Førsteordens difflikning

Generelt

$$\dot{x} = f(x,t), \qquad x(0) = x_0$$

Eksempel (separabel)

$$\dot{x} = x\cos t, \qquad x(0) = x_0.$$

$$\int_{x_0}^{x} \frac{dx'}{x} = \int_0^t \cos t'\, dt'.$$

$$\ln \frac{x}{x_0} = \sin t$$

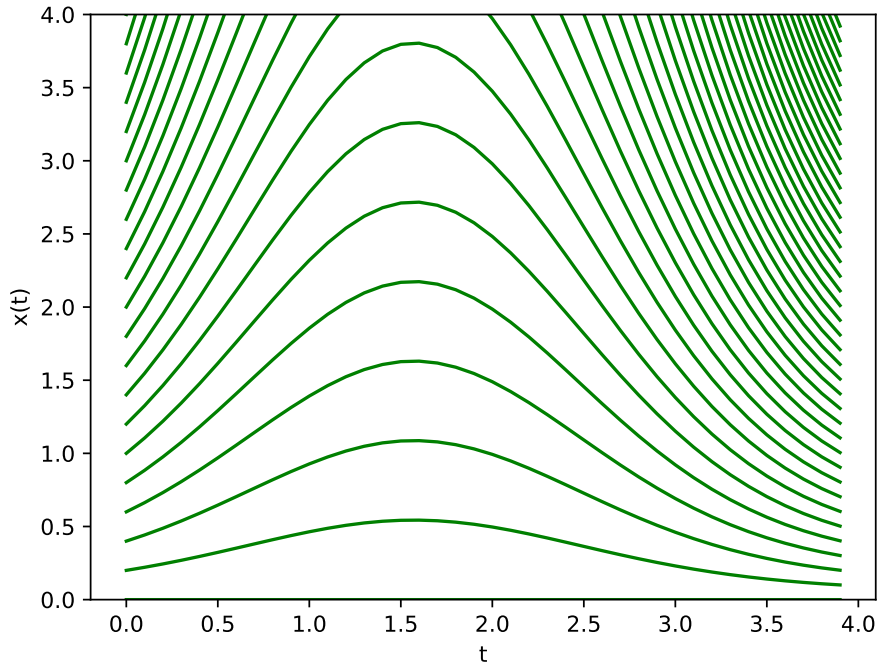
eller

$$x(t) = x_0 e^{\sin t}.$$

Geometrisk bilde

$$\dot{x} = f(x,t), \qquad x(0) = x_0$$

Beskriver en familie av løsninger $x(t)$, en for hver x_0.



Gjennom ethvert punkt (t,x) går det en entydig bestemt løsning.

