AST1100 Lecture Notes

19: Nuclear reactions in stellar cores

Before embarking on the details of thermonuclear reactions in stellar cores, we need to discuss a few topics...

1 Some particle physics

Nature is composed of three kinds of elementary particles: *leptons*, *quarks*, and *gauge bosons*. Nature also has four forces acting on these elementary particles: the strong and weak nuclear forces, the electromagnetic force and the force of gravity (from the point of view of general relativity the latter is not a force, from the point of view of particle physics, it is). Actually, it has been discovered that the weak nuclear force and the electromagnetic force are two aspects of the same thing. At higher energies they unify and are therefore together called the electroweak force.

The leptons can be divided in two groups, the 3 'heavy' (with much more mass than in the other group) leptons and 3 light leptons called neutrinos (with a very small mass). Each heavy lepton has a neutrino associated with it. In all there are thus 6 leptons

- the electron and the electron associated neutrino.
- the muon and the muon associated neutrino.
- the tau particle and the tau associated neutrino.

In collisions involving the electron, an electron (anti)neutrino is often created, in collisions involving the muon, a muon (anti)neutrino is often created and the same goes for the tau particle. Each lepton has *lepton number* +1 whereas an antilepton has lepton number -1. This is a property of the particle similar to charge: In the same way as the total charge is conserved in particle collisions, the total lepton number is also conserved.

There are also 6 kinds of quarks grouped in three generations. In the order of increasing mass these are

- the up (charge $+2/3e$) and down (charge $-1/3e$) quarks.
- the strange (charge $-1/3e$) and charm (charge $+2/3e$) quarks.

- the bottom (charge $-1/3e$) and top (charge $+2/3e$) quarks.

A quark has never been observed alone it is always connected to other quarks via the strong nuclear force. A particle consisting of two quarks is called a *meson* and a particle consisting of three quarks is called a *baryon*. Mesons and baryons together are called *hadrons*. A proton is a baryon consisting of three quarks, two up and one down quark. A neutron is another example of a baryon consisting of two down and one up quark.

In quantum theory, the forces of nature are carried by so-called gauge bosons. Two particles attract or repel each other through the interchange of gauge bosons. Normally these are *virtual gauge bosons*: Particles existing for a very short time, just enough to carry the force between two particles. The energy to create such a particle is borrowed from vacuum: The Heisenberg uncertainty relation

$$\Delta E \Delta t \leq \frac{h}{4\pi}, \tag{1}$$

allows energy ΔE to be borrowed from the vacuum for a short time interval Δt. The gauge bosons carrying the four forces are

- gluons in the case of the strong nuclear force
- W and Z bosons in the case of the weak nuclear force
- photons in the case of the electromagnetic force
- (gravitons in the case of the gravitational force: note that a quantum theory of gravity has not yet been successfully developed)

In quantum theory, the angular momentum or spin of a particle is quantized. Elementary particles can have integer spins or half integer spins. Particles of integer spins are called *bosons* (an example is the gauge bosons) and particles of half integer spin are called *fermions* (leptons and quarks are examples of fermions. Fermions and bosons have very different statistical properties, we will come to this in the next lecture.

Finally, all particles have a corresponding antiparticle: A particle having the same mass, but opposite charge. Antileptons also have opposite lepton number: -1. This is why a lepton is always created with an antineutrino in collisions. For instance, when a free neutron disintegrates (a free neutron only lives for about 12 minutes), it disintegrates into a proton and electron and an electron antineutrino. A neutron is not a lepton and hence has lepton number 0. Before the disintegration, the total lepton number is therefore zero. After the disintegration, the total lepton number is: 0 (for the proton) + 1 (for the electron) -1 (for the antineutrino) = 0, thus lepton number is conserved due to the creation of the antineutrino.

Now make a schematic summary of all the elementary particles and forces that have been observed in nature.

