Proposed solution
UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Midterm Exam in AST2110 — The Universe

Day of exam: Monday 26th March 2007

Exam hours: 13.30 – 16.30

This examination paper consists of 3 pages.

Appendices: None

Permitted materials: Rottmann: "Matematisk formelsamling"
Øgrim og Lian: "Størrelser og enheter i fysikk og teknikk"
or Angell og Lian: "Fysiske størrelser og enheter"
Approved calculator
Two A4 pages (can be written on both sides) with your own notes

Make sure that your copy of this examination paper is complete before answering.

Exercise 1

A ship is to sail from Oslo (59°54′48″N, 10°43′10″E) to New York (42°57′54″N, 76°01′00″W). Assume that the Earth is spherical with a radius of 6378 km.

a) Calculate the distance that the ship must sail in nautical miles or in km.

We draw a spherical triangle with the North Pole (NP) in one vertex, Oslo in another and New York in the third. We can now use the cosine formula of spherical trigonometry,

$$\cos(ON) = \cos(PO)\cos(PN) + \sin(PO)\sin(PN)\cos(A),$$

where ON **is the arclength between Oslo and New York,** PO **is the arc length from the North Pole to Oslo,** PN **is the arc length from the North Pole to New York and** A **is the angle between Oslo and New York, seen from the pole.**

From the given longitudes and latitudes, we find that $A = 86.7361$ **degrees,** $PO = 30.0867$ **degrees and** $PN = 47.0350$ **degrees. Putting this into the cosine formula, we find that** $ON = 52.3663$ **degrees. To get the length in nautical miles, we can just use the definition that one nautical mile is one arc minute along a great circle, i.e. 60 times the length in degrees, or 3142.0 nautical miles.**

To get km, one can tranform from nautical miles, or use simple geometry:

$$L = 2\pi r_{earth}\frac{ON}{360} = 5829.3km.$$

There is a small difference in the two numbers, partly because the earth is not exactly a sphere, partly because of rounding off in the radius of the earth.

b) What is the distance between Oslo and New York along a straight line through the Earth?

Simple geometry (draw the line dividing the angle in two) gives that the length of the hole is $2r_{earth}\sin(ON/2) = 5628.5$**km.**

Exercise 2

Cosmic rays consist of electrically charged particles that are accelerated by strong electromagnetic fields in supernova remnants etc. We will study a proton that is accelerating over a linear path with length L and reaches a final velocity $(1-\alpha)c$, where $\alpha = 10^{-10}$. We assume that the force on the proton is constant and will determine how the proton's energy, velocity etc. varies along its path. The answers are to be expressed by L, α, c and the (rest)mass m_0 of the proton. The answers are to be **approximate** answers, utilising the fact that $\alpha \ll 1$.

a) Show that at the end of the path, the momentum of the proton is

$$p = \frac{m_0 c}{\sqrt{2\alpha}}$$

and its kinetic energy

$$T = \frac{m_0 c^2}{\sqrt{2\alpha}}.$$

Solution:

$$p = \frac{m_0 u}{\sqrt{1-u^2/c^2}} = \frac{m_0 c(1-\alpha)}{\sqrt{2\alpha - \alpha^2}} \sim \frac{m_0 c}{\sqrt{2\alpha}}.$$

and

$$T = \frac{m_0c^2}{\sqrt{1-u^2/c^2}} - m_0c^2 = \frac{m_0c^2}{\sqrt{2\alpha-\alpha^2}} - m_0c^2 \sim \frac{m_0c^2}{\sqrt{2\alpha}}.$$

b) Show that the force on the proton is

$$F = \frac{m_0c^2}{L\sqrt{2\alpha}}.$$

Starting from zero kinetic energy, the kinetic energy of the proton must equal the work that has been done on it, i.e., $T = W = \int F dr$. However, here the force is constant, therefore $T = F\int dr = FL$. Therefore,

$$F = \frac{T}{L} = \frac{m_0c^2}{L\sqrt{2\alpha}}.$$

c) Explain why the momentum p at time t is given by $p = Ft$. How long does it take for the proton to reach the end of the path?

