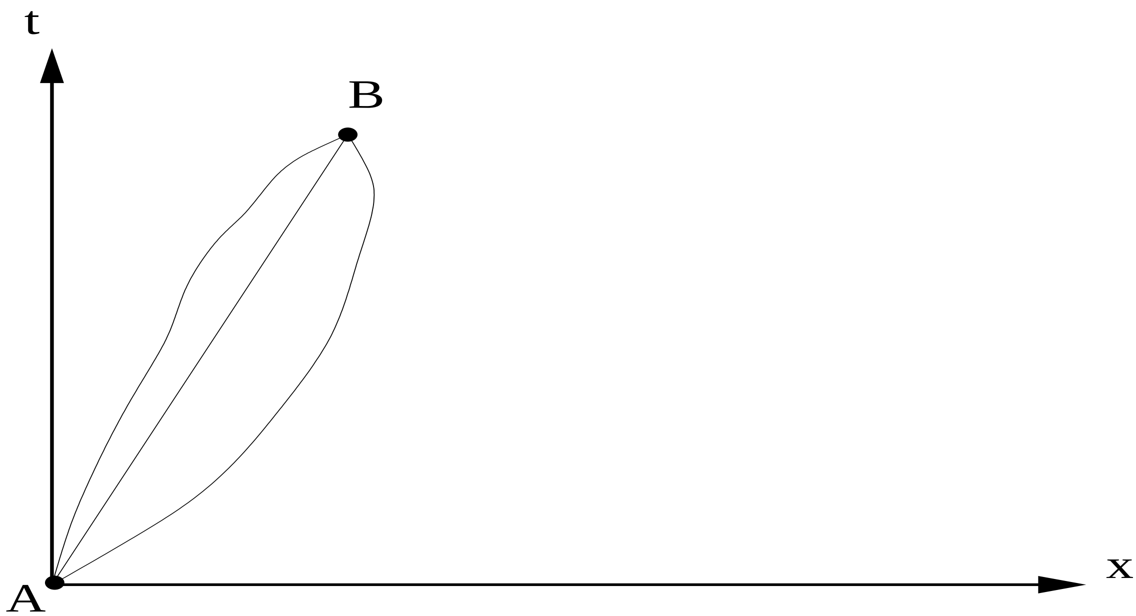

Figure 2: Different worldlines connecting the two events A and B.

position x_B in the laboratory frame. In the spacetime diagram we see three worldlines each corresponding to a car. The straight worldline must correspond to a car driving with constant speed $v = \Delta x/\Delta t =$ constant. The two other worldlines must correspond to cars accelerating (changing their speed and thereby changing the slope of the worldline) along the way from x_A to x_B but all cars reach point x_B at the same time (event B). All cars also passed point x_A at the same time (event A). Same time here means 'same time' for all frames of reference: all the cars meet at event A and B, so if they meet simultaneously in one frame of reference they must meet simultaneously in all other frames of reference (did you get this? If not, read the sentences again!).

We will now ask a question which answer may seem obvious in this case, but which might not be so obvious in other situations. The question is: Given a particle (or a car) going from event A to event B. If this particle is in free float (in special relativity this means that no forces act on the particle), which worldline will the particle take between event A and event B? Looking back at figure 2 we see three possible worldlines but in fact there is an infinite number of possible worldlines connecting the two events. The obvious answer in this case is that it will follow a straight line in spacetime, i.e. the straight worldline corresponding to constant velocity. This is just a modern way of saying Newton's first law: A body which is not under the influence of external forces will continue moving with constant velocity. But is there a deeper principle behind? In the theory of relativity there is, and this principle is called the *principle of maximal aging*. This is a fundamental principle in the special as well as in the general theory of relativity.

The principle of maximal aging says that a particle in free float (no forces act on the particle) will follow the worldline which corresponds to the longest possible proper time interval between the two events. We remember that proper time is the wristwatch time, the time measured on the clock attached to the particle. So let different particles take different paths in spacetime between the two events. Attach a wristwatch to each of the particles. At event B, you look at the time interval between event A and B measured on the wristwatch of each of the particles. The particle which measures the longest proper time, i.e. the particle with the wristwatch which made most ticks during the trip from event A to event B, is the particle taking the path that a particle in free-float would take.

How do we calculate the proper time interval that a given particle takes from event A to event B? The clue is to remember that the proper time interval $\Delta\tau$ between two events equals the spacetime interval, or the total length of the path in spacetime Δs taken between the two events. For the worldline of a particle with constant velocity, we know that the distance in spacetime traveled from event A to event B is just $\Delta s = \sqrt{\Delta t^2 - \Delta x^2}$ where Δx and Δt are space and time intervals measured in an arbitrary frame of reference. To measure the total spacetime interval along the worldline of a particle which does not move with constant velocity, we need to break the path up into small path lengths ds. This path length is so small that we can assume the velocity to be constant during the time it takes to travel this interval in spacetime. We can thus write $ds = \sqrt{dt^2 - dx^2}$ where dx and dt are the corresponding small space and time displacement measured in the arbitrary frame of reference. To obtain the total length of the path in spacetime traveled between two events A and B, we need to integrate all these tiny spacetime intervals ds giving

$$ds = \int_A^B \sqrt{dt^2 - dx^2}. \tag{1}$$

This equals measuring the length s of a curved path between two points A and B in the x-y plane:

$$ds = \int_A^B \sqrt{dx^2 + dy^2}.$$

Note again a huge difference here: The minus sign in the spacetime interval. We know from Euclidean geometry that the shortest path s between two points A and B in the plane, is the straight line. The minus sign in the line element for Lorentz geometry gives rise to the opposite result (which we will not derive here): The **longest** path s between two events A and B in spacetime is the straight worldline. Therefore, if we measure the length of the spacetime path for all the three worldlines in figure 2 using the integral in (1), we find that the longest path in spacetime is the straight worldline, i.e. the worldline of the car driving with constant velocity. Remember again that the length of the spacetime interval Δs equals the total proper time $\Delta\tau$ measured on the wristwatch of the particle. So the longest proper time interval between two events is measured on the particle taking the straight line in spacetime, i.e. the particle which has constant velocity. We have just deduced Newton's first law from the principle of maximal aging. When we come to the general theory of relativity, we will see that the spacetime geometry and hence the form of the line elements Δs is different in a gravitational field. We will need the principle of maximal aging to tell us how a free float particle is moving in this case.

