Merk:
Den matematiske utledningen av Corioliskraften m.m. er gjennomført på en ryddig og nokså lett forståelig måte i læreboka til Arya. Her er det scannet inn de sidene som er aktuelle for FYS-MEK/F 1110. Denne delen er også med som pensum i kurset.

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Introduction to Classical Mechanics

Second Edition

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11.3 ROTATING COORDINATE SYSTEMS

Let us consider a fixed reference frame S with coordinate axes XYZ and a rotating frame S' with coordinate axes $X'Y'Z'$. Thus S is an inertial system, while S' is a noninertial system. The origins of the two coordinate systems always coincide. The unit vectors in the two coordinate systems are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, and $\hat{\mathbf{k}}$, respectively, as shown in Fig. 11.2. Let the position of

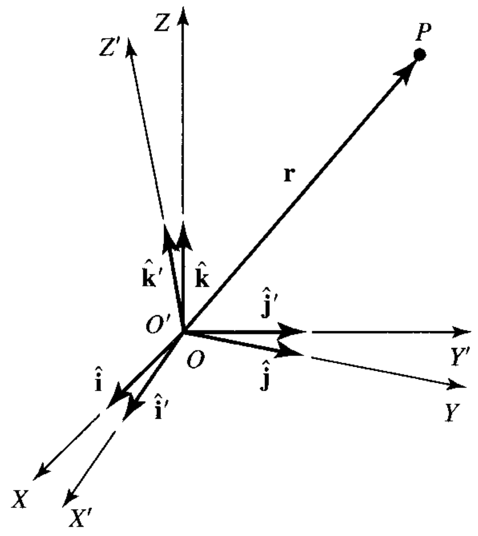

Figure 11.2 Coordinates X', Y', and Z' of system S' are rotating with respect to the stationary coordinates X, Y, and Z of system S.

the point P in space shown in Fig. 11.2 be represented by a vector $\mathbf{r}$. Since the origins of the two coordinate systems coincide, $\mathbf{r}$ is the same in both systems; only the components are different along different axes. (Also, $\mathbf{r}$ does not have to be measured from the origin.) Thus $\mathbf{r}$ in terms of components in either coordinate system may be written as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{11.12}$$

$$\mathbf{r} = x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}' + z'\hat{\mathbf{k}}' \tag{11.13}$$

The relation between the components in the two coordinate systems may be obtained by using the dot product. Thus

$$\begin{aligned} x &= \mathbf{r}\cdot\hat{\mathbf{i}} = x'(\hat{\mathbf{i}}'\cdot\hat{\mathbf{i}}) + y'(\hat{\mathbf{j}}'\cdot\hat{\mathbf{i}}) + z'(\hat{\mathbf{k}}'\cdot\hat{\mathbf{i}}) \\ y &= \mathbf{r}\cdot\hat{\mathbf{j}} = x'(\hat{\mathbf{i}}'\cdot\hat{\mathbf{j}}) + y'(\hat{\mathbf{j}}'\cdot\hat{\mathbf{j}}) + z'(\hat{\mathbf{k}}'\cdot\hat{\mathbf{j}}) \\ z &= \mathbf{r}\cdot\hat{\mathbf{k}} = x'(\hat{\mathbf{i}}'\cdot\hat{\mathbf{k}}) + y'(\hat{\mathbf{j}}'\cdot\hat{\mathbf{k}}) + z'(\hat{\mathbf{k}}'\cdot\hat{\mathbf{k}}) \end{aligned} \tag{11.14}$$

These equations may also be written as

$$\begin{aligned} x &= \lambda_{11}x' + \lambda_{21}y' + \lambda_{31}z' \\ y &= \lambda_{12}x' + \lambda_{22}y' + \lambda_{32}z' \\ z &= \lambda_{13}x' + \lambda_{23}y' + \lambda_{33}z' \end{aligned} \tag{11.15}$$

where $\lambda_{11}, \lambda_{21}, \ldots, \lambda_{33}$ are the cosines of the angles between the prime and the unprimed unit vectors. For example,$(\hat{\mathbf{i}}'\cdot\hat{\mathbf{i}}) = |1||1|\cos\theta_{11} = \cos\theta_{11} = \lambda_{11}$, where θ_{11} is the angle between $\hat{\mathbf{i}}'$ and $\hat{\mathbf{i}}$. Similarly, $(\hat{\mathbf{j}}'\cdot\hat{\mathbf{i}}) = |1||1|\cos\theta_{21} = \lambda_{21}$, where θ_{21} is the angle between $\hat{\mathbf{j}}'$ and $\hat{\mathbf{i}}$, with similar meanings for the other terms. Since the prime axes are rotating with respect to the unprimed axes, the cosines of the angles are a function of time.