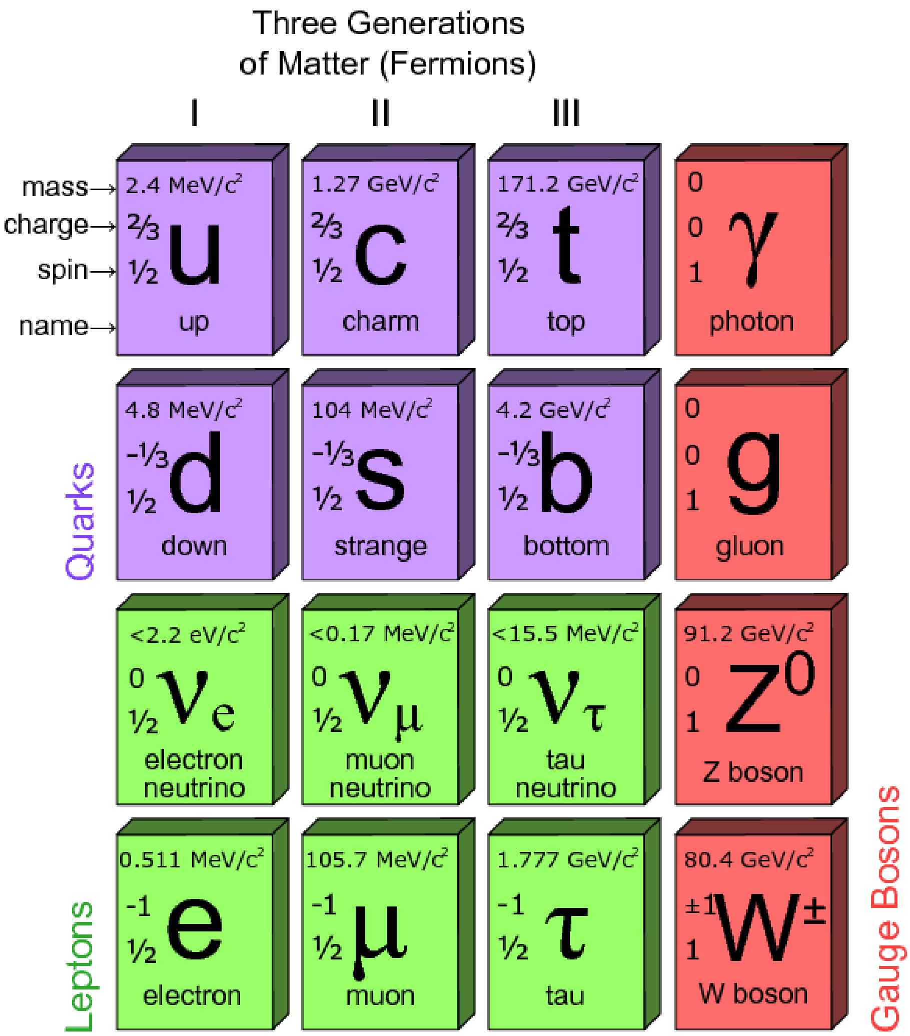

Figure 1: Info-figure: The Standard Model of particle physics is a theory concerning the electromagnetic, weak, and strong nuclear interactions, which mediate the dynamics of the known subatomic particles. The model includes 12 fundamental fermions and 4 fundamental bosons. The 12 elementary particles of spin 1/2 (6 quarks and 6 leptons) known as fermions are classified according to how they interact, or equivalently, by what charges they carry. Pairs from each classification are grouped together to form a generation, with corresponding particles exhibiting similar physical behavior. Fermions respect the Pauli exclusion principle, and each fermion has a corresponding antiparticle. Gauge bosons (red boxes) are defined as force carriers that mediate the strong, weak, and electromagnetic interactions. (Note that the masses of certain particles are subject to periodic reevaluation by the scientific community. The values in this graphic are as of 2008 and may have been adjusted since.) (Figure:Wikipedia)

2 Mass in special relativity

Another topic which we need to discuss before studying nuclear reactions is the notion of mass in the special theory of relativity. We have already seen that the scalar product of the momenergy four-vector equals the mass of a particle,

$$P_\mu P^\mu = E^2 - p^2 = m^2. \tag{2}$$

Imagine we have two particles with mass m_1 and m_2, total energy E_1 and E_2 and momenta p_1 and p_2. Assume that they have opposite momenta $p_1 = -p_2 = p$,

$$P^1_\mu = (E_1, p), \quad P^2_\mu = (E_2, -p)$$

with $E_1 = \sqrt{m_1^2 + p^2}$ and $E_2 = \sqrt{m_2^2 + p^2}$ These two particles could for instance constitute the proton and the neutron in a deuterium nucleus. The question now is, what is the total mass of the two-particle system (deuterium nucleus)? Let us form the momenergy four-vector for the nucleus

$$P_\mu = P^1_\mu + P^2_\mu = (E_1 + E_2, 0).$$

Using equation 2 we can now find the total mass of the two-particle system (the nucleus),

$$\begin{aligned} M^2 = P_\mu P^\mu = (E_1 + E_2)^2 &= E_1^2 + E_2^2 + 2E_1E_2 \\ &= m_1^2 + m_2^2 + 2p^2 + \sqrt{(m_1^2 + p_1^2)(m_2^2 + p_2^2)} \end{aligned}$$

where M is the total mass of the nucleus. We have two important observations: (1) Mass is *not* an additive quantity. The total mass of a system of particles is *not* the sum of the mass of the individual particles. (2) The mass of a system of particles depends on the total energy of the particles in the system. The energy of particles in an atomic nucleus includes the potential energy between the particles due to electromagnetic and nuclear forces.

Consider an atomic nucleus with mass M. This nucleus can be split into two smaller nuclei with masses m_1 and m_2. If total mass of the two nuclei m_1 and m_2 is smaller than the total mass of the nucleus, the rest energy is radiated away when the nucleus is divided. This is a nuclear fission process creating energy. Similarly if the total mass of m_1 and m_2 is larger than the total mass of the nucleus, then energy must be provided in order to split the nucleus. The same argument goes for nuclear fusion processes: Consider two nuclei with masses m_1 and m_2 which combine to form a larger nucleus of mass M. If M is smaller than the total mass of the nuclei m_1 and m_2 then the rest mass is radiated away and energy is 'created' in the fusion process. In some cases (particularly for large nuclei), the mass M is larger than the total mass of m_1 and m_2. In this case energy must be provided in order to combine the two nuclei to a larger nucleus. We will soon see that in order to produce atomic nuclei larger than iron, energy must always be provided.

3 Penetrating the Coloumb barrier

The strong nuclear force (usually referred to as the strong force) is active over much smaller distances than the electromagnetic force. The strong force makes protons attract protons and protons attract neutrons (and vice versa). For two atomic nuclei to combine to form a larger nucleus, the two nuclei need to be close enough to feel the attractive nuclear forces from each other. Atomic nuclei have positive charge and therefore repulse each other at larger distances due to the electromagnetic force. Thus for a fusion reaction to take place, the two nuclei need to penetrate the Coloumb barrier, the repulsive electromagnetic force between two equally charged particles. They need to get so close that the attractive strong force is stronger than the repulsive electromagnetic force. In figure 2 we show the combined potential from electromagnetic and nuclear forces of a nucleus. We clearly see the potential barrier at $r = R$. For a particle to get close enough to feel the attractive strong force it needs to have an energy of at least $E > E(R)$. We can make an estimate of the minimal temperature a gas needs in order to make a fusion reaction happen: The mean kinetic energy of a particle in a gas of temperature T is $E_K = (3/2)kT$ (see the exercises). The potential energy between two nuclei A and B can be written as

$$U = -\frac{1}{4\pi\epsilon_0}\frac{Z_A Z_B e^2}{r},$$

where ϵ_0 is the vacuum permittivity, Z_1 and Z_2 is the number of protons in each nucleus, e is the electric charge of a proton and r is the distance between the two nuclei. For nucleus A to reach the distance R (see figure 2) from nucleus B where the strong force starts to dominate, the kinetic energy must at least equal the potential energy at this point

$$\frac{3}{2}kT = \frac{1}{4\pi\epsilon_0}\frac{Z_A Z_B e^2}{R}.$$

The distance R is typically $R \sim 10^{-15}$ m. Considering the case of two hydrogen nuclei $Z = 1$ fusing to make helium $Z = 2$, we can solve this equation for the temperature and obtain $T \sim 10^{10}$ K. This temperature is much higher than the core temperature of the Sun $T_C \sim 15 \times 10^6$ K. Still this reaction is the main source of energy of the Sun. How can this be?

