Numerisk derivasjon

Den deriverte av en funksjon $f(x)$ i punktet a er gitt ved

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Numerisk vil vi velge en liten verdi for h og regner ut den deriverte tilnærmet

$$f'(a) \approx f'_h(a) = \frac{f(a+h) - f(a)}{h}.$$

Eksempel: $f(x) = \sin x, \qquad a = 0.5$

Analytisk: $f'(x) = \cos x \qquad \cos 0.5 = 0.877582561$

```
import numpy as np
import matplotlib.pyplot as plt

a = 0.5

print("h                  df_h                  df_h-cos(a)")

for h in 10.**(-np.arange(1,8)):
    d = (np.sin(a+h)-np.sin(a))/h
    print("%e     %e      %e"%(h,d,d-np.cos(a)))
```

```
h                  df_h                  df_h-cos(a)
1.000000e-01     8.521693e-01      -2.541321e-02
1.000000e-02     8.751708e-01      -2.411734e-03
1.000000e-03     8.773427e-01      -2.398590e-04
1.000000e-04     8.775586e-01      -2.397274e-05
1.000000e-05     8.775802e-01      -2.397147e-06
1.000000e-06     8.775823e-01      -2.396897e-07
1.000000e-07     8.775825e-01      -2.469499e-08
```

Trunkeringsfeil

Taylors formel med restledd

$$f(a+h) = f(a) + f'(a)h + \frac{h^2}{2}f''(u), \qquad u \in [a, a+h].$$

Dermed er feilen

$$f_h'(a) - f'(a) = \frac{h}{2}f''(u)$$

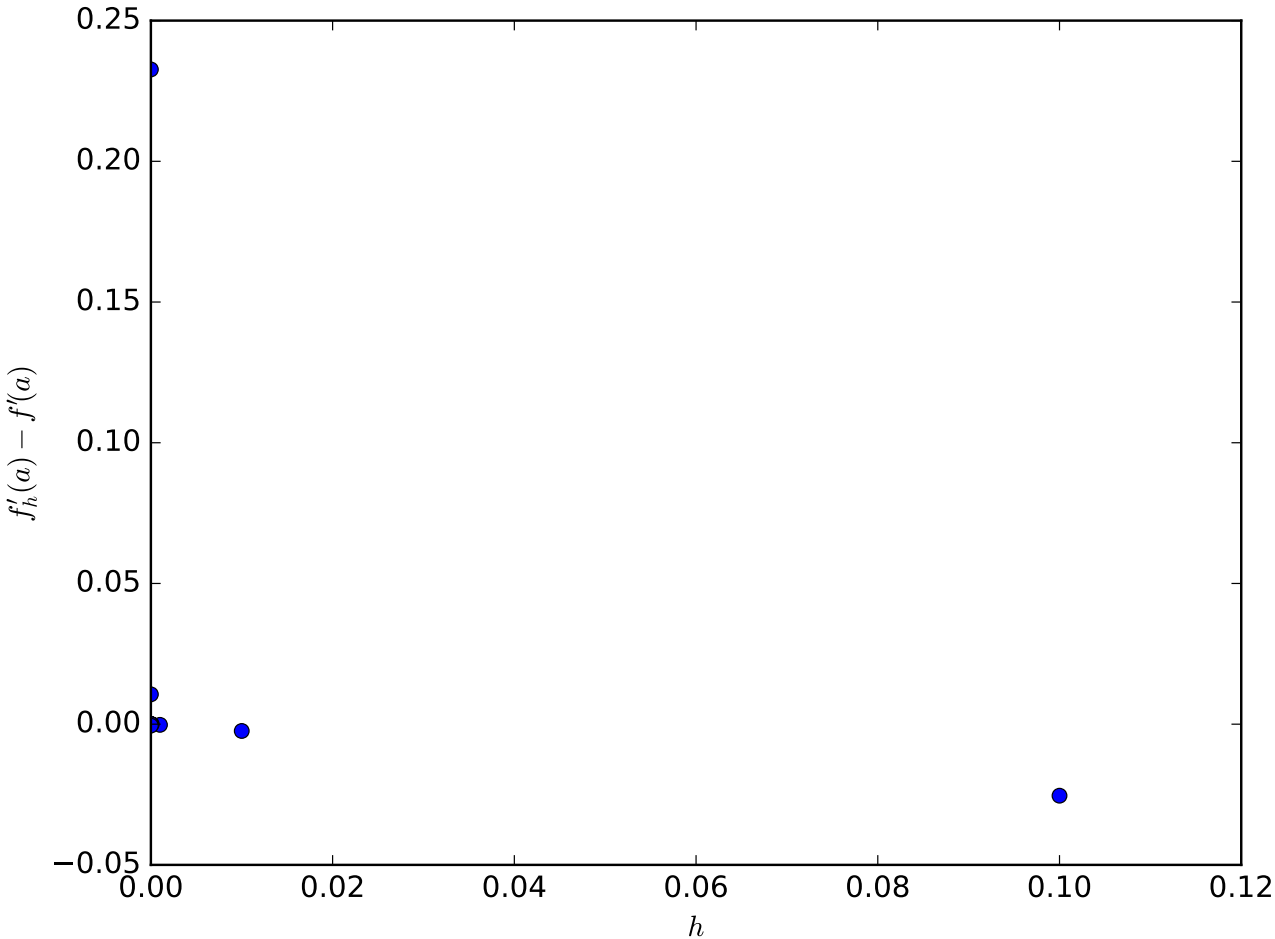
$$\frac{1}{2}f''(u) \approx \frac{1}{2}f''(a) = -\frac{1}{2}\sin(a) = -0,23971276.$$