2 Four-vectors

We are used to vectors in three-dimensional space giving the position of a point in space,

$$\vec{x} = (x_1, x_2, x_3),$$

where I have used (x_1, x_2, x_3) instead of (x, y, z) for the components in the three spatial dimensions. A 4-vector is similarly defined to give the position of an EVENT in four dimensional spacetime,

$$x = (x_0, x_1, x_2, x_3),$$

or if you wish (t, x, y, z). For components of a normal three dimensional vector, we use Latin letters, typically i and j, for the indices: The components of $\vec{x}$ are x_i where i goes from 1 to 3. For the components of a 4-vector, we use

Greek indices, typically μ and ν. The components of a four-vector x are x_μ where μ run from 0 to 3, 0 being the time component. If we wish to separate the time and space part of a four-vector we might also write it as $x = (t, x_i)$ where x_i refers to all three spatial components.

The four-vector x_μ points to an event in spacetime for a given frame of reference. We have already learned that in order to transform spacetime coordinates from one frame of reference to another, we need the Lorentz transformations. Thus, we may write the transformation of a four-vector x_μ in one frame of reference to x'_μ in another frame of reference by a matrix multiplication,

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Compare with the expression for the Lorentz transformation in the previous lecture notes. Check that the matrix multiplication gives you the correct equations. (Compare this equation with matrices which are used to rotate between coordinate systems in two spatial dimensions, do you see a similarity? Remember the analogy used in the previous lecture notes between a coordinate change in the (x, y) plane and the (x, t) diagram).

We can write this matrix equation as

$$x'_\mu = \sum_{\nu=0}^{3} c_{\mu\nu} x_\nu,$$

where $c_{\mu\nu}$ is the matrix above. This is the equation which transforms any four-vector from one frame of reference to another. We will now write this equation using the so-called Einstein conventions. This is just a rule which will save you from a lot of writing. Instead of writing the sum symbol, we simply say that when two factors in a term contain the same index, there is an implicit sum over this index. If the index is Latin, then there is a sum over the three spatial dimensions, if the index is Greek, there is a sum over the three spatial dimensions plus time. Using this convention we can write the previous equation simply as

$$x'_\mu = c_{\mu\nu} x_\nu \tag{2}$$

It can be shown that four-vectors follow the normal rules for summations and subtractions (see exercises). We will now look at the scalar product. For three dimensional vectors, the scalar product can be written as,

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{3} x_i y_i = x_i y_i,$$

where the Einstein convention was used in the last expression. We can also define a scalar product for four-vectors. Instead of writing a dot between the vectors, one usually writes the scalar product with one upper index and one lower index,

$$x^\mu y_\mu = x_0 y_0 - x_i y_i.$$

One index μ is written high and the other low to show that this is the scalar product and NOT a normal sum. Note that the scalar product is defined with a minus sign in front of the spatial part. If we had written both indices low, this would mean,

$$x_\mu y_\mu = x_0 y_0 + x_i y_i,$$

using the Einstein summation convention. This is different from the scalar product. It should be clear where the minus sign comes from, consider a spacetime interval Δx_μ (a spacetime interval is an interval between two points x_μ^1 and x_μ^2 in time and space such that $\Delta x_\mu = x_\mu^1 - x_\mu^2 = (\Delta t, \Delta x, \Delta y, \Delta z)$). The scalar product of a spacetime interval with itself gives,

$$\Delta x^\mu \Delta x_\mu = \Delta t^2 - \Delta x^2 = \Delta s^2$$

(assuming $\Delta y = \Delta z = 0$). The result is the *scalar* Δs^2. A scalar is a quantity which is invariant, which has the same value in all frames of reference. We already knew that the spacetime interval Δs^2 is a scalar (where did we learn this?). For infinitesimal distances between events, we may write this as,

$$ds^2 = dx^\mu dx_\mu.$$

We learned above that a four vector is a vector which transforms according to the Lorentz transformation (equation 2) when changing from one frame of reference to another frame of reference having velocity v with respect to the first. This has an important consequence: You cannot choose 4 numbers on random, put them together and call it a 4-vector! The numbers entering in a four-vector need to be physical quantities which are such that the 4-vector transforms accoring to equation 2. We thus need to take care when performing mathematical operations with 4-vectors: The result may not necessarily be a 4-vector.

As an example we will now investigate what happens with a 4-vector when multiplying it with some number. Say that you for some reason need to multiply a spacetime distance $\Delta x_\mu = (\Delta t, \Delta x, \Delta y, \Delta z)$ with the corresponding time interval Δt forming

$$\Delta y_\mu = \Delta t \Delta x_\mu.$$

Is Δy_μ a 4-vector? We can easily check this by checking whether it transforms according to equation 2 when changing frame of reference. We know that Δx_μ follows this transformation. We also now that $\Delta t' = (1/\gamma)\Delta t$ when changing frame of reference. We thus have for $\Delta y'_\mu$ in a new frame of reference

$$\Delta y'_\mu = \Delta t' \Delta x'_\mu = (1/\gamma)\Delta t c_{\mu\nu} \Delta x_\nu = (1/\gamma) c_{\mu\nu} \Delta y_\nu.$$

Because of the factor $1/\gamma$ we see that Δy_μ does not transform according to equation 2 and Δy_μ is therefore NOT a 4-vector. We thus cannot multiply a 4-vector with a time interval and obtain a 4-vector.

A four-vector which is multiplied by a scalar however, is itself a four-vector. If instead of multiplying Δx_μ with Δt, we multiply it with the corresponding spacetime interval Δs we get

$$\Delta y_\mu = \Delta s \Delta x_\mu.$$

Transforming to a different frame of reference we have again $\Delta x'_\mu = c_{\mu\nu} \Delta x_\nu$ since Δx_μ is a four-vector and $\Delta s' = \Delta s$ since Δs is a scalar. We thus have

$$\Delta y'_\mu = \Delta s' \Delta x'_\mu = \Delta s c_{\mu\nu} \Delta x_\nu = c_{\mu\nu} \Delta y_\mu$$

which does follow equation 2. In this case Δy_μ is a four-vector. We thus have generally that when A_μ is a four vector and f is a scalar, the product

$$B_\mu = f A_\mu,$$

is a 4-vector. In the exercises you will show that the results of summing or subtracting 4-vectors are 4-vectors.

3 Four-velocity

Can we define a four dimensional velocity V_μ, that is, a four dimensional vector showing the direction of motion in spacetime of a particle with coordinates x_μ? By analogy to normal three dimensional velocity, the four-velocity V_μ should be the the rate of change of x_μ. A natural choice would be dx_μ/dt, but this is not a four-vector: As we discussed above, Δt or dt is not a scalar, it has different values in different frames of reference. Thus dx_μ/dt does not transform as a 4-vector, i.e. you cannot use the Lorentz transformation to transform it from one frame of reference to another. But in order to have velocity, we need the rate of change with respect to some time interval Δt. Which measure of time can we use?