The secret is hidden in the world of quantum physics. Due to the Heisenberg uncertainty relation (equation 1), nucleus A can borrow energy ΔE from vacuum for a short period Δt. If nucleus A is close enough to nucleus B, the time Δt might just be enough to use the borrowed energy to penetrate the Coloumb barrier and be captured by the potential well of the strong force. This phenomenon is called *tunneling*. Thus, there is a certain probability that nucleus A spontaneously borrows energy to get close enough to nucleus B in order for the fusion reaction to take place.

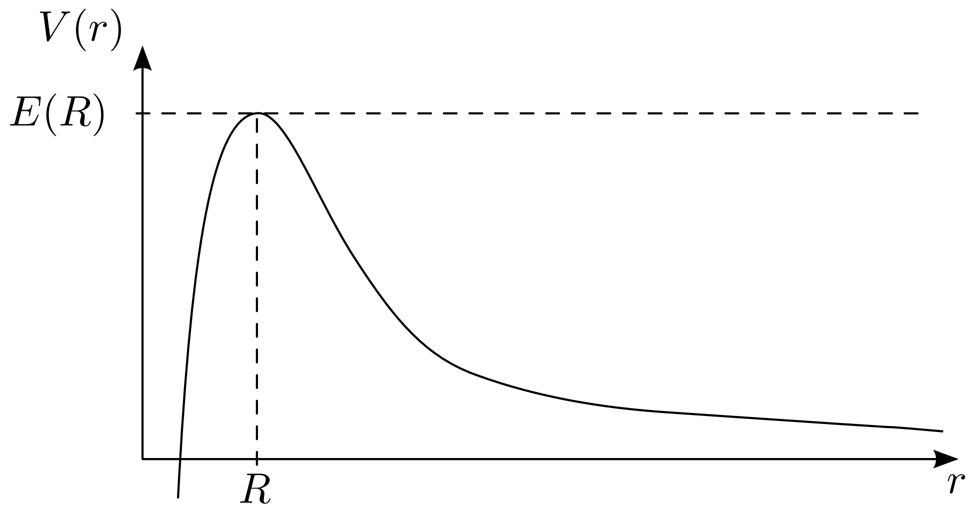

Figure 2: The repulsive Coloumb potential $V(r)$ as a function of distance between nuclei r. At small distances r we see the potential well from the attractive strong forces.

4 Nuclear reaction probabilities and cross sections

Quantum physics is based on probability and statistics. Nothing can be predicted with 100% certainty, only statistical probabilities for events to happen can be calculated. When nucleus A is at a certain distance from nucleus B we cannot tell whether it will borrow energy to penetrate the Coloumb barrier or not, we can only calculate the probability for the tunneling to take place. These probabilities are fundamental for understanding nuclear reactions in stellar cores. These probabilities are usually represented as *cross sections* σ.

The definition of the cross section is based on an imaginary situation which is a bit different from the real situation but gives an intuitive picture of the reaction probabilities and, most importantly, makes the calculations easier. It can be proven that the calculations made for this imaginary picture gives exact results for the real situation. Instead of the real situation where we have one nucleus A and one nucleus B passing each other at a certain distance (and we want to know the probability that they react), one imagines the nucleus B to be at rest and a number of nuclei of type A approaching it. One imagines nucleus B to have a finite two dimensional extension, like a disk, with area σ. Towards this disk there is a one dimensional flow of A particles (see figure 3). If a nucleus A comes within this disk, it is captured and fusion takes place, if not the nuclei do not fuse. It is important to understand that this is not really what happens: fusion can take place with any distance r between the nuclei. It might also well be that A is within the disk and the fusion reaction is not taking place. But in order to make calculations easier one makes this imaginary disk

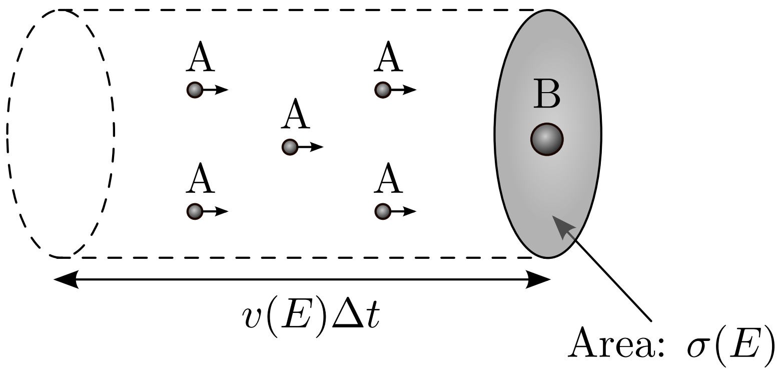

Figure 3: A particles streaming towards the disk with cross section $\sigma(E)$ around the B nucleus. A particles of energy E within the volume $v(E)\Delta t\sigma(E)$ will react with the nucleus B within time Δt.

with an effective cross section σ saying that any nucleus A coming within this disk will fuse. It can be shown that calculations made with this representation gives correct reaction rates even though the model does not give a 100% correct representation of the physical situation. Because of the simplified mathematics, the cross section σ is the most common way of representing a probability for a reaction or collision process to take place. You will now see how this imaginary picture is used to calculate reaction rates.