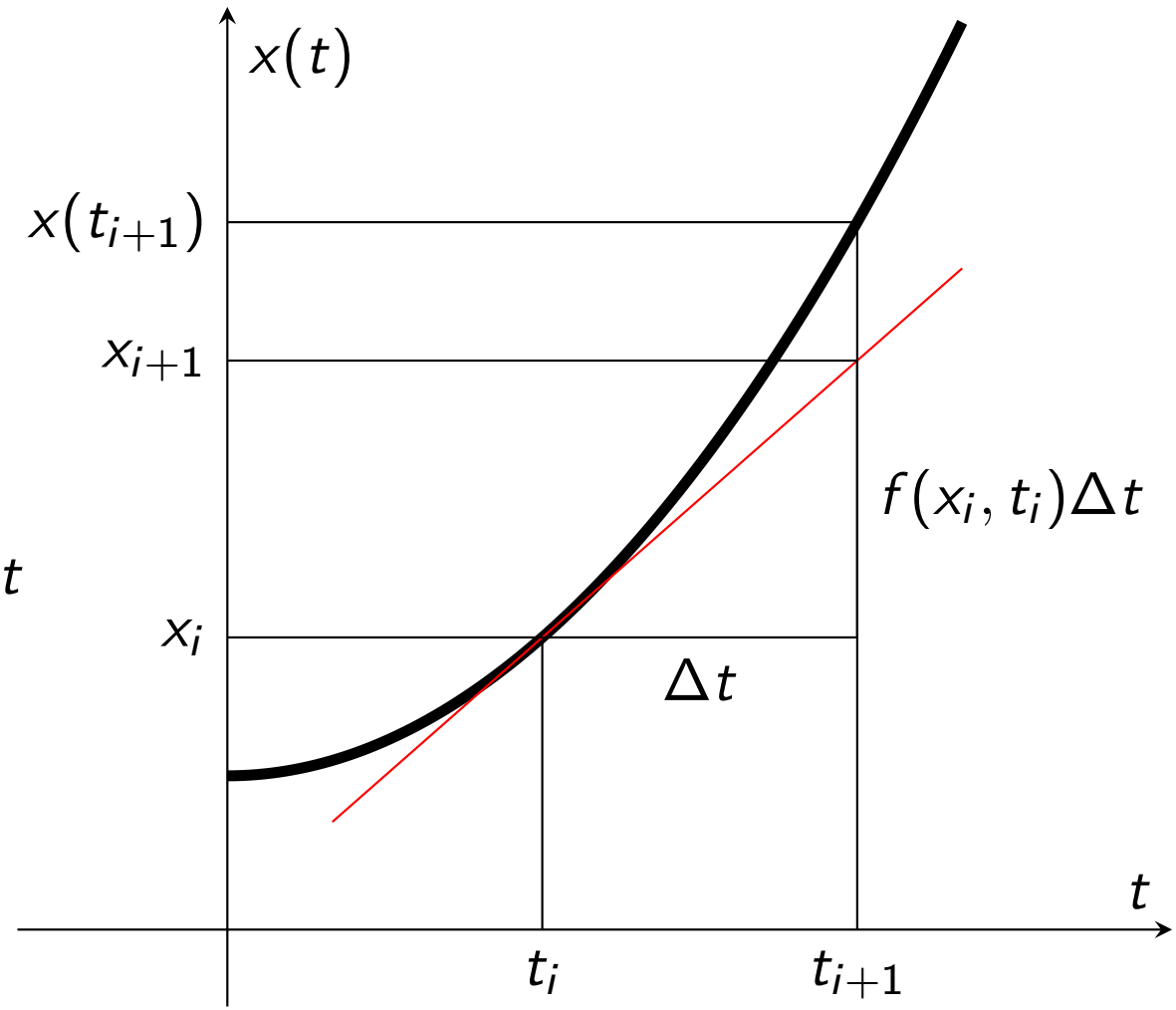
Eulers metode

$$\dot{x} = f(x,t), \qquad x(0) = x_0$$

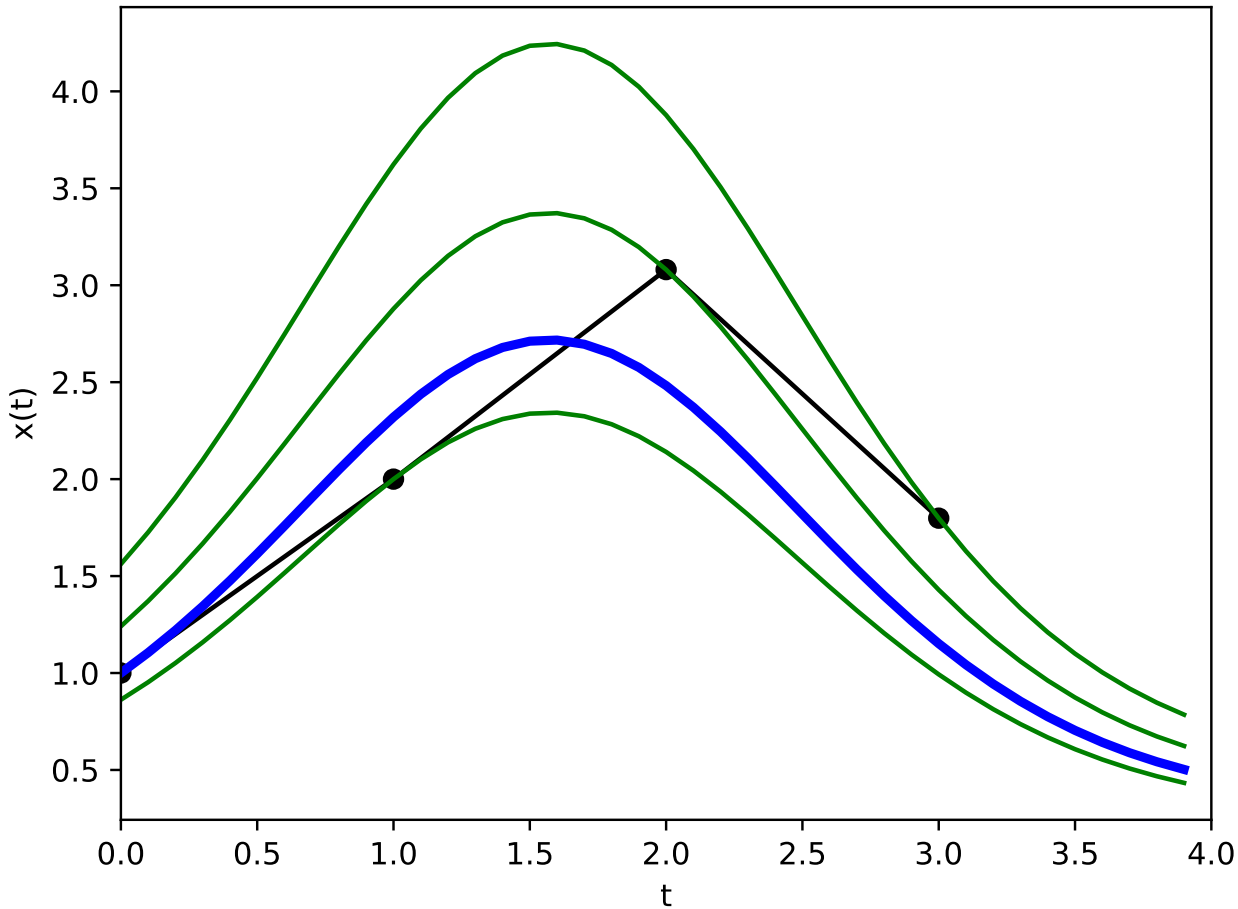
$$\dot{x} \approx \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$x(t+\Delta t) \approx x(t) + f(x,t)\Delta t$$