Remember that proper time τ is a scalar, it is defined as the time observed on the wristwatch of an observer. All observers will measure the same time interval $\Delta\tau$ between two events (how do they measure $\Delta\tau$?). Consider the example with the train and observer P who is jumping up and down. Measured on the wrist watch of observer P, one jump takes one second, thus one second of proper time for the frame of reference of the train. According to observer O's wristwatch, the jump takes 1.7 seconds, but this is not the proper time for the train (remember the definition of proper time!). But observer O can take his binoculars and read of the time between each jump on observer P's wristwatch. He will then find, in agreement with observer P, that in proper time units for the train, each jump takes one second.

Note that proper time needs to be defined with respect to some frame of reference (in this case the train), but once this is defined, everybody agrees on the proper time interval between two events taking place at the same spot in that frame. In the case of four-velocity, there is no doubt about which proper time we are speaking about: Four-velocity is the velocity of a particle or an object (for instance a train) and the proper time $\Delta\tau$ which we use to define four velocity is the time measured in the rest frame of this object. So four-velocity can be defined as

$$V_\mu = \frac{dx_\mu}{d\tau}.$$

Let us find the length (absolute value) of the four-velocity (the square root of the scalar product of the vector with itself). The square of the length is

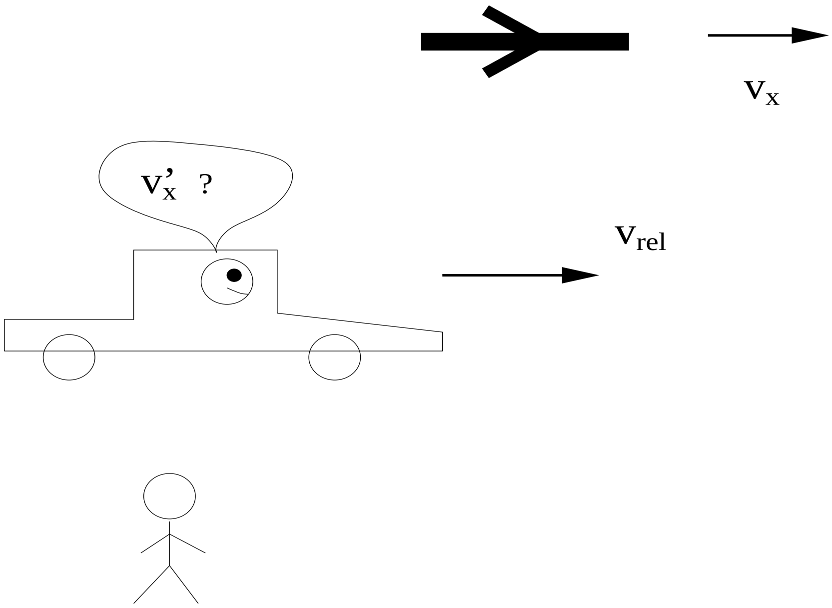

Figure 3: The observer on the ground measuring a velocity v_x for the airplane, wondering which velocity v'_x the driver of the car measures for the same airplane.

(as for normal vectors) given by

$$V_\mu V^\mu = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{dx_\mu dx^\mu}{d\tau^2} = \frac{ds^2}{d\tau^2} = \frac{d\tau^2}{d\tau^2} = 1.$$

(did you understand every step here?) Taking the square root of this we still get 1. The length of the four-velocity is thus always one. Remember that a velocity of one means the velocity of light. All particles move with the velocity of light in spacetime! For each proper time interval $\Delta\tau$ a particle moves an equal interval Δs in spacetime.

We can write the four-velocity in terms of normal 3-velocity as

$$V_\mu = (\frac{dt}{d\tau}, \frac{dx_i}{d\tau}) = (\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{dx_i}{dt}) = \frac{dt}{d\tau}(1, \vec{v}) = \gamma(1, \vec{v}),$$

where we have used that $\Delta t/\Delta\tau = dt/d\tau = \gamma$ from the previous lecture notes (go back and check how you derived this, it is important!). Now we are ready to answer a question that has bothered us all the time since we learned about the Lorentz transformations: We know how to transform between coordinates (x,t) and (x',t') in two different frames of reference. But how do you transform a velocity v_x from one frame to the other? Say that you stand on the ground and look at a passing airplane. You measure the velocity of the airplane along the x-axis to be v_x. A car is passing you on the street with velocity $v_{\rm rel}$ along the same x-axis and you note that the driver is also watching the airplane. You start to wonder which velocity v'_x that the driver is measuring for the airplane. The situation is depicted in figure 3. In normal non-relativistic physics you know that the answer should read $v'_x = v_x - v_{\rm rel}$, but we have learned that this does not work for velocities close to the velocities of light (for instance, look back at the Michelson-Morley experiment). Assuming that there are no motions in the y and z direction, we can now write the four velocity of the airplane from our laboratory frame as $V_\mu = \gamma(1,v_x)$ and from the car as $V'_\mu = \gamma'(1,v'_x)$ where $\gamma = 1/\sqrt{1-v_x^2}$ and $\gamma' = 1/\sqrt{1-(v'_x)^2}$. We know that four-velocity is a four-vector and that four-vectors by definition transform from one frame of reference to the other under the Lorentz transformation,

$$V'_\mu = c_{\mu\nu}V_\nu,$$

or written in terms of matrices as

$$\begin{pmatrix} \gamma' \\ \gamma' v'_x \\ \gamma' v'_y \\ \gamma' v'_z \end{pmatrix} = \begin{pmatrix} \gamma_{\rm rel} & -v_{\rm rel}\gamma_{\rm rel} & 0 & 0 \\ -v_{\rm rel}\gamma_{\rm rel} & \gamma_{\rm rel} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

where $\gamma_{\rm rel} = 1/\sqrt{1-v_{\rm rel}^2}$. From this matrix equation, we obtain two equations for the velocity v_x and v'_x,

$$\begin{aligned} \gamma' &= (\gamma_{\rm rel} - v_{\rm rel}\gamma_{\rm rel}v_x)\gamma \\ \gamma' v'_x &= (-v_{\rm rel}\gamma_{\rm rel} + \gamma_{\rm rel}v_x)\gamma. \end{aligned}$$

Dividing the second equation by the first, we obtain

$$v'_x = \frac{v_x - v_{\rm rel}}{1 - v_{\rm rel}v_x}, \tag{3}$$

which is the Lorentz transformation for velocities. Note that when the speed of the airplane approaches the speed of light, $v_x \to 1$ then $v'_x \to 1$ showing that the laboratory observer and the observer in the car will both measure the speed of light for the airplane. This solves the weird result obtained by Michelson and Moreley: The speed of light is the same from all frames of reference.