The disk cross section (tunneling probability) $\sigma(E)$ depends on the energy E of the incoming nucleus A. Thus the size of the immaginary disk (for the nucleus B at rest) depends on the energy E of the incoming particle A. We will now make calculations in the center of mass system. In problem 5 in the lectures on celestial mechanics, you showed that the total kinetic energy of a two-body system can be written as (ignoring gravitational forces)

$$E = \frac{1}{2}\hat{\mu}v^2,$$

where $\hat{\mu}$ is the reduced mass $\hat{\mu} = (m_1 m_2)/(m_1 + m_2)$. We showed that the two-body problem is equivalent to a system where a particle with mass $M = m_1 + m_2$ is at rest and a particle with the reduced mass $\hat{\mu}$ is moving with velocity v. In this case we imagine the nucleus B to be at rest and the particle A is approaching with velocity v.

We have deferred the full calculation of the reaction rate between A and B nuclei in a plasma using the cross-section to problem 4. In order to be able to do that calculation, we need to recall an expression which we have seen before. In the lectures on electromagnetic radiation we learned that the number density of particles with velocity between v and $v + dv$ in an ideal gas of temperature T with molecules of mass m can be written as

$$n(v)dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}} 4\pi v^2 dv. \tag{3}$$

You will use this in problem 4 when you need to multiply with the number of A and B nuclei in the gas.

The result you will find in problem 4 is the energy produced per kilogram of gas per second from nuclear reactions between A and B nuclei, ε_{AB}. You will show that it is given by

$$\varepsilon_{AB} = \frac{\varepsilon_0}{\rho}\left(\frac{2}{kT}\right)^{3/2}\frac{n_A n_B}{\sqrt{\hat{\mu}\pi}}\int_0^\infty dE E e^{-E/kT}\sigma(E), \tag{4}$$

where ε_0 is the energy released in each nuclear reaction between an A and a B nucleus, ρ is the total density of the gas and n_A and n_B are number densities of A and B nuclei. We will not do the integral here but note that the solution can be Taylor expanded around given temperatures T as

$$\varepsilon_{AB} = \varepsilon_{0,\text{reac}} X_A X_B \rho^\alpha T^\beta,$$

where ρ is the density, X_A and X_B are the mass fractions of the two nuclei

$$X_A = \frac{n_A m_A}{nm} = \frac{\text{total mass in type A nuclei}}{\text{total mass}},$$

and α and β depend on the temperature T around which the expansion is made.

Here, $\varepsilon_{0,\text{reac}}$, α and β will depend on the nuclear reaction (calculated from the integral 4). The constant $\varepsilon_{0,\text{reac}}$ includes the energy per reaction ε_0 for the given reaction as well as several other constants. If we have $\varepsilon_{0,\text{reac}}$, α and β for different nuclear reactions, we can use this expression to find the nuclear reactions which are important for a given temperature T in a stellar core.

The energy release per mass per time, ε, can be written as luminosity per mass

$$\frac{dL}{dm} = \varepsilon$$

The luminosity at a shell at a distance r from the center of a star can therefore be written as

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r), \tag{5}$$

which is another of the equations used together with the equation of hydrostatic equilibrium in the stellar model building described in the exercises of lecture 13–14.

5 Stellar nuclear reactions

For main sequence stars the most important fusion reaction fuses four $^1_1$H atoms to $^4_2$He. When writing nuclei, ^A_ZX, A is the total number of nucleons (protons and neutrons), Z is the total number of protons and X is the chemical symbol. There are mainly two chains of reaction responsible for this process. One is the pp-chain,

$$\begin{aligned} {}^1_1\mathrm{H} + {}^1_1\,\mathrm{H} &\rightarrow {}^2_1\mathrm{H} + {}^0_0\,\bar{\mathrm{e}} + {}^0_0\,\nu \\ {}^2_1\mathrm{H} + {}^1_1\,\mathrm{H} &\rightarrow {}^3_2\mathrm{He} + {}^0_0\,\gamma \\ {}^3_2\mathrm{He} + {}^3_2\,\mathrm{He} &\rightarrow {}^4_2\mathrm{He} + 2 \times {}^1_1\,\mathrm{H} \end{aligned}$$

Here ${}^0_0\nu$ is the electron associated neutrino, ${}^0_0\gamma$ is a photon and the bar represents antiparticles: ${}^0_0\bar{\mathrm{e}}$ is the antiparticle of the electron called the positron. This is the pp-I chain, the most important chain reactions in the solar core. There are also other branches of the pp-chain (with the first two reactions equal) but these are less frequent. The pp-chain is most effective for temperatures around 15 millions Kelvin for which we can write the reaction rate for the full pp-chain as

$$\varepsilon_{\mathrm{pp}} \approx \varepsilon_{0,\mathrm{pp}} X_H^2 \rho T_6^4,$$

where $T = 10^6 T_6$ with T_6 being the temperature in millions of Kelvin. This expression is valid for temperatures close to $T_6 = 15$. For this reaction $\varepsilon_{0,\mathrm{pp}} = 1.08 \times 10^{-12}$ $\mathrm{Wm^3/kg^2}$. The efficiency of the pp-chain is 0.007, that is only 0.7% of the mass in each reaction is converted to energy.

