Project 9.1: Stick-slip friction

In this project we will study a phenomenon called stick-slip friction. If you pull a block along a flat table with a soft spring, you will find that the block does not move continuously with a constant velocity, instead it moves in small jumps. Let us examine the process in detail: The block starts from rest. As you pull the spring, the spring will eventually pull at the block with a force larger than the maximum static friction force. Consequently, the block starts sliding, but it stops after a short distance, and the process repeats itself. This intermittent motion is called stick-slip friction, and it is the origin of the high-frequency vibrating tone you often hear from wheels that are not well lubricated. It is also one of the basic mechanisms leading to the wide distribution of earthquake sizes.

Here, we will introduce and study a model for stick-slip friction for a block pulled by a spring sliding over a flat, horizontal surface, as illustrated in figure 9.27.

The block has mass m. A massless spring (with spring constant k and equilibrium length b) is attached to the block at the point x. The free end (the right-hand end in figure 9.27) of the spring is at the point x_b. We move x_b, the free end of the spring, with a constant velocity u. The static and dynamic coefficients of friction for the contact between the block and the bottom surface are μ_s and μ_d respectively. The acceleration of gravity is $g = 9.8\text{m/s}^2$.

The block starts at the position $x(t_0) = 0$ at the time $t_0 = 0$. The position x_b of the free end of the spring is $x_b(t_0) = x(t_0) + b$ at t_0.

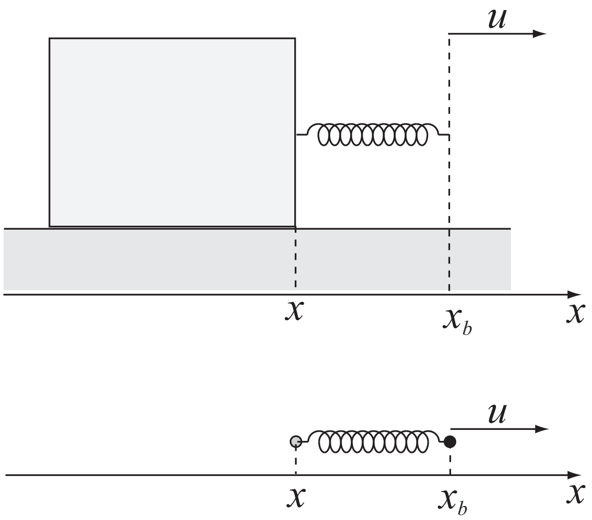

Figure 9.27: *Illustration of a block pulled by a spring.*

(a) Draw a free-body diagram for the block.

(b) Find the position of the spring attachment point $x_b(t)$ as a function of time.

(c) Show that the force, $\vec{F}$ on the block from the spring is:

$$\vec{F} = k(x_b - x - b)\hat{i} \; . \tag{9.128}$$

First, let us characterize the stationary state, where the block is moving at a *constant* velocity.

(d) Identify the forces acting on the block and draw a free-body diagram for the block in the stationary state.

(e) Introduce force models for all the forces acting on the block. Find the normal force, N, on the block.

(f) Find the acceleration of the block in the stationary state.

(g) Find the elongation ΔL of the spring in the stationary state.

(h) Find the position $x(t)$ of the block as a function of time in the stationary state.

Let us now address the situation where the block starts at rest. That is, we assume that the block starts at $x(t_0) = 0\text{m}$ with $v(t_0) = 0\text{m/s}$ at the time $t_0 = 0\text{s}$.

(i) Identify the force acting on the block and draw a free-body diagram of the block before the block starts moving. Introduce force models for all the forces.

(j) Assume that the block starts at rest. Find the elongation ΔL of the spring at the instant the block starts moving.

(k) Assume that the block starts at rest. Find the friction force on the block as a function in the period before the block starts moving. Sketch the friction force as a function of time until some time after the block has started moving.

Finally, let us address the motion of the block immediately after it starts moving.

(l) Show that the acceleration of the block immediately after it starts moving is:

$$a = \frac{k}{m}(x_b - x - b) - \mu_d g \; , \tag{9.129}$$

Explain why you cannot use this relation for the acceleration to determine the subsequent motion of the block.

Now, we will develop a general method to find the motion of the block. That is, we want to find $x(t)$. We will do this stepwise, checking our results along the way.

First, we study the case when $u = 0\text{m/s}$ and the coefficients of friction are zero, $\mu_s = \mu_d = 0$.

(m) Identify the forces acting on the block and draw a free-body diagram of the block. Introduce force models for all the forces.

(n) Find an expression for the horizontal acceleration of the block.

(o) If the block starts with the velocity $v(t_0) = v(0) = v_0$, show that

$$x(t) = \frac{v_0}{\omega}\sin\omega t \; , \tag{9.130}$$

where

$$\omega = \sqrt{\frac{k}{m}} \; , \tag{9.131}$$

describes the motion of the block – in the case when $u = 0\text{m/s}$, and $\mu_s = \mu_d = 0$.