4 Relativistic momentum and energy

What about momentum and energy? We have learned that the velocity v of an object as measured from two different frames of reference transform according to the Lorentz transformation (equation 3). This must necessarily have consequences for how we measure momentum $p = mv$ and energy $E = 1/2mv^2$ from two different frames of reference. There must be some corresponding Lorentz transformations for momentum and energy. We have learned a simple and easy recipe for finding the transformation equations between different frames: Construct a four-vector and use the transformation properties for four-vectors. This worked for velocity so let's try with momentum and energy.

We start with momentum. In order to construct a four-vector P_μ for momentum, let's try a form which is as similar as possible to the Newtonian form $\vec{p} = m\vec{v}$. Rest mass (the mass measured in the rest frame of the object) is a scalar quantity, so

$$P_\mu = mV_\mu$$

is a four-vector. Using that $V_\mu = \gamma(1, \vec{v})$, we can write momentum as

$$P_\mu = m\gamma(1, \vec{v}) = \gamma(m, \vec{p}),$$

where $\vec{p}$ is the Newtonian momentum. Taking the spatial part of this equation we see that relativistic momentum can be written in three dimensions simply as

$$\vec{p}_{\text{relativistic}} = \gamma m\vec{v}, \tag{4}$$

where $\vec{v}$ is the normal 3-velocity of an object. What is the meaning of the time component $P_0 = \gamma m$ of the momentum 4-vector? In order to investigate this let us write it in the Newtonian limit. For $v << 1$ (velocity much lower than the velocity of light) we can make a Taylor expansion in v,

$$P_0(v) = P_0(v = 0) + \frac{dP_0}{dv}(v = 0)v + \frac{1}{2}\frac{d^2P_0}{dv^2}(v = 0)v^2,$$

where the derivatives taken at $v = 0$ are (check it!) $P_0(v = 0) = m$, $dP_0/dv(v = 0) = 0$ and $d^2P_0/dv^2(v = 0) = m$. We get

$$P_0 = m + \frac{1}{2}mv^2.$$

The last term is just the expression for Newtonian kinetic energy. The first term is the *rest energy* of a particle, converted to normal units it can be written as the more well known $E = mc^2$. The rest energy is the energy of a particle at rest, it is the energy in the mass of the particle. Thus, the time component of the momentum four-vector is relativistic energy,

$$E_{\text{relativistic}} = m\gamma, \tag{5}$$

which in the Newtonian limit reduces to the Newtonian kinetic energy plus an energy term which did not exist in Newtonian physics, the energy of the mass of the particle. So the 4-vector P_μ is not just a momentum 4-vector, it is *the momentum-energy 4-vector* which time component is energy and space component is momentum. It means that energy and momentum are related in the same way as space and time are. In the same manner as we talk about spacetime, indicating that space and time are basically two aspects of the same thing, we can call energy and momentum collectively as *momenergy*. The four-vector P_μ is simply the *momenergy four-vector*.

What is the length of the momenergy four-vector? Using that $P_\mu = mV_\mu$ we have for the square of the length

$$P_\mu P^\mu = m^2 V_\mu V^\mu = m^2.$$

The length is the square root of m^2 which is m. The length of the momenergy four-vector is an invariant and it is thus simply the mass. We have seen that we can write $P_\mu = \gamma(m, \vec{p})$ giving (using equations 4 and 5)

$$P_\mu = (E_{\text{relativistic}}, \vec{p}_{\text{relativistic}}).$$

From now on we will drop the subscript 'relativistic' and always refer to the relativistic energy and relativistic momentum using E and p. But how can we be so sure? How can we know that this is the correct expression for energy and momentum? What is the criterion for a quantity to be energy or momentum? We know that energy and momentum are conserved quantities. The total energy and momentum of particles after a collision should always be the same as the total energy and momentum before the collision. So this is easy to check: Measure the total energy and momentum of particles before and after a collision, if they are the same we have found the correct expressions for momenergy. This has been tested thousands of times in particle accelerators with particles moving close to the speed of light. It turns out that the Newtonian energy and momentum are **not** conserved in these collisions. The relativistic energy and momentum defined as we have done above however, **are** conserved.

By now we have got used to measure time and space in the same units and therefore we have also got used to add these quantities $\Delta x + \Delta t$ without hesitating. We see that the result of measuring time and space in the same units is that momentum and energy are also measured in the same units, the units of mass. We remember that since space and time are measured in the same units, the speed v is a dimensionless number. The factor γ is clearly also dimensionless, so the momentum $p = m\gamma v$ can be measured in the units of mass (kg). The same goes for energy $E = m\gamma$, which also has dimension mass. So both energy and momentum are measured in kg and these quantities can therefore be added, just as we can add intervals in time and distances in space. The momenergy four-vector is $P_\mu = (E, \vec{p})$, taking the scalar product we have (remembering the result above that the length of P_μ is just m),

$$P_\mu P^\mu = E^2 - p^2 = m^2,$$

we can thus write energy in terms of momentum as

$$E = \sqrt{m^2 + p^2}.$$

A photon is massless, so for photons this relation is just

$$E = p,$$

or by using normal units $E = pc$ which is a more known form of this expression.

We return to the above example with the airplane and the passing car. You measure the relativistic energy and momentum of the airplane from the laboratory frame (the ground) and you wonder what energy and momentum the driver of the car measures for the same airplane. The momenergy four-vector is a four-vector which means that it can be transformed from one frame of reference to the other by the Lorentz transformation,

$$P'_\mu = c_{\mu\nu} P_\nu,$$

or in matrix form (remember that there were no movements in the y and z direction)

$$\begin{pmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma_{\rm rel} & -v_{\rm rel}\gamma_{\rm rel} & 0 & 0 \\ -v_{\rm rel}\gamma_{\rm rel} & \gamma_{\rm rel} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Giving the following transformation equations for momentum and energy

$$\begin{aligned} E' &= \gamma_{\rm rel} E - v_{\rm rel}\gamma_{\rm rel} p_x \\ p'_x &= \gamma_{\rm rel} p_x - v_{\rm rel}\gamma_{\rm rel} E \end{aligned}$$

where $v_{\rm rel}$ is the relative velocity between the two frames of reference, the observer on the ground and the car (see figure 4).

We will now use these equations to answer the following question: What energy and momentum (E', p'_x) does a person passing you in his car with a velocity v (relative to you) measure that you have? From your frame of reference in which you are at rest, your momentum is by definition zero $p = 0$ and you energy equals your mass $E = m$. We will now transform these quantities to the driver of the car measuring your energy and momentum to be E' and p'. The relative velocity of the car with respect to you is simply $v_{\rm rel} = v$. Then the energy and momentum that the driver in the car measures that you have is simply (using the equations above, check that you get the same result),

$$E' = \gamma E \qquad p'_x = -v\gamma E$$

Note that $\gamma > 1$ so the driver in the car measures, not only a larger absolute momentum, but also larger energy.