Let us now consider a vector $\mathbf{A}$ in space. Since the coordinates are rotating, if $\mathbf{A}$ is constant in time in one coordinate system, it will not be constant in another coordinate system. That is, the time derivatives of a vector will be different in the two coordinate systems. This is because, by the time $\mathbf{A}(t)$ changes to $\mathbf{A}(t + \Delta t)$ in one system in time Δt, $\mathbf{A}(t)$ has changed in the other coordinate system due to the rotation. Let $\mathbf{A}$ have the components (A_x, A_y, A_z) and (A'_x, A'_y, A'_z) in the unprimed and the primed coordinates respectively. Thus

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{11.16}$$

$$\mathbf{A} = A'_x\hat{\mathbf{i}}' + A'_y\hat{\mathbf{j}}' + A'_z\hat{\mathbf{k}}' \tag{11.17}$$

Let d/dt denote the derivative with respect to the unprime (fixed) coordinate system and d'/dt with respect to the prime (rotating) coordinate system. From Eqs. (11.16) and (11.17), we obtain (by denoting the derivative by an overdot)

$$\frac{d\mathbf{A}}{dt} = \dot{A}_x\hat{\mathbf{i}} + \dot{A}_y\hat{\mathbf{j}} + \dot{A}_z\hat{\mathbf{k}} \tag{11.18}$$

$$\frac{d'\mathbf{A}}{dt} = \dot{A}'_x\hat{\mathbf{i}}' + \dot{A}'_y\hat{\mathbf{j}}' + \dot{A}'_z\hat{\mathbf{k}}' \tag{11.19}$$

To obtain a relation between $d\mathbf{A}/dt$ and $d'\mathbf{A}/dt$, we proceed as follows:

$$\frac{d\mathbf{A}}{dt} = \frac{d}{dt}(A'_x\hat{\mathbf{i}}' + A'_y\hat{\mathbf{j}}' + A'_z\hat{\mathbf{k}}')$$

$$= \dot{A}'_x\hat{\mathbf{i}}' + \dot{A}'_y\hat{\mathbf{j}}' + \dot{A}'_z\hat{\mathbf{k}}' + A'_x\frac{d\hat{\mathbf{i}}'}{dt} + A'_y\frac{d\hat{\mathbf{j}}'}{dt} + A'_z\frac{d\hat{\mathbf{k}}'}{dt} \tag{11.20}$$

Using the result of Eq. (11.19) in Eq. (11.20),

$$\frac{d\mathbf{A}}{dt} = \frac{d'\mathbf{A}}{dt} + A'_x\frac{d\hat{\mathbf{i}}'}{dt} + A'_y\frac{d\hat{\mathbf{j}}'}{dt} + A'_z\frac{d\hat{\mathbf{k}}'}{dt} \tag{11.21}$$

Thus we are left with evaluating the last three terms. To do this, let us suppose that the primed coordinate system is rotating about some axis ON through an origin with an angular velocity $\boldsymbol{\omega}$, as shown in Fig. 11.3. Let a vector $\mathbf{B}$ making an angle θ with the axis ON be at rest in the primed coordinate system so that its prime derivative $d'\mathbf{B}/dt = 0$, while we calculate the unprimed derivative $d\mathbf{B}/dt$. (Note that the time derivative of $\mathbf{B}$ depends on the component $\mathbf{B}$ along the axis, and not on the position of $\mathbf{B}$ in space.) At time t, vector $\mathbf{B}(t)$ is along OP. At a later time $t + \Delta t$, $\mathbf{B}(t + \Delta t)$ is along OQ. Thus $\mathbf{B}(t + \Delta t) - \mathbf{B}(t) = \Delta\mathbf{B}$. The magnitude of $\Delta\mathbf{B}$ from Fig. 11.3. is