$$x_{i+1} = x_i + f(x_i, t_i)\Delta t.$$



Eulers metode følger tangenten i startpunktet for hvert intervall



```
# Eulers metode for \dot x = x cos t. Vi sammenlikner numerisk og analytisk los

import numpy as np
import matplotlib.pyplot as plt


x0 = 1
dt = 0.1
tMax = 20

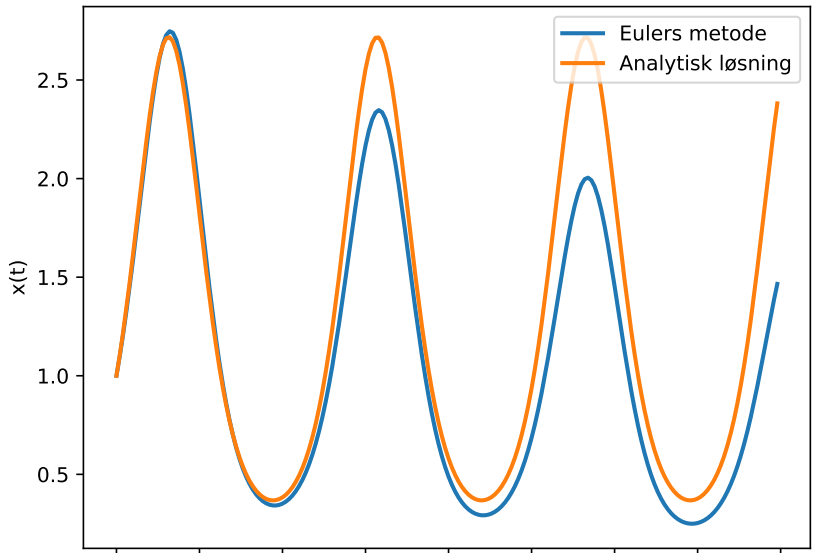
n = int(tMax/dt)
t = np.zeros(n)
dx = np.zeros(n)
x = np.zeros(n)

x[0] = x0
t[0] = 0

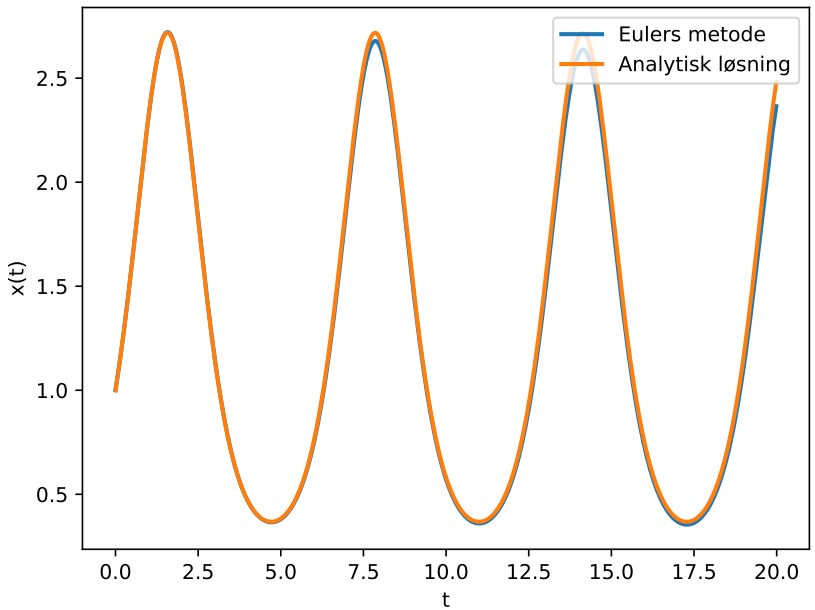
for i in range(n-1):
    t[i+1] = t[i] + dt
    dx[i] = np.cos(t[i])*x[i]
    x[i+1] = x[i] + dx[i]*dt

plt.plot(t,x,linewidth=2, label='Eulers metode')
plt.plot(t,x0*np.exp(np.sin(t)),linewidth=2, label='Analytisk løsning')
plt.legend(loc='upper right')
plt.xlabel('t')
plt.ylabel('x(t)')
```

Mindre Δt gir mindre feil



$\Delta t = 0.1$



$\Delta t = 0.01$

Feil som funksjon av Δt

Taylors formel med restledd

$$x(t+\Delta t) = x(t)+\dot{x}(t)\Delta t+\frac{1}{2}\ddot{x}(u)\Delta t^2 = x(t)+f(x,t)\Delta t+\frac{1}{2}\ddot{x}(u)\Delta t^2$$

med $u \in [t, t+\Delta t]$. Eulers metode med $x_i = x(t)$ og $t_i = t$ tilnærmer $x(t+\Delta t)$ med