From the point of view of Newtonian mechanics this was to be expected: with respect to the driver you have a non-zero velocity and kinetic energy, thus both your momentum and energy are clearly larger with respect to him than with respect to your rest frame. But from the point of view of geometry it might seem strange: In your rest frame the four-vector P_μ only has a time

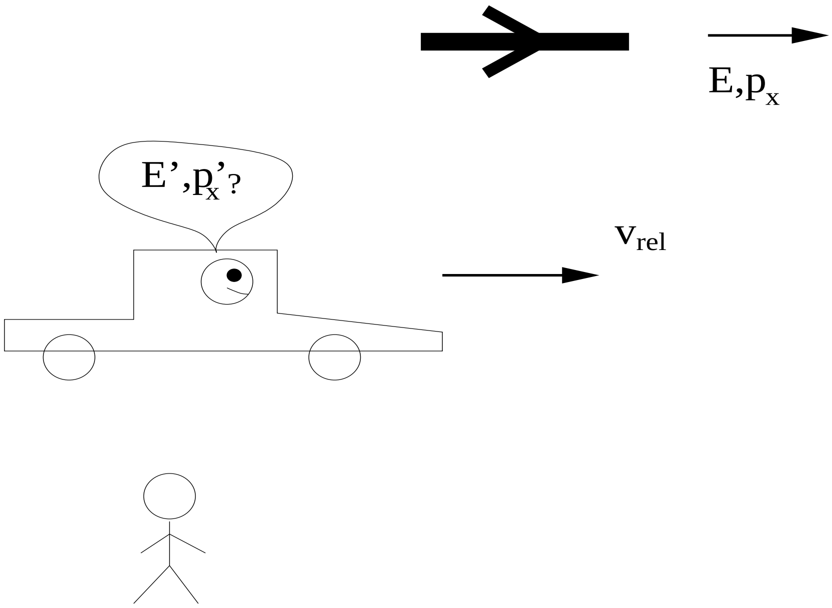

Figure 4: The observer on the ground measuring a velocity v_x for the airplane, wondering which velocity v'_x the driver of the car measures for the same airplane.

component and no space component. In the frame of the driver, both the time and space component of the vector are larger than in your frame. But the length of the momenergy vector P_μ is always the same, equal to m. Going back to normal 3D geometry this would not be possible. Imagine a vector $\vec{a} = (f, g, 0)$ and another vector $\vec{b} = (2f, h, 0)$. If the length of these vectors are the same, then we have that $h < g$. We see that from normal geometry you would expect that if the length of a vector is constant, then if you increase one component of the vector the other should decrease. The reason for this discrepancy with normal geometry is that spacetime has Lorentz geometry whereas 3D space has Euclidean geometry. Lorentz geometry has a minus sign in the definition of the scalar product (which also defines the length of the vector) making such an effect possible.

Now you know the basics of the special theory of relativity and you have got the necessary preparations to start studying the general theory of relativity. In the general theory of relativity we will study how masses curve spacetime, making the expression for the line element Δs different close to a large mass. This change in the line element changes the dynamics and gives rise to what we in Newtonian terms call the force of gravity.

5 List of expressions you should know by now

world line → page 1
timelike → page 1
lightlike → page 1
spacelike → page 2
principle of maximal aging → page 4
wrist watch time → page 4
scalar → page 7
four vector → page 7
four velocity → page 9
momenergy → page 13

6 Problems

Problem 1 (2 - 3 hours) Before embarking on the problems with four vectors and relativistic dynamics, we have one more important case to study from the previous lecture. This is the so-called 'twin paradox'. This long and detailed exercise is very important to gain some basic understanding for the underlying physics of many of the so-called paradoxes in the theory of relativity.

You are an astronaut traveling to the star Rigel, 800 light years from Earth. You start at $x = x' = 0$ and $t = t' = 0$ where (x, t) are Earth frame coordinates and (x', t') are spaceship coordinates. You travel in your spaceship at a velocity of $v = 0.99995$. We assume that Earth and Rigel do not move with respect to each other and that they therefore are in the same frame of reference.

1. Event A is you departing from Earth. Event B is you arriving at Rigel. In the Earth frame it took $800/0.99995 \approx 800.04$ years to arrive at Rigel. We know that for you it took a factor $\gamma = 1/\sqrt{1-v^2}$ less ($\Delta t = \gamma\Delta t'$, Δt is measured in Earth frame, $\Delta t'$ is measured in spaceship