h	df_h	df_h-cos(a)
1.000000e-01	8.521693e-01	-2.541321e-02
1.000000e-02	8.751708e-01	-2.411734e-03
1.000000e-03	8.773427e-01	-2.398590e-04
1.000000e-04	8.775586e-01	-2.397274e-05
1.000000e-05	8.775802e-01	-2.397147e-06
1.000000e-06	8.775823e-01	-2.396897e-07
1.000000e-07	8.775825e-01	-2.469499e-08

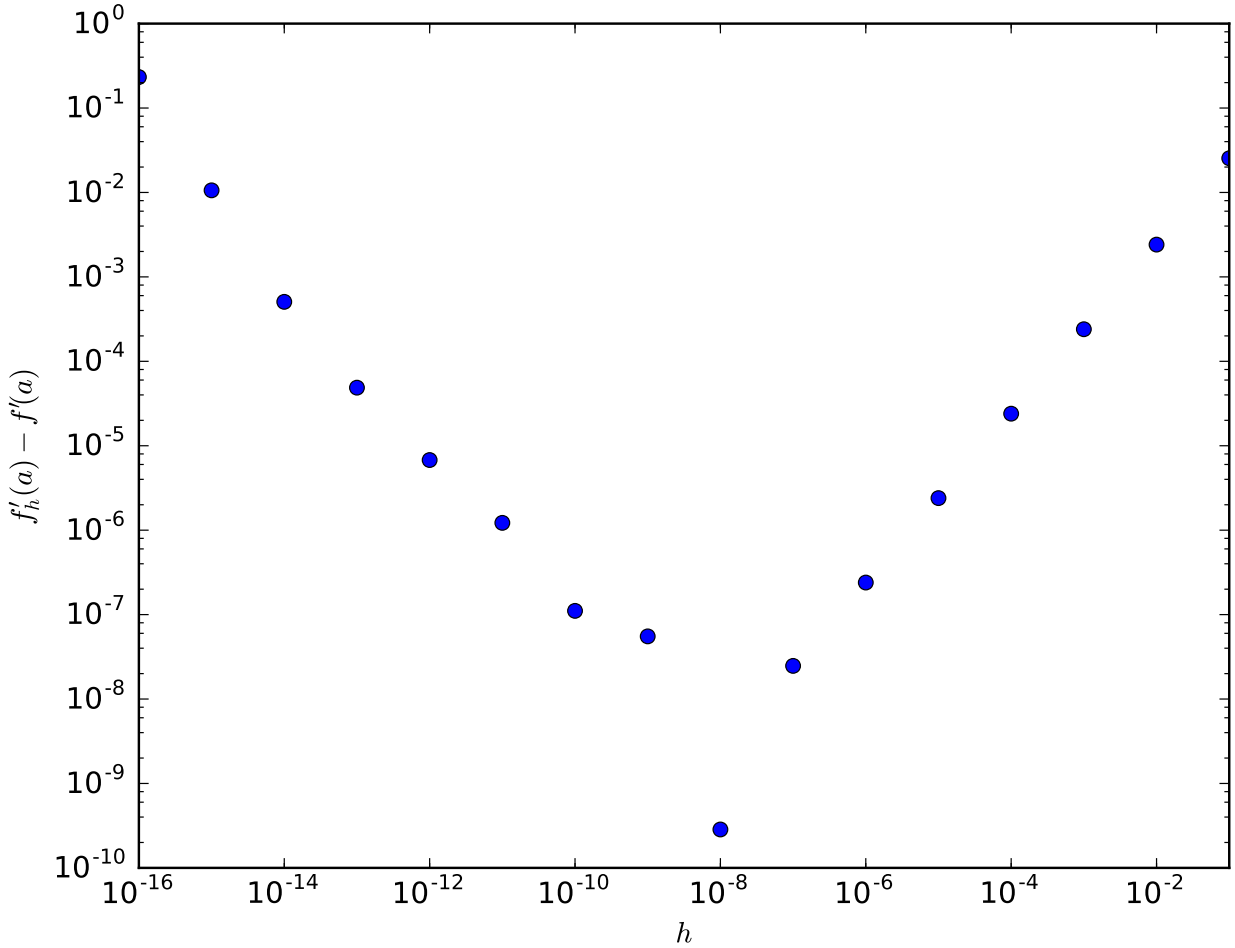
Avrundingsfeil

h	df_h	df_h-cos(a)
1.000000e-01	8.521693e-01	-2.541321e-02
1.000000e-02	8.751708e-01	-2.411734e-03
1.000000e-03	8.773427e-01	-2.398590e-04
1.000000e-04	8.775586e-01	-2.397274e-05
1.000000e-05	8.775802e-01	-2.397147e-06
1.000000e-06	8.775823e-01	-2.396897e-07
1.000000e-07	8.775825e-01	-2.469499e-08
1.000000e-08	8.775826e-01	2.850324e-10
1.000000e-09	8.775825e-01	-5.522612e-08
1.000000e-10	8.775825e-01	-1.107373e-07
1.000000e-11	8.775813e-01	-1.220960e-06
1.000000e-12	8.775758e-01	-6.772075e-06
1.000000e-13	8.776313e-01	4.873908e-05
1.000000e-14	8.770762e-01	-5.063724e-04
1.000000e-15	8.881784e-01	1.059586e-02
1.000000e-16	1.110223e+00	2.326405e-01

Vanlig plot



Logaritmisk plot



Avrundingsfeil

$f(x) = \sin x, \qquad a = 0.5, \qquad h = 10^{-13}$

$$\begin{aligned} f(a+h) &= 0.4794255386042908 \\ f(a) &= 0.4794255386042030 \\ f(a+h) - f(a) &= 0.0000000000000878 \end{aligned}$$