By definition, force is the time derivative of momentum, both in classical mechanics and in relativity. Therefore momentum is the integral of the force with respect to time, $p = \int F dt$. Here the force is constant, and therefore it can be set outside the integral, $p = Ft$.

In this case, we therefore have

$$t = \frac{p}{F} = \frac{L}{c}.$$

With a very small α there is an insignificant time in which the proton travels with a speed significantly less than c.

d) Show that the velocity of the proton during its acceleration is given by (x is the position of the proton):

$$\frac{1}{\sqrt{1-(u/c)^2}} - 1 = \frac{x}{L\sqrt{2\alpha}}.$$

We go back to the work, i.e., $W = T = Fx$ and set in the full expression,

$$\frac{m_0c^2}{\sqrt{1-u^2/c^2}} - m_0c^2 = \frac{m_0c^2}{L\sqrt{2\alpha}}x.$$

Dviding by m_0c^2, we get the sought equation.

e) Find x when the velocity is $0.99c$.

Using the equation from d, we find that the proton reaches 0.99c after $8.6 \times 10^{-5}L$.

f) Draw a graph showing how kinetic energy T and velocity u increases with x.

T of course rises linearly with x, since we have $T = Fx$ and F is a constant. The velocity will rise extremely fast and reach 0.99c after less than 1/10000 of L, and in a graph it will be indistinguishable from C after that.

g) From your knowledge of p as function of t and T as function of x, use the expression for the relation between kinetic energy and momentum to find x as a function of t.

From above, we know that $T = Fx$. The relation between energy and momentum in special relativity is

$$E = \sqrt{m_0^2 c^4 + p^2 c^2},$$

and $T = E - m_0 c^2$. Hence,

$$T = Fx = \sqrt{m_0^2 c^4 + p^2 c^2} - m_0 c^2.$$

Above, we also found $p = Ft$, which we can insert, i.e.,

$$Fx = \sqrt{m_0^2 c^4 + F^2 t^2 c^2} - m_0 c^2.$$

Using $F = m_0 c^2/(L\sqrt{2\alpha})$, we find

$$\frac{m_0 c^2}{L\sqrt{2\alpha}} x = \sqrt{m_0^2 c^4 + \frac{m_0^2 c^6 t^2}{2\alpha L^2}} - m_0 c^2.$$

We divide by what is to the left of x and find

$$x = L\left(\sqrt{2\alpha + \frac{c^2 t^2}{L^2}} - \sqrt{2\alpha}\right).$$

Exercise 3

This exercise is given 50% of the weight of exercises 1 and 2. Give short answers to the following questions:

a) What is the absolute magnitude of a star that has a distance of 100 pc and apparent magnitude 6.0?

We have the distance formula $m - M = 5\log d - 5$. Therefore, $M = m - 5\log d + 5$, where d is measured in pc. Hence $M = 6.0 - 5 \times 2 + 5 = 6 - 10 + 5 = +1.0$.

b) Use one of Kepler's laws to find the orbital period of Venus, which has an average distance from the sun of 0.723 AU.

Kepler's third law says that $P^2 = a^3$, when the period P and the average distance a is given in units of terrestrial years and AU, respectively. The period of Venus is therefore $0.723^{3/2}$ years, or 224.5 days.

c) Sirius has $B - V = 0.01$ while the Sun has $B - V = 0.65$. Which has the highest surface temperature? Why?

The magnuitude is related to flux by $m = -2.5 \log F + const$. Therefore, a low value of $B-V$ means that the flux is large in the B band relative to the V band. A star which has a $B-V$ smaller than another star, therefore has a larger fraction of its energy radiated at small wavelengths compared to the other star. For a black body (Planck) spectrum, this ratio is higher, the higher the temperature is. Therefore, a star with low $B-V$ has higher temperature than a star with high $B-V$, and Sirius is hotter than the Sun.