$$x_{i+1} = x_i + f(x_i, t_i)\Delta t.$$

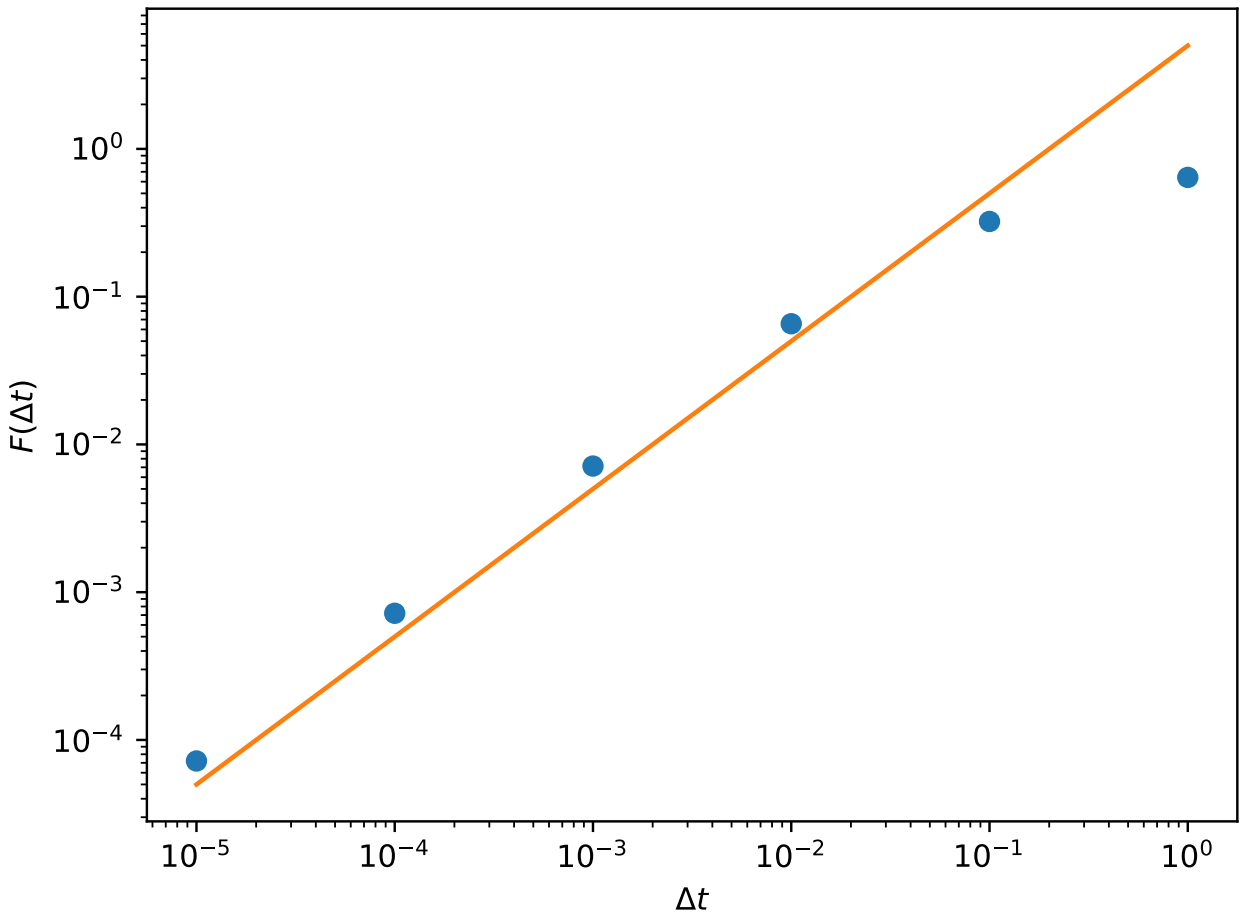
Feilen i ett tidssteg er da

$$F_1(\Delta t) = x(t+\Delta t) - x_{i+1} = \frac{1}{2}\ddot{x}(u)\Delta t^2.$$

Hvis vi numerisk løser likninga fra starttida $t = 0$ til en sluttid T er antallet steg vi gjør $N = T/\Delta t$