(p) Write a numerical algorithm to find the position and velocity of the block at a time $t_i + \Delta t$, $x(t_i + \Delta t)$ and $v(t_i + \Delta t)$, given the position and velocity of the block at a time t_i, $x(t_i)$ and $v(t_i)$.

(q) Implement the numerical algorithm in a program to find the position of the block as a function of time for $m = 0.1$kg, $k = 100$N/m, $b = 0.1$m and $v_0 = 0.1$m/s. Plot the behavior for a simulation of 2s, and compare the result of your program with with exact solution. Ensure that you choose a time-step Δt the reproduces the exact solution with sufficient accuracy. What happens if you choose a too large time-step Δt?

Let us now address the situation when the block is pulled at a finite velocity, u.

(r) Modify your program to find the position of the block when $u = 0.1$m/s and the block starts at rest. In this case, the exact solution is:

$$x(t) = ut - \frac{u}{\omega} \sin \omega t \; . \tag{9.132}$$

Compare your result with the exact solution by plotting both the simulated x and the exact x in the same plot.

Finally, we address the full complexity of the situation, and introduce non-zero friction forces.

(s) Modify your program to include friction using $\mu_s = 0.6$, $\mu_d = 0.3$. Show a plot of $x(t)$ for $m = 0.1$kg and for $m = 1.0$kg.

(t) You should also plot the spring force F on the block as a function of time for both cases in the previous exercise. Can you explain the differences?

(u) What happens if you instead decrease k to $k = 10$N for $m = 0.1$kg. Can you explain the behavior?

The rest of the project is **optional** and **difficult**, but interesting.

The model we have studied so far introduces the concept of stick-slip friction in a simple setting. We will now use the understanding we have developed so far to modify our model to address the slip occuring along a fault plane in the Earth's crust, which is the origin of earthquakes. Earthquakes typically occur along pre-existing fracture planes seperating two rock plates. As a simplified model, we assume that the forces acting on one of the plates is slowly building up due to large scale tectonic effects such as the motion of the continental plates. The two rock plates are pressed together, but the forces building up in one of the plates will eventually lead to a small part of the two surfaces starting to slip – when the local static friction threshold is exceeded. Since the rock is elastic, this will lead to larger forces on the surrounding parts of the plate, which may again slip. Thus a single slip can lead to an avalance of slips, which is what constitutes an earthquake.

How can we model the behavior of the two plates? Let us model one of the plates as a flat, stationary plate – the bottom plate. The other plate we model as an elastic system, consisting of blocks of mass m connected by springs with spring constant K with an equilibrium length, b, as illustrated in figure 9.28.

The top plate, which consists of the blocks, is loaded externally by tectonic forces that slowly build up. We model the horizontal forces acting on the blocks by attaching a horizontal, external spring to each of the blocks. The other end of the spring is pulled with a constant velocity u – just as we did for the spring-block system presented above. The spring constant for these external springs are k and the equilibrium length of the spring is zero. The vertical forces acting on the blocks from above is represented by a constant vertical force, N, acting on each block from above. This force is much larger than gravity, so we can neglect the effects of gravity. There is no friction at the top of the blocks, but there are frictional forces between the bottom surface and each of the blocks. The static and dynamic coefficients of friction for these contacts are μ_s and μ_d respectively.

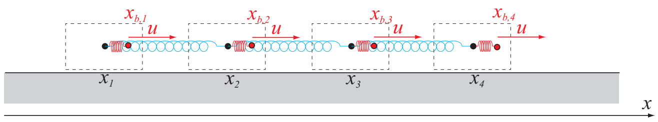

Figure 9.28: *Illustration of a series of connected blocks squeezed between an upper plate moving with a constant velocity u and a stationary lower plate.*

The initial position of block i is $x_i = bi$, and the initial position of attachment point i is $x_{b,i} = x_i$. Each of the attachment points $x_{b,i}$ are moving with a constant velocity u in the horizontal direction.

(v) Draw a free-body diagram of one of the blocks.

(w) Show that the force from the springs on block i (which has a block on both its left and its right side) is:

$$F = k(x_{b,i} - x_i) + K(x_{i+1} - x_i - b) - K(x_i - x_{i-1} - b) \; . \tag{9.133}$$

(x) Write a program to model the behavior of $M = 10$ blocks using $k = 0.01$N/m, $K = 0.1$N/m, $b = 0.1$m, $N = 1.0$N, $u = 0.1$m/s, $\mu_s = 0.6$, and $\mu_d = 0.5$. Run the simulation using a small random perturbation of the initial positions of each block, so that block i is at an initial position $x_{i,0}$:

$$x_{i,0} = x_{b,i}(0) + b * 0.2 * (1 - 2 * \Delta) \; , \tag{9.134}$$

where Δ is a random number uniformly distributed from 0 to 1. You can use `random.rand` in Python or `rand` in Matlab. Experiment with slighly different values of the parameters and provide plots of the various behaviors you observe. Can you still explain what is happening, and can you see how your results relate to earthquakes?