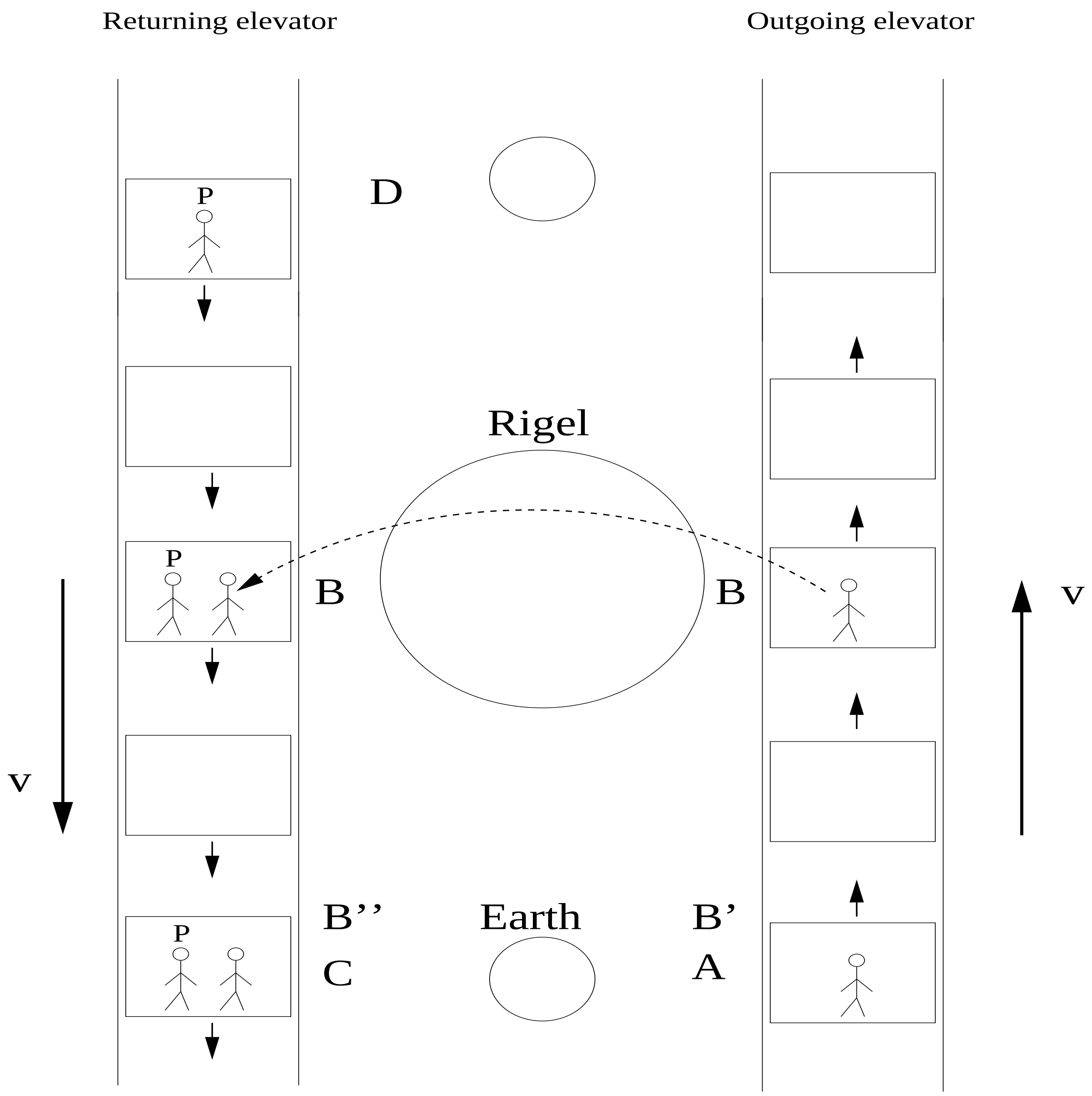

Figure 5: The elevator between Earth and Rigel.

frame). How long time did it take for you (on your wristwatch) to arrive at Rigel?

2. After arriving at Rigel, you make the necessary scientific measurements (which takes very little time and can therefore be ignored) and start the return flight. You fly with exactly the same speed $v = 0.99995$ towards Earth. Event C is when you arrive back on Earth. Use the same arguments (or symmetry arguments) to find the time Δt and $\Delta t'$ it took from Rigel and back to Earth in the two frames of reference.

3. If you have done your calculations correct, here is a summary of the situation: In the Earth frame, it took you 1600.08 years to go to Rigel and return. On your wristwatch it took you 16 years to go to Rigel and back. So while hundreds of generations have passed on Earth, you return only 16 years older.

4. We will now make the same calculations again, but just switch the frames: The laboratory frame (x, t) is now the frame of the spaceship and the moving frame (x', t') is the Earth frame. Because of the principle of relativity we are allowed to switch the roles and we should arrive at exactly the same result using the same laws of physics. From this point of view, this is what is happening: You sit in you spaceship which is now the laboratory frame defined to be at rest and at $x = x' = 0$ at $t = t' = 0$ (event A), the Earth starts moving away from you with velocity $v = 0.99995$ and Rigel starts approaching you with the same velocity. After a time Δt Rigel arrives at your position (event B). We know from previous calculations that the trip took 8 years in your frame of reference which is now the laboratory frame. Using again that $\Delta t = \gamma \Delta t'$ (and make sure not to confuse Δt and $\Delta t'$) show that the clocks on Earth at the moment when Rigel arrives at your position show 0.08 years. Only 0.08 years had passed on Earth during the 8 year (on your watch) trip to Rigel.

5. Now, this might look like a paradox, but we will show further down that it is not. No matter how strange this might sound, it is consistent. The paradox is still to come. After making your investigations of Rigel, Rigel departs and Earth approaches you again at the speed of $v = 0.99995$. Making the same calculations again you will find that it takes the Earth 8 years to return to you. Let's again check carefully how

long it takes on the Earth clocks for Earth to return at your position: At the moment you have finished your investigations, the Earth clocks show $t' = 0.08$ years and your clock shows $t = 8$ years. It takes Earth again $\Delta t = 8$ years to arrive at your position. We have as always that $\Delta t = \gamma \Delta t'$. How long did it take for Earth to return to your position measured on Earth clocks?

6. If you made the last calculation correct, this is now the situation: It took you 16 years from Earth departed to Earth returned. However, on Earth clocks it took 0.16 years. So while you are 16 years older, only two months have passed on Earth. Above we found that 1600 years had passed on Earth. Now, this is a paradox!

7. Clearly we made an error somewhere in the calculations. Or maybe we simply forgot some basic principles from special relativity? It appears at first sight that the two roles are equal,that we can choose whether we consider the Earth frame as the laboratory frame or the spaceship frame as the laboratory frame. But are the two roles really identical? What is the difference between the two observers, the Earth observer and the spaceship observer?

8. Don't read on until you have found an answer to the previous question. Here comes the solution: The difference is that whereas the Earth observer always stays in the same frame of reference, the spaceship observer changes frames of reference: The spaceship needs to accelerate at Rigel in order to change direction and return to Earth. The Earth does not undergo such an acceleration. The expression $\Delta t = \gamma \Delta t'$ was derived for constant velocity (look back at its derivation). It is not valid when the velocity is changing. In order to solve this problem properly one needs to either use general relativity which deals with accelerations or we can view the acceleration as an infinite number of different free float frames, frames with constant velocity, and apply special relativity to each of these frames. We will not do the exact calculation here, but we will do some considerations giving you some more understanding of what is happening. We will now consider three frames of reference. The Earth frame (x, t), the outgoing spaceship frame (x', t') and the returning spaceship frame (x'', t''). Instead of spaceships we will look at it as elevators going between Earth and Rigel. There are boxes going in both directions. At $x = x' = 0$ and $t = t' = 0$ you jump

into one of these boxes leaving for Rigel. There are other observers in other boxes before you and after you. The situation is depicted in figure 5. In the following use the Lorentz transformations to transform between the coordinate systems. We write the distance between Earth and Rigel in the Earth frame as L_0. Event A happens at $x_A = x'_A = 0$ and $t_A = t'_A = 0$. Event B is again the moment when you arrive at Rigel. At what time t_B in the Earth frame do you arrive at Rigel? (express the answer in terms of L_0 and v)

9. Use the Lorentz transformations to find t'_B, the time on your wristwatch when you arrive at Rigel. Insert numbers and check that you still find that the trip takes 8 years for you.

10. We now define event B'. At the same time as you arrive at Rigel (in your frame of reference which is now the frame of the outgoing elevator),another observer in another box in your elevator (thus another observer in your frame of reference using clocks synchronized with yours) passes the Earth at position $x_{B'} = 0$. Event B' is the event that he looks out and checks what time it is on Earth. So event B' takes place at the position of the Earth, but at the same time as you arrive at Rigel (same time in the outgoing reference frame). Show that the time $t_{B'}$ he reads on the Earth clocks can be written as $t_{B'} = L_0/v - vL_0$. Insert numbers. **hint:** You first need to find the position $x'_{B'}$ of event B' in the outgoing elevator frame.