$$F_T(\Delta t) \propto N\Delta t^2 = T\Delta t.$$

Feil som funksjon av Δt



Implisitt Eulers metode

$$\dot{x} = f(x,t) \qquad x(t+\Delta t) \approx x(t) + f(x,t)\Delta t$$

Euler

$$x_{i+1} = x_i + f(x_i, t_i)\Delta t.$$

Implisitt Euler

$$x_{i+1} = x_i + f(x_{i+1}, t_{i+1})\Delta t.$$

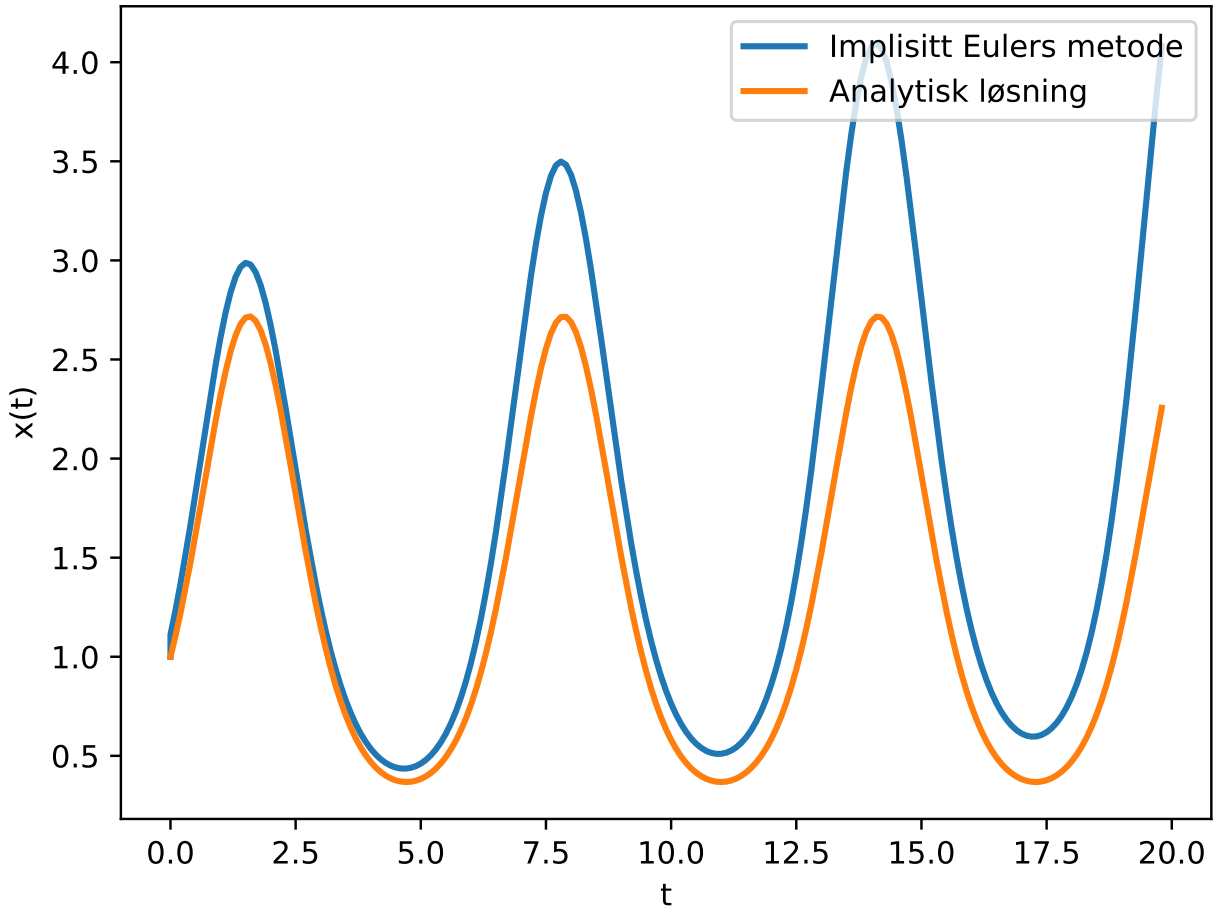
Eksempel

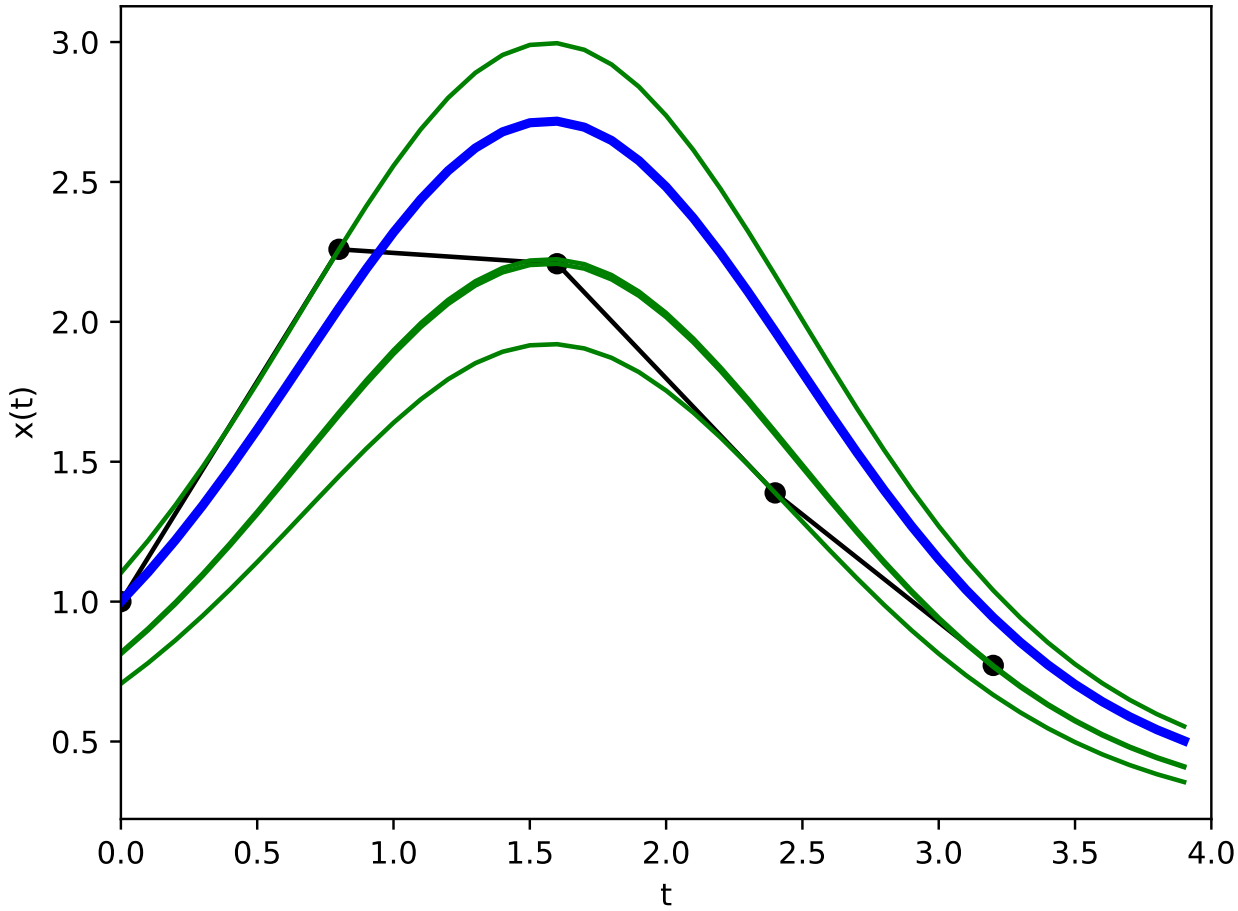
$$\dot{x} = x\cos t,$$

$$x_{i+1} = x_i + x_{i+1}\cos t_{i+1}\Delta t$$

som vi løser og finner

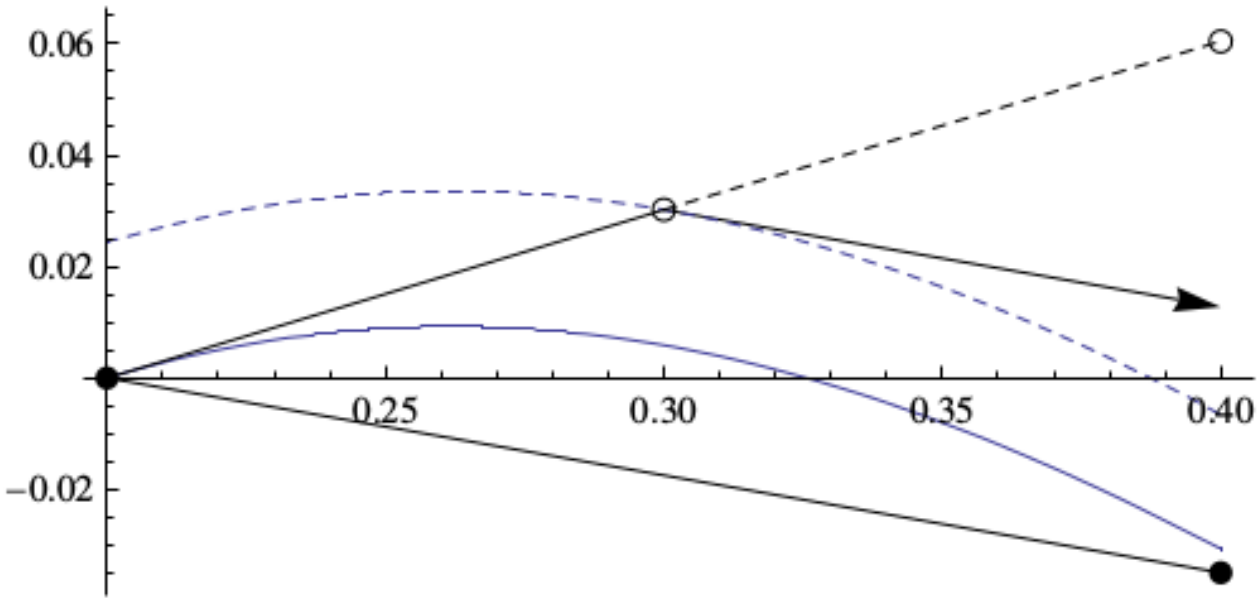
$$x_{i+1} = \frac{x_i}{1 - \cos t_{i+1}\Delta t}.$$