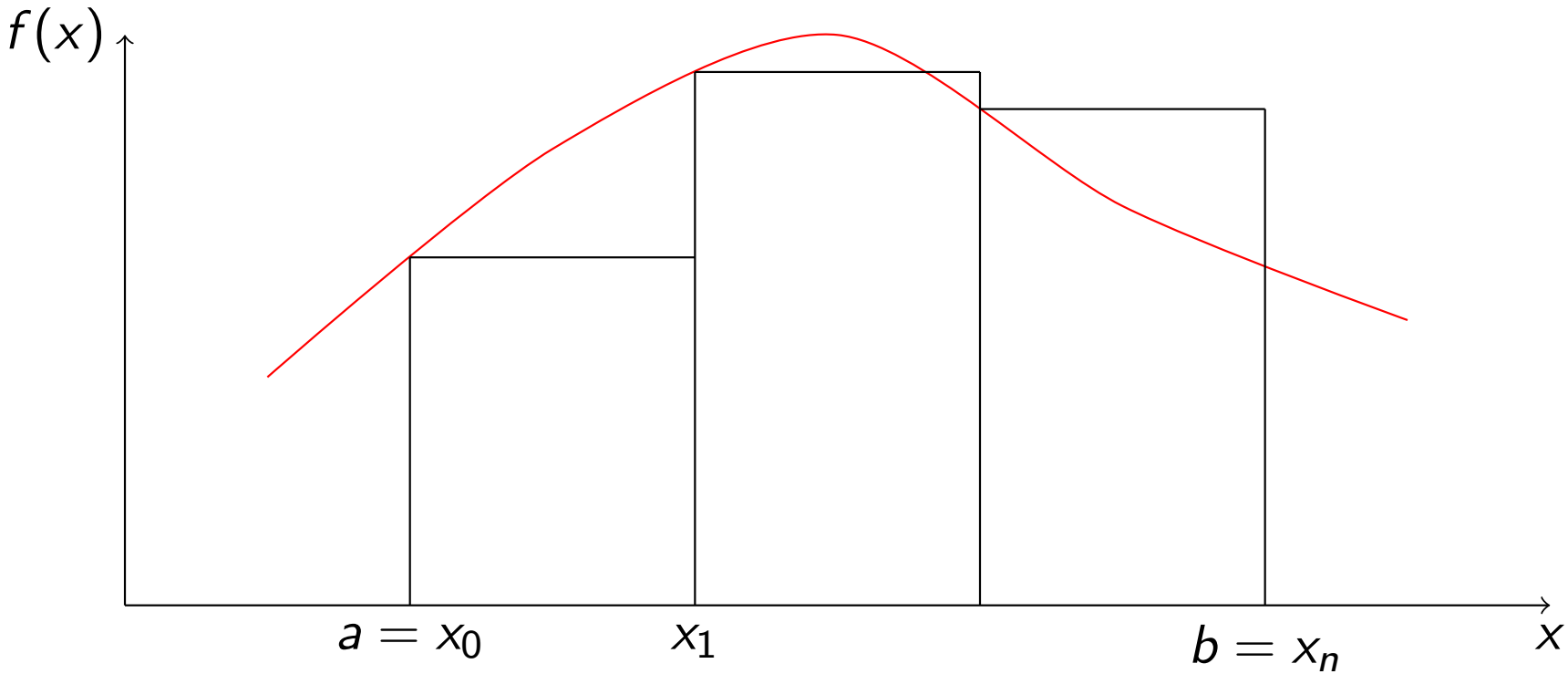
Dette representeres i datamaskina som

$$8.78???????????? \cdot 10^{-14}$$

$$f'_{10^{-13}}(a) = 0.8776313009661862$$

men bare de første 3-4 sifrene har noen reell betydning, og resten er "funnet på" av maskina vår.

Numerisk integrasjon ved rektangelmetoden (venstre trappesum)