11. Insert numbers in your previous result. Explain the result which we found earlier when using the spaceship as the laboratory frame: Namely that when Rigel arrived at the spaceship, we calculated that on the Earth clocks only 0.08 years had passed. Why is this not a surprise? Those who were surprised earlier, do you know understand which error you made when you got surprised? Which basic principle of relativity had you forgotten?

12. We learned in the previous question that even if the Earth clocks were observed at the same moment as the spaceship/elevator arrived at Rigel (in the outgoing frame), these two events (the observation of the Earth clocks and the arrival at Rigel) were NOT simultaneous in the Earth frame. For you, sitting in the outgoing elevator, only 0.08 years have passed on Earth when you arrive at Rigel. For observers on the Earth

on the other hand, you arrived at Rigel when 800 years had passed. At Rigel you meet a box in the returning elevator. You jump over to the box in the returning elevator at event B where you meet person P who has been traveling in the elevator from far away. Actually, at the same time (in the Earth frame) as you started your journey from Earth, person P started his journey from the other side of Rigel. We call the event when person P started his journey for event D. Event A and event D are simultaneous in the Earth frame. In order for you and person P to meet at event B, person P must have started on a planet a distance $2L_0$ from Earth (a distance L_0 from Rigel) as measured in the Earth frame. In that way you both cover a distance L_0 with the same speed v and therefore you can both meet at Rigel at time L_0/v as measured on Earth clocks. We call the coordinate system of the returning elevator (x'', t''). The clocks in the system of the returning elevator are set to zero at the moment when person P starts his journey. In the following, we will use spacetime intervals instead of the Lorentz transformation: The reason for this is that the returning elevator is not synchronized with the Earth frame at $x = 0, t = 0$. This was assumed when we deduced the form of the Lorentz transformation which we use in this course. Therefore, we will now again use invariance of the spacetime interval to obtain our answers. We will first check what the wristwatch of person P shows when he meets you at event B. In analogy to your own travel, it should intuitively show the same as your wristwatch: Both of you started at $t = 0$ on Earth clocks as well as on your own wristwatch. Both of you travel a distance L_0 (as measured in the Earth frame) at velocity v. But we have learned not to trust our intuition when working with relativity, so let's check. We will now consider the spacetime interval Δs_{BD} in order to find t''_B, the time on the wristwatch of person P at event B. Write down the space and time intervals Δx_{BD}, Δt_{BD}, $\Delta x''_{BD}$ and $\Delta t''_{BD}$. Show that invariance of the spacetime interval gives

$$\frac{L_0^2}{v^2} - L_0^2 = (t''_B)^2,$$

which gives $t''_B = L_0/(v\gamma)$, exactly as we thought. Your wristwatches agree at event B. Reassuring to see that our intiution still gives som reasonable results every now and then.

13. We will now try to find out what the time is on Earth for persons in

the returning elevator. In the frame of the outgoing elevator, we used a person who was situated in an elevator box at the same position of the Earth and looked out at the clocks on Earth exactly at the same time as event B happened (in the frame of the outgoing elevator). We called this event B' (looking at the clocks on Earth). We found that only 0.08 years had passed on Earth when you arrived at Rigel. We will now make the same check from the returning elevator. A person in an elevator box of the returning elevator being at the position of the Earth exactly at the same time as event B happens (now from the frame of the returning elevator) looks at the clocks on Earth. We call this event B" (the person in the box at the position of the Earth looking at the Earth clocks). We will now try to find out what he saw, i.e. which time $t_{B''}$ he observed on the Earth clocks. For this we will use spacetime interval $\Delta s_{DB''}$. Show that the space and time intervals from each frame are the following:

$$\begin{aligned}
\Delta x_{DB''} &= 2L_0 \\
\Delta t_{DB''} &= t_{B''} \\
\Delta x''_{DB''} &= L_0/\gamma \\
\Delta t''_{DB''} &= L_0/(\gamma v)
\end{aligned}$$

You might be a bit surprised by one of these results, but if you have doubts, do the following: Make one drawing for event D and one for event B". Show the position of the zero-point (the position of person P is the zero point of the x'' axis) of each of the x-axes in both plots and find the distances between events. Did it make it clearer?

14. Use invariance of the spacetime interval to show that

$$t_{B''} = \frac{L_0}{v} + L_0 v$$

Setting in for numbers this gives you $t''_{B''} = 1600.00$ years. Surprised? What has happened? You are still at event B, you made a very fast jump so almost no time has passed since you were in the outgoing elevator. But just before the jump, only 0.08 years had passed on Earth since you started your journey. Now, less than the fraction of a second later, 1600 years have passed on Earth. So in the short time that you jump lasted, 1599.92 years passed on Earth! This is were the solution to the twin paradox is hidden: When you jump, you change reference frame: You are accelerated. Special relativity is not valid for accelerated frames (actually one could solve this looking at the acceleration as an infinite sum of reference frames with different constant velocities). When you are accelerated, you experience fictive forces. This does not happen at Earth, the Earth does not experience the same acceleration. This is the reason for the asymmetry: If you speed had been constant, you and Earth could exchange roles and you would get consistent results. But since you are accelerated in the jump while the Earth is not, there is no symmetry here, you and the Earth cannot switch roles.

15. Let's summerize the situation: In your frame, you started your journey at $t = t' = 0$ and arrived at Rigel at $t' = 8$ years. In the Earth frame you arrived at Rigel after 800.04 years of travel. In your frame, the clocks on Earth show 0.08 years when you arrive at Rigel. Only 0.08 years have passed on Earth at the time you arrive at Rigel, seen from your frame. Then you jump to the returning elevator. Your watch still shows $t' = t'' = 8$ years. But now you have switched frame of reference. Now suddenly 1600 years have passed on Earth, clocks on Earth went from 0.08 years to 1600 years during the jump, as seen from your frame. As seen from Earth, the clock showed 800.04 years during your jump.

16. Seen from the Earth, you need 800.04 years to return, so the total time of your travel measured in the frame of reference of the Earth is $t = 1600.08$ years. In your own frame, the return trip took 8 years (by symmetry to the outgoing trip), so the total travel time for yourself is 16 years. But according to your frame of reference, the Earth clocks again aged 0.08 years during your return trip (by symmetry to the outgoing trip). When you were at Rigel, the observer in your frame of reference saw that the Earth clocks showed 1600 years. In your frame, 0.08 years passed on Earth during your return trip. So consistenly you find the Earth clocks to show 1600.08 years when you set your feet on the Earth again. This is also what we find making the calculation in the Earth frame $800.04 \times 2 = 1600.08$. But hundreds of generations have passed, and you have only aged 16 years. But after all these strange findings I'm sure you find this pretty normal by now. Everything clear? Read through one more time.