Implisitt Eulers metode
Analytisk løsning
x(t)
t
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0


3.0
2.5
2.0
1.5
1.0
0.5
x(t)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
t

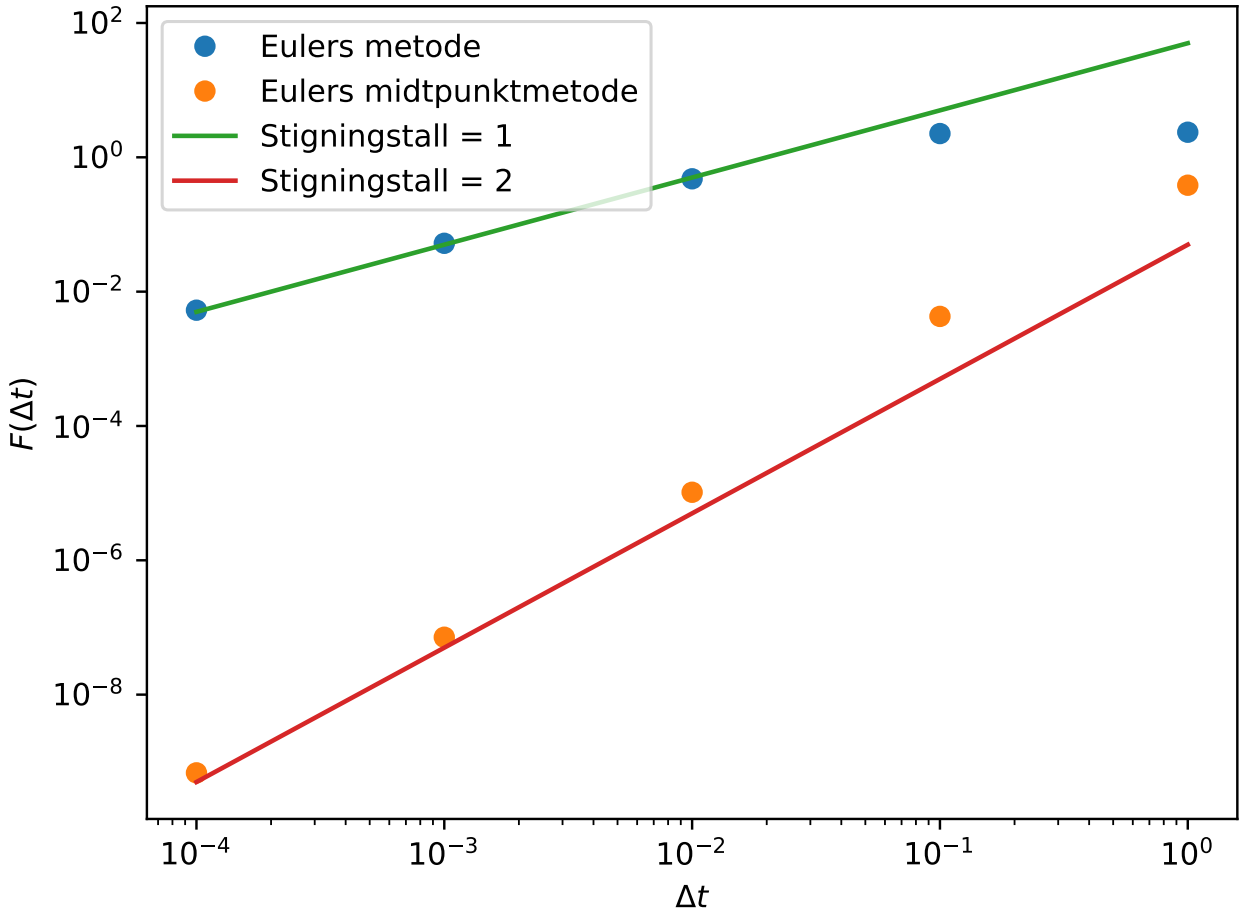
Eulers midtpunktmetode

$$\dot{x} \approx \frac{x(t + \Delta t/2) - x(t - \Delta t/2)}{\Delta t}.$$



$$x_{i+1/2} = x_i + f(x_i, t_i)\frac{\Delta t}{2}$$

$$x_{i+1} = x_i + f(x_{i+1/2}, t_i + \Delta t/2)\Delta t$$


Eulers metode
Eulers midtpunktmetode
Stigningstall = 1
Stigningstall = 2
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
$F(\Delta t)$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
Δt

Høyere ordens difflikninger kan omgjøres til sett av koblede første ordens likninger