The other reaction converting four ${}^1_1\mathrm{H}$ to ${}^4_2\mathrm{He}$ is the CNO-cycle,

$$\begin{aligned} {}^{12}_6\mathrm{C} + {}^1_1\,\mathrm{H} &\rightarrow {}^{13}_7\mathrm{N} + {}^0_0\,\gamma \\ {}^{13}_7\mathrm{N} &\rightarrow {}^{13}_6\mathrm{C} + {}^0_0\,\bar{\mathrm{e}} + {}^0_0\,\nu \\ {}^{13}_6\mathrm{C} + {}^1_1\,\mathrm{H} &\rightarrow {}^{14}_7\mathrm{N} + {}^0_0\,\gamma \\ {}^{14}_7\mathrm{N} + {}^1_1\,\mathrm{H} &\rightarrow {}^{15}_8\mathrm{O} + {}^0_0\,\gamma \\ {}^{15}_8\mathrm{O} &\rightarrow {}^{15}_7\mathrm{N} + {}^0_0\,\bar{\mathrm{e}} + {}^0_0\,\nu \\ {}^{15}_7\mathrm{N} + {}^1_1\,\mathrm{H} &\rightarrow {}^{12}_6\mathrm{C} + {}^4_2\,\mathrm{He} \end{aligned}$$

with a total reaction rate

$$\varepsilon_{\mathrm{CNO}} = \varepsilon_{0,\mathrm{CNO}} X_H X_{\mathrm{CNO}} \rho T_6^{20},$$

where $\varepsilon_{0,\mathrm{CNO}} = 8.24 \times 10^{-31}$ $\mathrm{Wm^3/kg^2}$ and

$$X_{\mathrm{CNO}} = \frac{M_{\mathrm{CNO}}}{M}$$

is the total mass fraction in C, N and O. These three elements are only catalysts in the reaction, the number of C, N and O molecules do not change in the reaction. This expression is valid for $T_6 \approx 15$. We see that when the temperature increases a little, the CNO cycle becomes much more effective because of the power 20 in temperature. In the exercises you will find how much. Thus, the CNO cycle is very sensitive to the temperature. Small changes in the temperature may have large influences on the energy production rate by the CNO cycle.

For stars with an even hotter core, also ${}^4_2\text{He}$ may fuse to heavier elements. In the triple-alpha process three ${}^4_2\text{He}$ nuclei are fused to form ${}^{12}_6\text{C}$.

$$ {}^4_2\text{He} + {}^4_2\text{He} \rightarrow {}^8_4\text{Be} + {}^0_0\,\gamma $$

$$ {}^8_4\text{Be} + {}^4_2\text{He} \rightarrow {}^{12}_6\text{C}^* + {}^0_0\,\gamma $$

Here the reaction rate can be written as

$$ \varepsilon_{3\alpha} = \varepsilon_{0,3\alpha}\rho^2 X_{He}^3 T_8^{41}. $$

Here $T = 10^8 T_8$, T_8 is the temperature in hundred millions of Kelvin and $\varepsilon_{0,3\alpha} = 3.86 \times 10^{-18}$ Wm^6/kg^3. This expression is valid near $T_8 = 1$. We see an extreme temperature dependence. When the temperature is high enough, this process will produce much more than the other processes.

For higher temperatures, even heavier elements will be produced for instance with the reactions

$$ {}^4_2\text{He} + {}^{12}_6\text{C} \rightarrow {}^{16}_8\text{O} + {}^0_0\,\gamma \tag{6} $$

$$ {}^{12}_6\text{C} + {}^{12}_6\text{C} \rightarrow {}^{24}_{12}\text{Mg} + {}^0_0\,\gamma \tag{7} $$

There is a limit to which nuclear reactions can actually take place: The mass of the resulting nucleus must be lower than the total mass of the nuclei being fused. Only in this way energy is produced. This is not always the case. For instance the reactions

$$ {}^{12}_6\text{C} + {}^{12}_6\text{C} \rightarrow {}^{16}_8\text{O} + 2{}^4_2\text{He} \tag{8} $$

and

$$ {}^{16}_8\text{O} + {}^{16}_8\text{O} \rightarrow {}^{24}_{12}\text{Mg} + 2{}^4_2\text{He} \tag{9} $$

require energy *input*, that is the total mass of the resulting nucleus is larger than the total mass of the input nuclei. It is extremely difficult to make such reactions happen: Only in extreme environments with very high temperatures is the probability for such reactions large enough to make the processes take place.

In figure 4 we show the mass per nucleon for the different elements. We see that we have a minimum for ${}^{56}_{26}\text{Fe}$. This means that for lighter elements (with less than 56 nucleons), the mass per nucleon decreases when combining nuclei to form more heavier elements. Thus, for lighter elements, energy is usually released in a fusion reaction (with some exceptions, see equation 8 and 9). For elements heavier than iron however, the mass per nucleus increases with increasing number of nucleons. Thus, energy input is required in order to make nuclei combine to heavier nuclei. The latter processes are very improbable and require very high temperatures.

We see that we can easily produce elements up to iron in stellar cores. But the Earth and human beings consist of many elements much heavier