Problem 2 (30 min. - 45 min.) You are in the laboratory frame watching two cars passing from position $x = 0$ at $t = 0$ (event 1) and arriving simultaneously at position $x = L$ some time $t = T_L$ (event 2) later (all coordinates taken in the laboratory frame). Car A moves with constant velocity $v_A = c/2$ whereas car B accelerates from $v = 0$ at $x = 0$ and accelerates such that it reaches $x = L$ simultaneously with car A. In the following you will draw some spacetime diagrams. We are not interested in exact numbers in this exercise, only roughly correct relative distances and slopes on the worldlines showing that you have understood the basic principles.

1. Make a spacetime diagram in the laboratory frame showing the worldlines of yourself and the two cars.
2. Make a spacetime diagram in the reference frame of car A showing the three same worldlines.
3. Make a spacetime diagram in the reference frame of car B showing the three same worldlines.
4. Return to the first spacetime diagram, the diagram for the laboratory frame. The wristwatch of the driver of car A makes exactly 10 ticks from event 1 to event 2. The first tick happens at event 1 and the last tick happens at event 2. Draw a dot on the worldline of car A at roughly the position of each of the ticks. The important point here is to have correct relative spacings between each tick.
5. The driver of car B has an identical wristwatch making ticks with exactly the same frequency in the rest frame of the watch. Use the principle of maximal aging to judge whether driver B will experience more or less ticks on his watch from event 1 to event 2.
6. Again, draw a dot on the worldline of car B at the positions where the wristwatch of the driver makes a tick. Again, the exact position is not important, but the relative distances between the dots should be correct.**hint:** For each dot you draw, look at the slope of the worldline.

Problem 3 (10 min. - 30 min.) A four vector is defined to be a vector in spacetime which transforms from one frame of reference to another (from x_μ to x'_μ) using the Lorentz transformation

$$x'_\mu = c_{\mu\nu} x_\nu.$$

To check if a four dimensional vector is a four-vector, you need to check whether this relation is true or not. We will now test if four-vectors follow the normal rules of addition, that the sum of two four-vectors is really a four-vector. Assume you have two four-vectors A_μ and B_μ. You sum the two to make a vector D_μ,

$$D_\mu = A_\mu + B_\mu.$$

You now need to show that the result, D_μ, is also a 4-vector. Use the transformation properties of A_μ and B_μ to obtain these vectors in a different frame A'_μ and B'_μ. Find an expression for the sum of the two vectors, D'_μ, in the other frame expressed by D_μ in the laboratory frame and show that D_μ is indeed a four vector.

Problem 4 (90 min. - 2 hours) A free neutron has a mean life time of about 12 minutes after which it disintegrates into a proton, an electron and a neutrino. We will ignore the neutrino here, assuming that the only products of disintegration are a proton and an electron. A neutron moves along the positive x axis in the laboratory frame with a velocity $v = 0.99$. It disintegrates spontaneously and a proton and an electron is seen to continue in the same direction as the neutron. Use tables to find the mass of the electron, proton and neutron. We will try to calculate the speed of the proton and the electron in the lab-frame. The easiest way to do this is in the rest frame of the neutron where the neutron has a very simple expression for energy and momentum. In the lab frame this would have been a lot more work since all three particles have velocities.

1. In the rest frame of the original neutron (which has now disintegrated), what was the total energy and momentum of the neutron before disintegration? Write the answer in terms of a momenergy four-vector P'_μ(neutron).

2. In the rest frame of the original neutron, write the momenergy four-vector P'_μ(proton) of the proton expressed in terms of the proton mass m_p and the unknown proton velocity v'_p in the neutron rest frame.

3. Still in the neutron frame, write the expression for the momenergy four-vector P'_μ(electron) in terms of the electron mass m_e and the unknown electron velocity v'_e measured in the neutron frame.

4. Use conservation of momenergy

$$P'_\mu(\text{neutron}) = P'_\mu(\text{proton}) + P'_\mu(\text{electron}),$$

to find the velocity of the proton and the electron in the rest frame of the original neutron. (insert numbers). **hint:** This can be ugly if you don't do it right: Write the momentum part of the equation in terms of γ-factors only, then substitute for one of the γ from the energy part of the eqaution. Then you will avoid second order equations.

5. Use the transformation properties for four-vectors

$$P'_\mu(\text{electron}) = c_{\mu\nu} P_\nu(\text{electron}),$$

to find the energy and momentum of the electron and proton in the laboratory frame. (insert numbers:what units do your results have if you keep $c = 1$)

6. Use the numbers you have obtained for energy or momentum to obtain the speed of the electron and proton in the laboratory frame.

7. As an independent check (and to see an alternative way of doing it), use the relativistic formula for addition of velocities to obtain the speed of the proton and electron in the lab frame, using only the speed you have obtained for the proton in the neutron frame as well as the speed of the neutron seen from the lab frame.

8. For those who like long and ugly calculations only: Do everything from the beginning, but use only the lab-frame to obtain the same results. Do you see the advantage of using 4-vectors and change of frames?

Problem 5 (90 min. - 2 hours) An electron and a positron (the anti particle of the electron having the same mass) are approaching each other with the same velocity $v = 0.995$ in opposite directions in the laboratory frame. In the collision, both particles are annihilated and two photons are produced. One photon travels in the positive x direction, the other in the negative x direction. Use tables to find the mass of an electron.

1. What is the velocity of the positron in the rest frame of the electron?

2. Write down the momenergy four-vectors $P_\mu(\text{electron})$ and $P_\mu(\text{positron})$ of the positron and the electron in the laboratory frame (use numbers).

3. Use the transformation properties of four-vectors to write down the momenergy four-vectors P'_μ(electron) and P'_μ(positron) of the positron and the electron in the rest frame of the electron (again use numbers).

4. Show that the momenergy four-vector of a photon traveling in the positive x-direction can be written

$$P_\mu^\gamma = (E, E, 0, 0),$$

where E is the energy of the photon.

5. Use conservation of momenergy in the laboratory frame to argue that the two photons must have the same energy seen from the laboratory frame.

6. What is the energy of the photons and thereby the wavelength in the laboratory frame?

7. Use transformation properties for four-vectors to show that the energy of a photon in a frame moving with velocity v with respect to the laboratory frame is

$$E' = E\gamma(1 - v)$$

8. What is the energy of each of the two photons in the rest frame of the electron?

9. Use the expression for E' in terms of E to derive the relativistic Doppler formula

$$\frac{\Delta\lambda}{\lambda} = \left(\sqrt{\frac{1+v}{1-v}} - 1\right)$$

10. Show that the relativistic Doppler formula is consistent with the normal Doppler formula for low velocities. **hint:** Make a Taylor expansion of $f(v) = \sqrt{(1+v)/(1-v)}$ for small v.