$$\Delta B = RQ\,\Delta\phi = (B\sin\theta)(\omega\,\Delta t)$$

$$\frac{\Delta B}{\Delta t} = \omega B\sin\theta \tag{11.22}$$

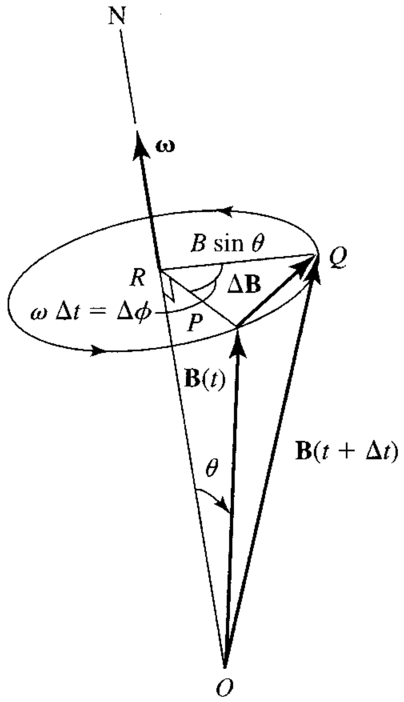

Figure 11.3 Primed coordinate system with vector **B** in it is rotating about an axis *ON* with an angular velocity **ω**.

or, in the limit $\Delta t \to 0$, we get

$$\frac{dB}{dt} = \omega B \sin\theta \tag{11.23}$$

or

$$\frac{d\mathbf{B}}{dt} = \boldsymbol{\omega} \times \mathbf{B} \tag{11.24}$$

This equation correctly gives the direction of $d\mathbf{B}/dt$ by using the definition of the cross product. The direction of $d\mathbf{B}/dt$ is perpendicular to the plane containing **ω** and **B**. We can make use of Eq. (11.24) to evaluate the last three terms in Eq. (11.21). Each of the three terms takes the form

$$\frac{d\hat{\mathbf{i}}'}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{i}}', \qquad \frac{d\hat{\mathbf{j}}'}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{j}}', \qquad \frac{d\hat{\mathbf{k}}'}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{k}}' \tag{11.25}$$

Substituting these in Eq. (11.21), we obtain

$$\frac{d\mathbf{A}}{dt} = \frac{d'\mathbf{A}}{dt} + A'_x(\boldsymbol{\omega} \times \hat{\mathbf{i}}') + A'_y(\boldsymbol{\omega} \times \hat{\mathbf{j}}') + A'_z(\boldsymbol{\omega} \times \hat{\mathbf{k}}')$$

or

$$\frac{d\mathbf{A}}{dt} = \frac{d'\mathbf{A}}{dt} + \boldsymbol{\omega} \times \mathbf{A} \tag{11.26}$$

Thus, knowing the time derivative of **A** in the unprimed (fixed) coordinate system, we can calculate the time derivative of **A** in the prime (rotating) coordinate system by using Eq. (11.26). In general, we may write the relation between d/dt and d'/dt as

$$\frac{d}{dt} = \frac{d'}{dt} + \boldsymbol{\omega} \times \tag{11.27}$$

When applied to a position vector **r**, this gives

$$\frac{d\mathbf{r}}{dt} = \frac{d'\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \tag{11.28}$$

or

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} \tag{11.29}$$

Equation (11.28) or (11.29) gives the relation between **v** in a fixed coordinate system and velocity **v**′ in a rotating coordinate system. This relation applies even when $\boldsymbol{\omega}$ is changing in both magnitude and direction. We are interested in getting the second derivative of **r**. This is achieved by differentiating Eq. (11.28) once again. That is,

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}\left(\frac{d'\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}\right) \tag{11.30}$$

Substituting for *d*/*dt* from Eq. (11.27), we have

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d'}{dt} + \boldsymbol{\omega} \times \quad\right)\left(\frac{d'\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}\right)$$

$$= \frac{d'^2\mathbf{r}}{dt^2} + \boldsymbol{\omega} \times \frac{d'\mathbf{r}}{dt} + \frac{d'\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d'\mathbf{r}}{dt} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

or

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d'^2\mathbf{r}}{dt^2} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} + 2\boldsymbol{\omega} \times \frac{d'\mathbf{r}}{dt} + \frac{d'\boldsymbol{\omega}}{dt} \times \mathbf{r} \qquad \text{Coriolis theorem} \tag{11.31}$$

Also, we may replace $d'\omega/dt$ by $d\omega/dt$ because the prime and unprime derivatives of any vector parallel to the axis of rotation are the same. We can verify this by using Eq. (11.26), in which we substitute $\mathbf{A} = \boldsymbol{\omega}$; that is,

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{d'\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \frac{d'\boldsymbol{\omega}}{dt}, \qquad \text{that is, } \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}' \tag{11.32}$$

since $|\boldsymbol{\omega} \times \boldsymbol{\omega}| = \omega^2 \sin 0° = 0$; $\boldsymbol{\omega} \times \boldsymbol{\omega}$ is a null vector.