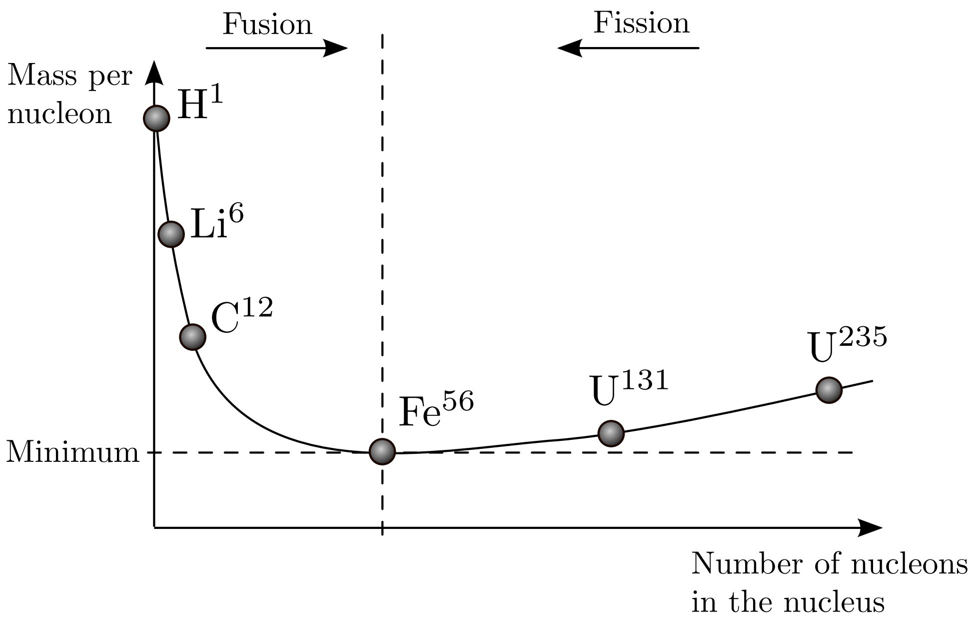

Figure 4: Schematic diagram of mass per nucleon as a function of the number of nucleons in the nucleus. Note that we are only illustrating the general trends. There are for instance a few light elements for which the mass per nucleon increases with increasing number of nucleons in the nucleus.

than iron. How were these produced? In the Big Bang only hydrogen and helium were produced so the heavier elements must have been created in nuclear reactions at a later stage in the history of the universe. We need situations were huge amounts of energy are available to produce these elements. The only place we know about where such high temperatures can be reached are supernova explosions. We will come back to this later.

6 The solar neutrino problem

If you look back at the chain reactions above you will see that neutrinos are produced in the pp-chain and the CNO cycle. We have learned in earlier lectures that neutrinos are particles which hardly react with matter. Unlike the photons which are continuously scattered on charged particles on they way from the core to the stellar surface, the neutrinos can travel directly from the core of the Sun to the Earth without being scattered even once. Thus, the neutrinos carry important information about the solar core, information which would have otherwise been impossible to obtain without being at the solar core. Using the chain reactions above combined with the theoretical reaction rates, we can calculate the number of neutrinos with a given energy we should observe here at Earth. This would be an excellent test of the theories for the composition of the stellar interiors as well as of our understanding of the nuclear reactions in the stellar cores. The procedure is as follows

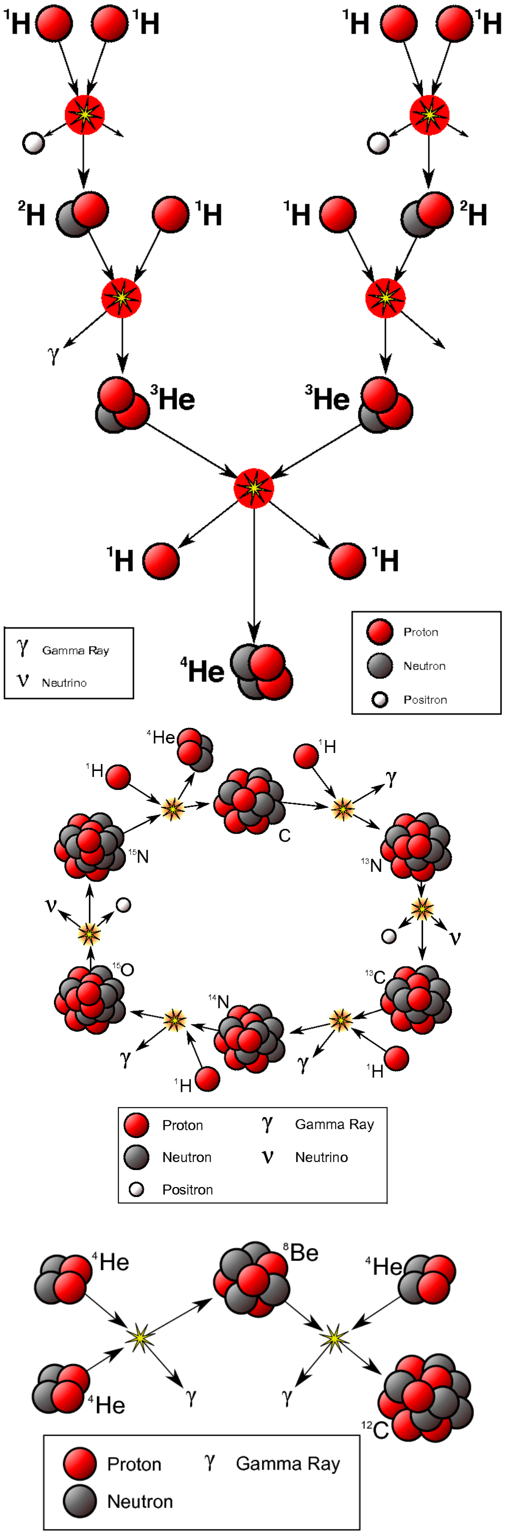

Figure 5: Info-figure: The protonproton (pp) chain reaction, The carbon-nitrogen-oxygen (CNO) cycle (the helium nucleus is released at the top-left step) and the triple-alpha process.(Figure:Wikipedia)

1. Stellar model building: Solve the coupled set of equations consisting of the equation of hydrostatic equilibrium, equation 5 as well as several equations from thermodynamics describing the transport of energy within the Sun. The solutions to these equations will give you the density $\rho(r)$ and temperature $T(r)$ of the Sun as a function of distance r from the center.

2. The temperature $T(r)$ at a given distance r combined with the above expressions for stellar reaction rates gives the number of neutrinos produced in the different kinds of chain reactions and what energies E these neutrinos should have.

3. Measure the flux of neutrinos for different energy ranges E that we receive on Earth and compare to theoretical predictions.

4. If there is agreement, it means we have obtained the correct model for the Sun. If the agreement is not satisfactory, we need to go back to the first step and make the stellar model building with different assumptions and different parameters.

