Fys-Mek1110 – 2013 – Oblig 1

Modeling a 100m race

In this project we will develop an advanced model for the motion of a sprinter during a 100m race. We will build the model gradually, adding complications one at a time to develop a realistic model for the race.

(a) A sprinter is accelerating along the track. Draw a free-body diagram of the sprinter, including only horizontal forces. Try to make the length of the vectors correspond to the relative magnitudes of the forces.

Let us assume that the sprinter is accelerated by a constant horizontal driving force, $F = 400\text{N}$, from the ground all the way from the start to the 100m line (averaged over a few steps). The mass of the sprinter is $m = 80\text{kg}$.

(b) Find the position, $x(t)$, of the sprinter as a function of time.

(c) Show that the sprinter uses $t = 6.3\text{s}$ to reach the 100m line.

This is a bit fast compared with real races. However, real sprinters are limited by air resistance. Let us introduce a model for air resistance by assuming that air resistance force is described by a square law:

$$D = (1/2)\rho C_D A(v - w)^2 \tag{1}$$

where ρ is the density of air, A is the cross-sectional area of the runner, C_D is the drag coefficient, v is the velocity of the runner, and w is the velocity of the air. At sea level $\rho = 1.293\text{kg/m}^3$, and for the runner we can assume $A = 0.45\text{m}^2$, and $C_D = 1.2$. You can initially assume that there is no wind: $w = 0\text{m/s}$.

Assume that the runner is only affected by the constant driving force, F, and the air resistance force, D.

(d) Find an expression for the acceleration of the runner.

(e) Use Euler's method to find the velocity, $v(t)$, and position, $x(t)$ as a function of time for the runner. The runner starts from rest at the time $t = 0\text{s}$. Plot the position, velocity and acceleration of the runner as a function of time. How did you decide on the time-step Δt? (Your answer should include the program used to solve the problem and the resulting plots).

(f) Use the results to find the race time for the 100m race.

(g) Show that the (theoretical) maximum velocity of a runner driven by these forces is:

$$v_T = \sqrt{\frac{2F}{\rho C_D A}}\,. \tag{2}$$

The runner may have to run more than 100m to reach this velocity. (We often call this maximum velocity the terminal velocity – "terminal" because the velocity increases until it reaches the terminal velocity, where the acceleration becomes zero). Find the numerical value of the terminal velocity for the runner. Do you think this is realistic?

So far the model only includes a constant driving force and air resistance. This is clearly a too simplified model to be realistic. Let us make the model more realistic by adding a few features.

First, there is a physiological limit to how fast you can run. The driving force from the runner should therefore decrease with velocity, so that there is a maximum velocity at which the acceleration is zero even without air resistance. While we do not know the detailed physiological mechanisms for this effect, we can make a simplified force model to implement the effect by introducing a driving force, F_D, with two terms: a constant term, F, and a term that decreases with increasing velocity, F_V:

$$F_V = -f_V v \ , \tag{3}$$

so that the driving force is:

$$F_D = F + F_V = F - f_V v \ . \tag{4}$$

Reasonable values for the parameters are $F = 400$N, and $f_v = 25.8$sN/m. (These values are chosen to make the maximum velocity reasonable – they are not based on a physiological consideration).

(h) If you assume that the runner is subject only to these two driving forces, what is his maximum velocity? (You can ignore the drag term, D, in this calculation).

In addition, during the first few seconds the runner is crouched and accelerating rapidly. In this phase, his cross-sectional area is smaller because he is crouched, and the driving force exerted by the runner is larger than later. Let us also introduce these aspects into our model.

First, let us assume that the crouched phase lasts approximately for a time, t_c. We do not expect this phase to end abruptly at a specific time. Instead, we expect the driving force to decrease gradually (and the cross-sectional area to increase gradually) as the runner is going from a crouched to an upright running position. A common way to approximate such a change is through an exponential function that depends on the time and the characteristic time, t_c. For example, by introducing an initial driving force, F_C:

$$F_C = f_c \exp(-(t/t_c)^2) \ . \tag{5}$$

When $t = 0$, the force is equal to f_c, but as time increases, the force decreases rapidly. When the time has reached t_c, the force has dropped to $1/e \simeq 0.37$ of the value at $t = 0$, and after a time $4t_c$ this contribution to the driving force has dropped to less than 2% of its initial value.

Notice that we do not have any experimental or theoretical reason to use this particular form for the time dependence. We have simply chosen a convenient form as a first approximation, and then we use this form and try to get reasonable results with it. A better approach would be to have experimental data on how the force varied during the first few seconds, but unfortunately we do not know this. Making rough estimates that you can subsequently improve by better measurements, calculations, or theory will be an important part of how you apply physics in practice.

The total *driving* force is then:

$$F_D = F + f_c \exp(-(t/t_c)^2) - f_v v \ . \tag{6}$$

where reasonable values for the parameters are $f_c = 488$N and $t_c = 0.67$s. (These values are chosen so that the total race-time becomes reasonable).

In addition, we need to modify the air resistance force because the runner is crouched in the initial phase, so that the cross-sectional area is reduced. We therefore need to replace the cross-sectional area A in the expression for D with a time-dependent

expression, $A(t)$, with the properties that: 1) when time is zero, the area should be reduced to 75% of the area during upright running (again, we guess reasonable values); and 2) after a time much larger than t_c, the runner is upright, and the cross-sectional area should be A. Again, we introduce a modification to the area that depends on the exponential factor used above:

$$\begin{aligned} A(t) &= A - 0.25A\exp(-(t/t_c)^2) \\ &= A\left(1 - 0.25\exp(-(t/t_c)^2)\right) \ . \end{aligned} \tag{7}$$

The air resistance force therefore becomes:

$$\begin{aligned} D &= \frac{1}{2}A(t)\rho C_D(v-w)^2 \\ &= \frac{1}{2}A\left(1 - 0.25\exp(-(t/t_c)^2)\right)\rho C_D(v-w)^2 \end{aligned} \tag{8}$$

The total force on the runner is:

$$\begin{aligned} F_{\text{net}} &= F + F_C - F_V - D \\ &= F + f_c\exp(-(t/t_c)^2) - f_v v - D \ , \end{aligned} \tag{9}$$

where $F = 400\text{N}$ is a constant driving force, and the other terms have been addressed above.

(i) Modify your numerical method to include these new forces. Find and plot $x(t)$, $v(t)$, and $a(t)$ for the motion.

(j) How fast does he run 100m?

(k) Compare the magnitudes of the various forces acting on the runner by plotting F (which is constant), F_C, F_V and D as a function of time for a 100m race. Discuss how important the various effects are.

(l) Use the model to test how the resulting time on 100m would change if the runner had a hind wind with a wind speed of $w = 1\text{m/s}$. What if he was running into a wind with a wind speed of $w = 1\text{m/s}$?

End of Oblig 1