$$I = \int_a^b f(x)dx \approx \sum_{i=0}^{n-1} f(x_i)\Delta x$$

Eksempel

$$I = \int_0^1 \sin x dx = \left[-\cos x \right]_0^1 = 1 - \cos 1 = 0,459$$

```
import numpy as np
import matplotlib.pyplot as plt

a,b = 0.,1.

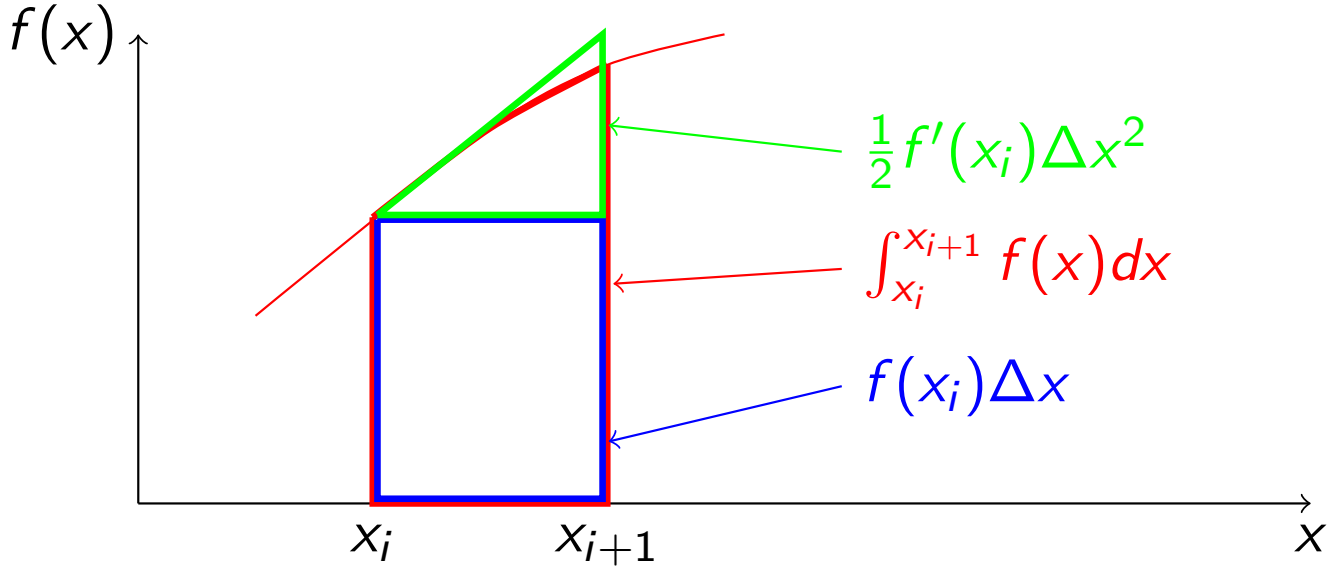
analytisk = 1-np.cos(1.0)


print("dx                  integral_dx                  feil")

for dx in 10.**(-np.arange(1,10)):
    x = np.arange(a,b,dx)
    integral = sum(np.sin(x))*dx
    print("%e     %.10e     %e"%(dx,integral,integral-analytisk))
```

dx	integral_dx	feil
1.000000e-01	4.1724099962e-01	-4.245669e-02
1.000000e-02	4.5548650839e-01	-4.211186e-03
1.000000e-03	4.5927692033e-01	-4.207738e-04
1.000000e-04	4.5965562020e-01	-4.207393e-05
1.000000e-05	4.5969348677e-01	-4.207359e-06
1.000000e-06	4.5969727340e-01	-4.207355e-07
1.000000e-07	4.5969765206e-01	-4.207361e-08
1.000000e-08	4.5969768992e-01	-4.207372e-09
1.000000e-09	4.5969769371e-01	-4.207485e-10

Trunkeringsfeil for ett rektangel



Taylorutvikler til 1. orden om x_i.

$$f(x_i + y) = f(x_i) + f'(x_i)y + \cdots$$

$$\int_{x_i}^{x_{i+1}} f(x)dx = \int_0^{\Delta x} f(x_i + y)dy$$

$$\approx f(x_i)\Delta x + f'(x_i)\int_0^{\Delta x} ydy = f(x_i)\Delta x + \frac{1}{2}f'(x_i)\Delta x^2.$$

Trunkeringsfeil for hele integralet

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx f(x_i)\Delta x + \frac{1}{2}f'(x_i)\Delta x^2.$$

$$I = \int_a^b f(x)dx \qquad I_{\Delta x} = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