For many years, there was a strong disagreement between the neutrino flux observed at Earth and the solar models. The observed number of neutrinos was much lower than predicted. Now the discrepancy is resolved and the solution led to an important discovery in elementary particle physics: It was discovered that the neutrinos have mass. It was previously thought that neutrions were massless like the photons. Elementary particle physics predicted that if the neutrinos have mass, they may oscillate between the three different types of neutrino. If neutrinos have mass, then an electron neutrino could spontaneously convert itself into a muon or tau neutrino. The first neutrino experiments were only able to detect electron neutrinos. The reason they didn't detect enough solar neutrinos was that they had converted themselves to different types of neutrinos on the way from the solar core to the Earth. Today neutrino detectors may also detect other kinds of neutrino and the observed flux is in much better agreement with the models. But it does not mean that the solar interior and solar nuclear reactions are completely understood. Modern neutrino detectors are now used to measure the flux of different kinds of neutrinos in different energy ranges in order to understand better the processes being the source of energy in the Sun as well as other stars.

But the neutrinos hardly react with matter, how are they detected? This is not an easy task and a very small fractions of all the neutrinos passing through the Earth are detected. One kind of neutrino detector consists of a tank of cleaning fluid C_2Cl_4, by the reaction

$$^{37}_{17}\mathrm{Cl} + ^{0}_{0}\nu \rightarrow ^{37}_{18}\mathrm{Ar} + ^{0}_{-1}\mathrm{e}.$$

The argon produced is chemically separated from the system. Left to itself the argon can react with an electron (in this case with its own inner shell electron) by the converse process

$$^{37}_{18}\mathrm{Ar} + ^{0}_{-1}\mathrm{e} \rightarrow ^{37}_{17}\mathrm{Cl} + ^{0}_{0}\nu.$$

The chlorine atom is in an excited electronic state which will spontaneously decay with the emission of a photon. The detection of such photons by a photomultiplier then is an indirect measurement of the solar neutrino flux.

7 Problems

Problem 1 (2–3 hours)

We will show that the mean kinetic energy of a particle in the gas is

$$K = \frac{3}{2}kT.$$

In statistics, if x is a stochastic random variable and we want to find the mean value of a function $f(x)$ of this random variable, we use the formula for the mean

$$< f(x) >= \int dx f(x) P(x),$$

where $P(x)$ is the probability distribution function describing the probability of finding a certain value for the random variable x. The probability distribution needs to be normalized such that

$$\int dx P(x) = 1.$$

All integrals over x are over all possible values of x.

Let's translate the last sentences into a more understandable language: physics. Our random variable x is simply the velocity v of particles in a gas. Why random? Because if you take a gas and choose randomly one particle in the gas, you do not know which value you will find for v, it is random. Thermodynamics gives us the *probability distribution* $P(x)$ of velocities. This probability distribution tells us the probability that our choosen gas particle has a given velocity v. In an ideal gas, the probability distribution is given by the Maxwell-Boltzmann distribution function in equation 3. Finally, the function $f(x)$ is any function of the velocity, of which we want the mean value. This could for instance be the kinetic energy $K(v) = (1/2)mv^2$. This is a function of the random variable v and we would indeed like to find the mean value of this function, that is, the mean kinetic energy of a particle in the gas. This mean kinetic energy would be the energy we would find if we measured the kinetic energy of a large number of particles in the gas and took the mean. It is that simple. So now we substitute x with v, $f(x)$ with $K(v)$ and the probability distribution $P(x)$ with $n(v)$. There is however one caveat: Above we mentioned that $P(x)$ needs to be normalized. The form of the Maxwell-Boltzmann distribution in equation 3 is not normalized. We call the normalized distribution $n_{\text{norm}}(v)$. Then we have

$$< K >= \int dv K(v) n_{\text{norm}}(v).$$

In the following you will need the following two integrals

$$\int_0^\infty dx e^{-x} x^{1/2} = \frac{\sqrt{\pi}}{2}$$

and

$$\int_0^\infty dx e^{-x} x^{3/2} = \frac{3\sqrt{\pi}}{4}.$$

1. First we need to find $n_{\rm norm}(v)$. We write

$$n_{\rm norm}(v) = \frac{1}{N} n(v).$$

   Use the normalization integral for $P(x)$ above to find N.

2. Now use the normalized distribution function to find the mean kinetic energy of a particle in an ideal gas.

3. Now we will check our result numerically: Note that the Maxwell-Boltzmann distribution function in equation 3 is the probability of finding a particle with absolute value v of the velocity. We now want to simulate gas particles using this distribution, but in order to create a realistic simulation we also need to take account the direction of the particles. The corresponding Maxwell-Boltzmann distribution function for the probability of finding a gas particle with velocity vector $\vec{v}$ can be written like this:

$$n(\vec{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}},$$

   which is the expression we need to use to simulate particles (in a later lecture you will learn how to go from this exression for $n(\vec{v})$ to the expression for $n(v)$ in equation 3). Note that this distribution is already normalized. Looking at this expression, you see that this is a Gaussian distribution function which can be written on the form

$$P(\vec{v}) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-(v_x^2+v_y^2+v_z^2)/(2\sigma^2)}$$

   (a) Comparing with the Maxwell-Boltzmann distribution for $\vec{v}$, what is σ here?

   (b) In Python there is a function `random.gauss(mean,sigma)` to produce random numbers with a Gaussian probability distribution. The v_x, v_y and v_z components of the velocity of gas particles are thus all distributed randomly with mean value 0 and standard deviation given by the σ which you just found. Now you will simulate 10000 gas particles with a temperature $T = 6000$ K (like on the solar surface), assuming that the atoms in the gas are hydrogen atoms. Now produce the random velocity components v_x, v_y and v_z of these particles using the `random.gauss` function in Python. Now you have an array which has the velocity in each direction for all your 10000 particles representing what you would really find if you had a look at the velocities of 10000 particles in a gas with this temperature. Now compute the kinetic energy for each of the particles and take the mean value over all your particles. Compare the number you get to what you obtain with the analytic expression you found above. Does it fit? If it does not fit precisely, and if you have the computer power to do it, repeat the code but now with 100000 particles. Does it fit better now? In most real situatuions, an analytic expression cannot be found and simulations like these have to be made.