Eksempel: Harmonisk oscillator

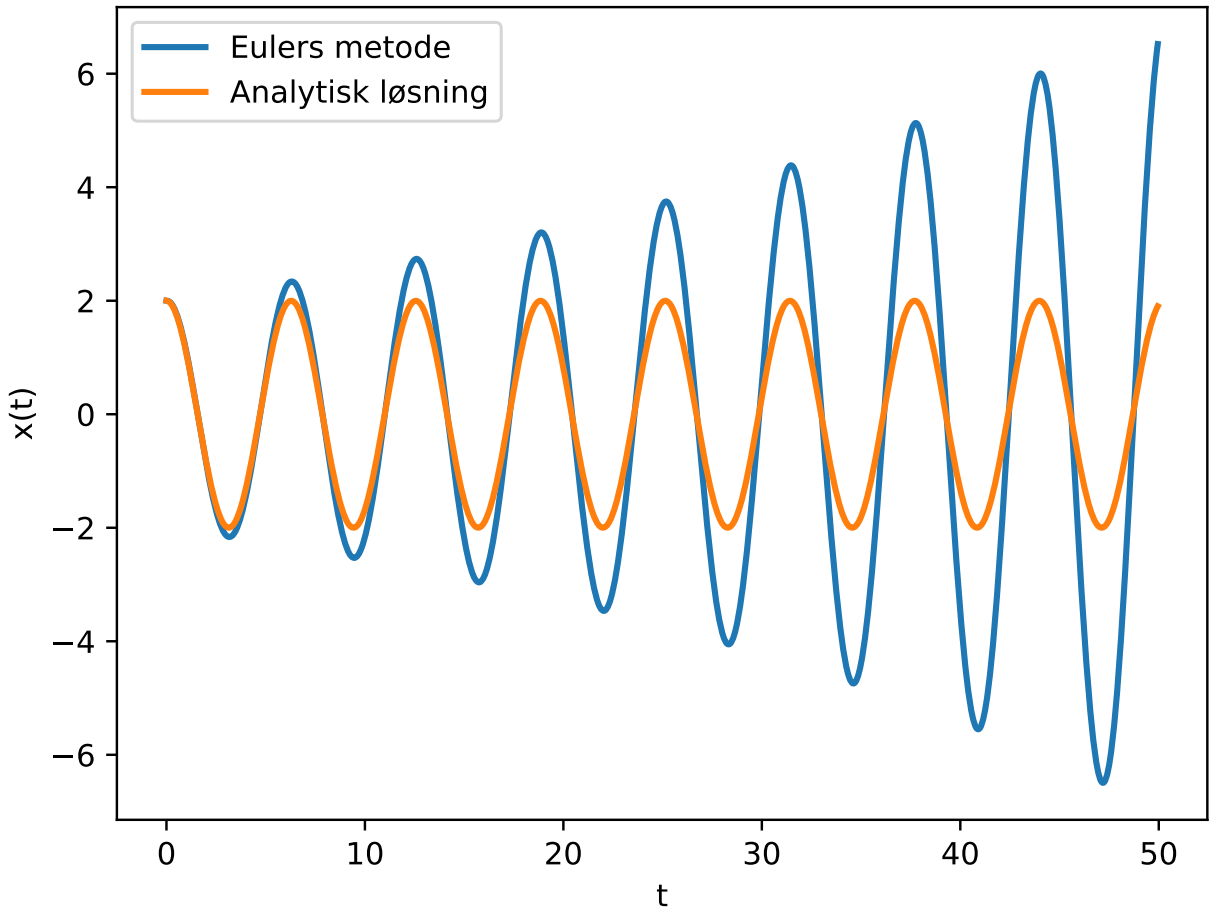
$$\ddot{x} = -\omega^2 x, \qquad x(0) = x_0, \qquad \dot{x}(0) = v_0.$$

$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x.$$

Eulers metode for disse gir uttrykkene

$$v_{i+1} = v_i - \omega^2 x_i \Delta t$$
$$x_{i+1} = x_i + v_i \Delta t.$$

Med steglengden $\Delta t = 0.05$



Euler-Cromer metoden

Vi bytter ut

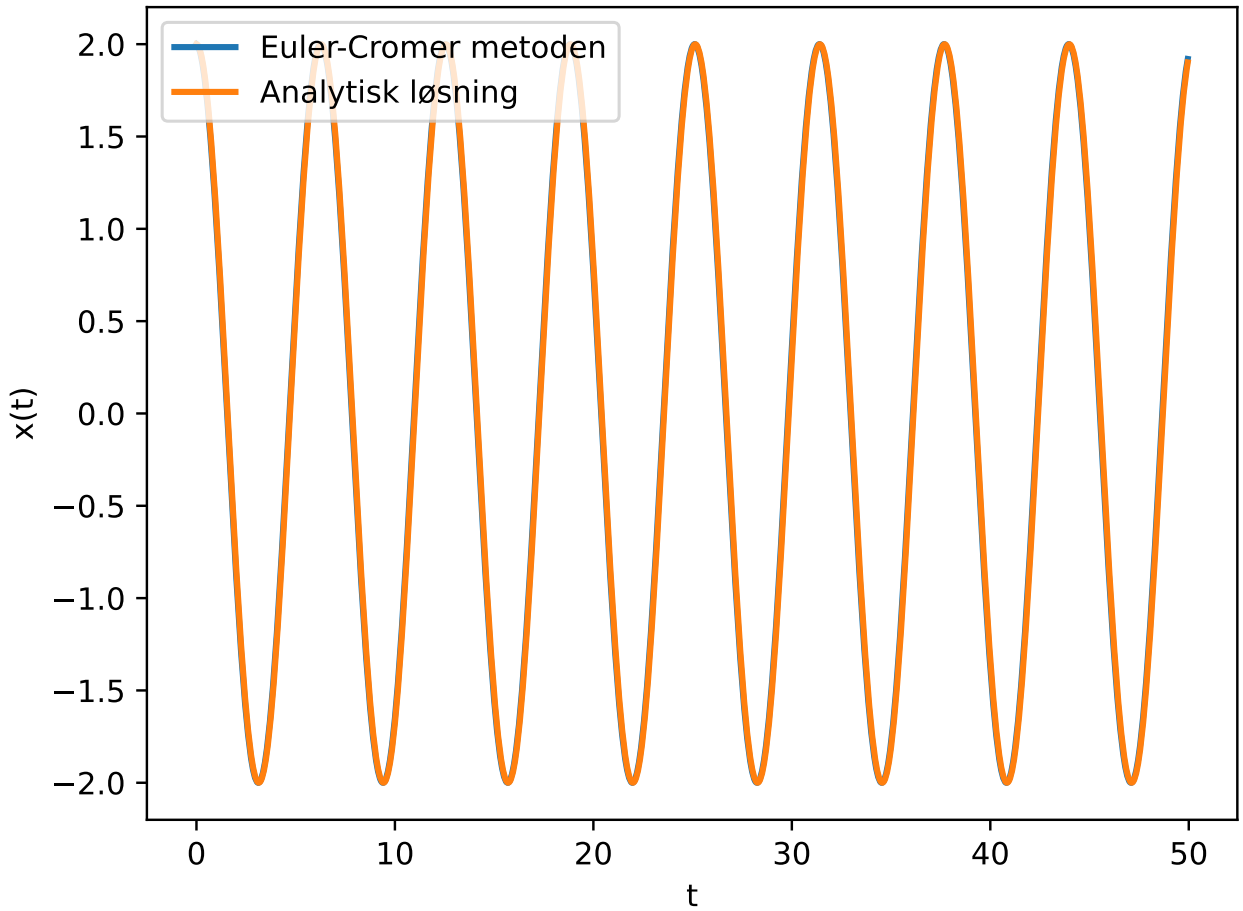
$$v_{i+1} = v_i - \omega^2 x_i \Delta t$$
$$x_{i+1} = x_i + v_i \Delta t.$$