In summary, we may say that if O and O' remain fixed, then any point in space in located by a position vector **r**, which is the same in both coordinate systems, while the velocity and acceleration are given by Eqs. (11.28) and (11.31), respectively.

Equation (11.31) is a statement of the *Coriolis theorem.* We briefly discuss each term of this equation.

$\dfrac{d^2\mathbf{r}}{dt^2}$ ≡ acceleration relative to the unprimed or fixed coordinate system

$\dfrac{d'^2\mathbf{r}}{dt^2}$ ≡ acceleration relative to the prime coordinate system

$\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$ ≡ centripetal acceleration of a point in rotation about an axis

$2\boldsymbol{\omega} \times \dfrac{d'\mathbf{r}}{dt}$ ≡ Coriolis acceleration, which is present when a particle is moving in the prime (rotating) coordinate system

$\dfrac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}$ ≡ nonuniform rotation, which vanishes if $\boldsymbol{\omega}$ is constant about a fixed axis

If we assume that Newton's second law is valid in the unprimed (fixed) coordinate system, using Eq. (11.31) we obtain

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} = m\frac{d'^2\mathbf{r}}{dt^2} + m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} + 2m\boldsymbol{\omega} \times \frac{d'\mathbf{r}}{dt} + m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} \quad \textbf{(11.33)}$$

while the effective force $\mathbf{F}'$, as observed in the rotating coordinate system acting on m, is given by

$$m\frac{d'^2\mathbf{r}}{dt^2} \equiv \mathbf{F}' = \mathbf{F} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \frac{d'\mathbf{r}}{dt} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} \quad \textbf{(11.34)}$$

where

$-m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$ ≡ centrifugal force acting away from the center **(11.35a)**

as we will explain shortly.

$-2m\boldsymbol{\omega} \times \dfrac{d'\mathbf{r}}{dt}$ ≡ Coriolis force **(11.35b)**

$-m\dfrac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}$ ≡ transverse force for the case of a nonuniform rotation, which is zero because we shall deal with uniform rotation only **(11.35c)**

In Eq. (11.33), $|\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}| = a_c$ is the centripetal acceleration because, as shown in Fig. 11.4, it is directed toward the center and perpendicular to the axis of rotation. As shown, $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ or $v = \omega r \sin\theta$, where v is the speed of the circular motion and $r \sin\theta$ is the distance from the axis. From Fig. 11.5, using $\omega = v/(r \sin\theta)$, we get

$$a_c = |\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}| = \omega^2 r \sin\theta = \frac{v^2}{r\sin\theta} \quad \textbf{(11.36)}$$

The quantity $-m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$ is called the *centrifugal force* and is equal to $-m\omega^2 r$ in the case where $\boldsymbol{\omega}$ is normal to the radius vector. The negative sign means that the centrifugal force is directed outward or away from the center of rotation, as shown in Fig. 11.5. According to classical mechanics, the centrifugal force is not a real force; it is a fictitious or noninertial force.

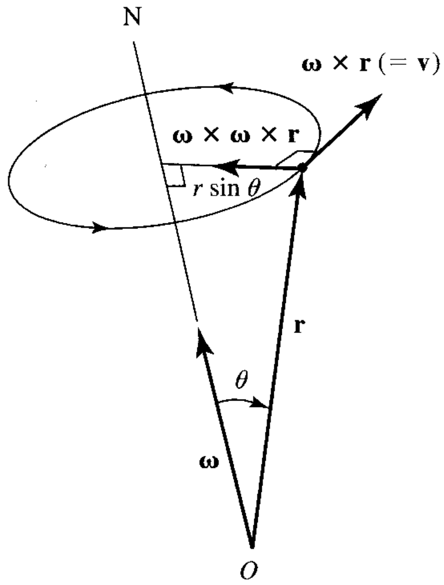

Figure 11.4 Centripetal acceleration $a_c = |\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}|$ resulting from the rotation of the primed coordinate system.

This force is present only if we refer to moving coordinates in space. Thus, for example, a particle moving in a circle has no centrifugal force acting on it. A force that is acting toward the center, producing centripetal acceleration, is present. On the other hand, if we observe this moving particle from a reference frame that is moving with the particle, the particle will be at rest in this system. There is a force acting toward the center, but the particle does not fall toward the center. This is possible only if the force toward the center is balanced by an outward force, the centrifugal force.