Problem 2 (60–90 min.)

One of the solar standard models predict the following numbers for the solar core: $\rho = 1.5 \times 10^5$ kgm^{-3}, $T = 1.57 \times 10^7$ K, $X_{\rm H} = 0.33$, $X_{\rm He} = 0.65$ and $X_{\rm CNO} = 0.01$. We will assume that the expressions for energy production per kilogram given in the text are valid at the core temperature of the Sun. We will make this approximation even for the expression for the triple-alpha reaction which is supposed to be correct only for higher temperatures.

1. Calculate the total energy produced per kilogram in the Sun by the pp-chain, CNO-cycle and the triple-alpha process.
2. Find the ratio between the energy production of the pp-chain and the CNO-cycle and between the pp-chain and the triple-alpha process. The energy produced by the CNO cycle is only about 1% of the total energy production of the Sun. If you got a very different number in your ratio between the pp-chain and the CNO-cycle, can you find an explanation for this difference? What would you need to change in order to obtain a more correct answer?
3. Now repeat the previous question using a mean core temperature of about $T = 13 \times 10^6$ K. Use this temperature in the rest of this exercise.
4. At which temperature T does the CNO cycle start to dominate?
5. Assume for a moment that only the pp-chain is responsible for the total energy production in the Sun. Assume that all the energy production in the Sun takes place wihin a radius $R < R_E$ inside the solar core.Assume also that the density, temperature and mass fractions of the elements are constant within the radius R_E. So all the energy produced by the Sun is produced in a sphere of radius R_E in the center of the solar core. Use the above numbers and the solar luminosity $L_\odot = 3.8 \times 10^{26}$ W to find the size of this radius R_E within which all the energy production takes place. Express the result in solar radii $R_\odot \approx 7 \times 10^8$ m. The solar core extends to about $0.2R_\odot$. How well did your estimate of R_E agree with the radius of

the solar core?

6. If the CNO-cycle alone had been responsible for the total energy production of the Sun, what would the radius R_E had been? (again express the result in solar radii)

Problem 3 (30 min.–1 hour)

1. Go through all the nuclear reactions in the pp-chain and CNO cycle. For each line in the chain, check that total charge and total lepton number is conserved. (there might be some printing errors here, if you spot one where is it?)

2. After having checked all these reactions you should have gained some intuition about these reactions and the principles behind them. So much that you should be able to guess the missing numbers and particles in the following reactions

$$\begin{aligned} {}^{27}_{14}\mathrm{Si} &\rightarrow {}^{27}_{?}\mathrm{Al} + e^{+} + ? \\ {}^{27}_{?}\mathrm{Al} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{24}_{12}\mathrm{Mg} + {}^{?}_{2}? \\ {}^{35}_{17}\mathrm{Cl} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{36}_{18}\mathrm{Ar} + ? \end{aligned}$$

Problem 4 (30–60 min.)

Before you can do this exercise, you need to read through section 4 again. In that section, we were considering a gas with a total number density of particles n per volume, a number density n_A per volume of A nuclei and a number density n_B per volume of B nuclei. We will now calculate the rate of reaction between A and B nuclei in the gas.

1. First we will try to find how many A nuclei with a given energy E will react with one B nucleus per time interval Δt. The answer is simple: All the A particles with energy E which are in such a distance from B that they will hit the disk with cross section $\sigma(E)$ around nucleus B within the time interval Δt (again, this is an imaginary situation: only one nucleus A can really react with B, the numbers we obtain are in reality probabilities). In figure 3 we illustrate the situation. Let $n_A(E)$ be the number density of A nuclei with energy E such that $n_A(E)dE$ is the number density of A nuclei with energies between E and $E + dE$. Then, show that the total number of nuclear reactions per nucleus B from A nuclei with energies in the interval E to $E+dE$ is given by
$$dN_A(E) = v(E)\, dt\, \sigma(E) n_A(E)\, dE. \tag{10}$$

2. Show that the velocity of an A-nuclei can be written as
$$v(E) = \sqrt{\frac{2E}{\hat{\mu}}}$$

(What is μ here?)

3. Use equation 3 to show that the number of A nuclei with energies between E and $E + dE$ is given by

$$n_A(E)dE = \frac{2n_A}{\sqrt{\pi}(kT)^{3/2}} E^{1/2} e^{-\frac{E}{kT}} dE.$$

4. Show now that the total reaction rate per B nucleus, i.e. the number of A nuclei reacting with each B nucleus (independent of the energy of the B nucleus, remember that the B nucleus is at rest) is given by

$$\frac{dN_A(E)}{dt} = \frac{1}{\sqrt{\pi\hat{\mu}}}(\frac{2}{kT})^{3/2}\sigma(E)n_A E e^{-\frac{E}{kT}} dE,$$

5. To obtain the total reaction rate r_{AB} between A and B nuclei having the number of reactions per B nucleus, we thus need to multiply with the total density of B nuclei n_B and integrate over all energies E. Show that the reaction rate, the total number of nuclear reactions per unit of volume per unit of time, is given by

$$r_{AB} = \frac{dN}{dt} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_A n_B}{\sqrt{\hat{\mu}\pi}} \int_0^\infty dE E e^{-E/kt} \sigma(E)$$

6. Use this expression to find the units of the reaction rate r_{AB} to check if you find the units that you would expect for r_{AB} being the total number of nuclear reactions per unit of volume per unit of time.

7. It is common to express the reaction rate using ε_{AB} which is the energy released per kilogram of gas per second. Assume that the energy released in each reaction between an A nucleus and a B nucleus is given by ε_0 (which is *not* the vacuum permittivity ϵ_0) and show that ε_{AB} can be written in terms of r_{AB} as

$$\varepsilon_{AB} = \frac{\varepsilon_0}{\rho} r_{AB}.$$

(Why does the density enter here?)