med

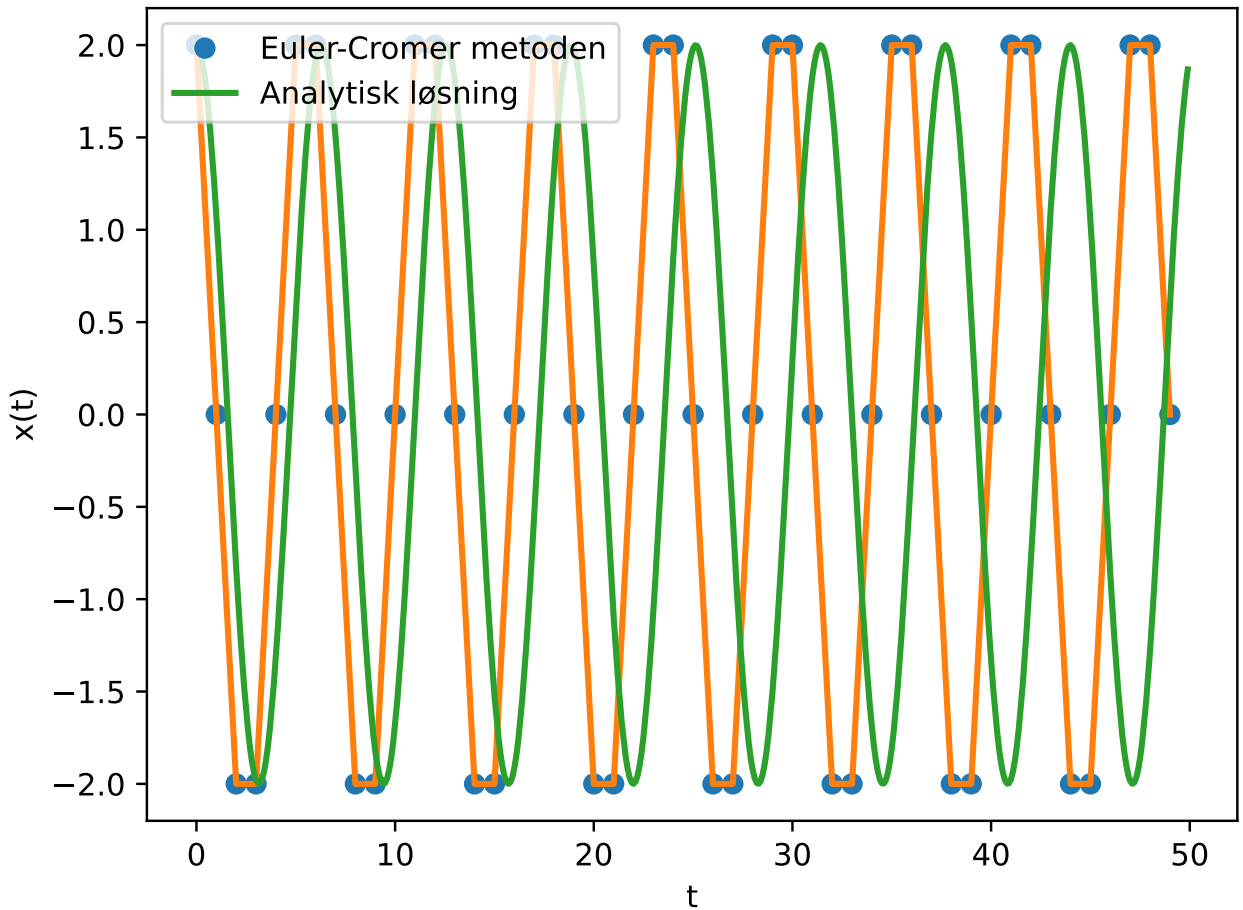
$$v_{i+1} = v_i - \omega^2 x_i \Delta t$$
$$x_{i+1} = x_i + v_{i+1} \Delta t.$$

Dette er som å regne ut v med Eulers metode og x med implisitt Eulers metode.

Med steglengden $\Delta t = 0.05$ som over får vi



Med steglengden $\Delta t = 1$ (som er altfor mye!)



Vi kan også undersøke hvor stor feil vi får som funksjon av Δt. Her har vi brukt Eulers metode, Eulers midtpunktmetode og Euler-Cromer metoden.