The term $-2m\boldsymbol{\omega} \times (d'\mathbf{r}/dt)$ is called the *Coriolis force* and results from the motion of a particle in a rotating coordinate system. This force is directly proportional to a velocity $\mathbf{v}'$ and

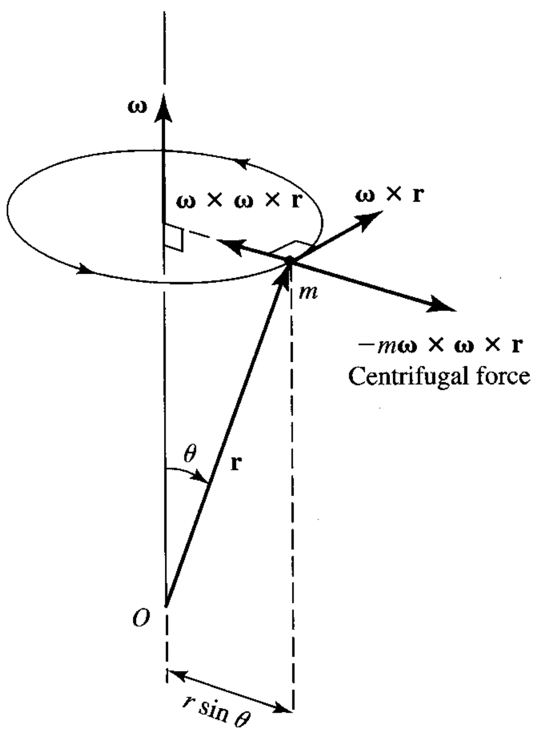

Figure 11.5 Centrifugal force resulting from rotational motion is shown directed away from the center.

will disappear if there is no motion. Once again, according to classical mechanics, it is not a real force; it is a noninertial force (or fictitious force).

It is essential to note that both the centripetal force and Coriolis force have been introduced for only one purpose: to write an equation that is similar to Newton's second law and that is still applicable to noninertial systems including rotational coordinate systems. Thus, if we write Newton's second law in the unprime (or fixed) coordinates as

$$\mathbf{F} = m\,\frac{d^2\mathbf{r}}{dt^2} = m\mathbf{a}$$

then the force $\mathbf{F}'$ acting on m in a rotating coordinate system will be

$$\begin{aligned}\mathbf{F}' &= m\,\frac{d'^2\mathbf{r}}{dt^2} = m\mathbf{a}' \\ &= \mathbf{F} + \text{noninertial forces} \\ &= \mathbf{F} + \mathbf{F}_{\text{cent}} + \mathbf{F}_{\text{Cor}} + \mathbf{F}_{\text{trans}} \qquad (11.37)\end{aligned}$$

where the noninertial (or fictitious) forces are the centrifugal and Coriolis forces. This modification serves a very useful purpose, as we shall see in the following sections.

If the prime coordinate system has both translational and rotational motion, the following equations give the relations between the displacement, velocity, and acceleration vectors in the two systems. If $\mathbf{r}_0$ is the distance of O' from O,

$$\mathbf{r} = \mathbf{r}' + \mathbf{r}_0 \qquad (11.38)$$

$$\frac{d\mathbf{r}}{dt} = \frac{d'\mathbf{r}'}{dt} + \boldsymbol{\omega} \times \mathbf{r}' + \frac{d\mathbf{r}_0}{dt} \qquad (11.39)$$

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d'^2\mathbf{r}'}{dt^2} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \frac{d'\mathbf{r}'}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}' + \frac{d^2\mathbf{r}_0}{dt^2} \qquad (11.40)$$

The centrifugal and Coriolis forces are not due to any physical interaction but are a result of kinematics; hence such forces are called *noninertial* or *fictitious forces.* For example, real forces always decrease with distance, whereas the centrifugal force increases with distance. It is true that it is convenient to use a rotational coordinate system to describe rotational motion of an object, but one must remember that the noninertial fictitious forces must be used only in a noninertial system or rotational coordinate system and not in an inertial system. For example, when a stone tied to a string is whirled in a circle, we feel as if a force is pulling the stone outward; we call this centrifugal force. For an observer in a rotating coordinate system with the stone, the stone is stationary and the outward centrifugal force balances the inward tension in the string. But in an inertial system there is no centrifugal force, only the tension in the string that causes the radial acceleration. Description in either coordinate system is correct provided the proper forces are taken into consideration. Similarly, when a car is going around a curve too fast, it will skid outward. According to an observer in an inertial system, the sideways force exerted by the road on the tires of a car is not sufficient to keep the car turning with the road. To an observer in the car (in the noninertial system), it may feel as if the car is being pushed outward by a centrifugal force.

The most important application of the above discussion is presented in the next two sections.