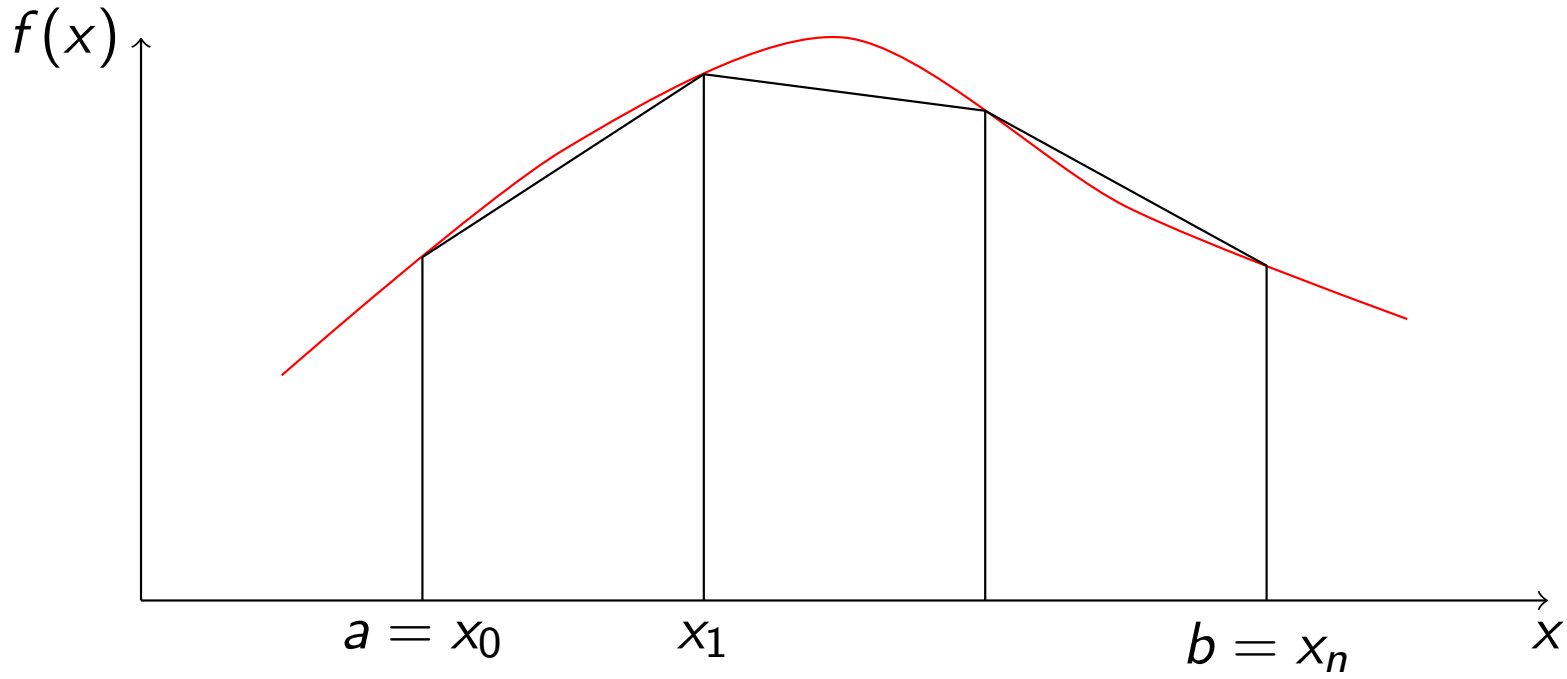
Feilen blir omtrent

$$E = I - I_{\Delta x} = \sum_i \frac{1}{2}f'(x_i)\Delta x^2 = \frac{1}{2}\Delta x \sum_i f'(x_i)\Delta x$$

$$\approx \int_a^b f'(x)dx = \frac{1}{2}\Delta x[f(b) - f(a)].$$

Feilen blir proporsjonal med Δx

Trapesmetoden



$$I = \int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \frac{1}{2}[f(x_i) + f(x_{i+1})]\Delta x$$

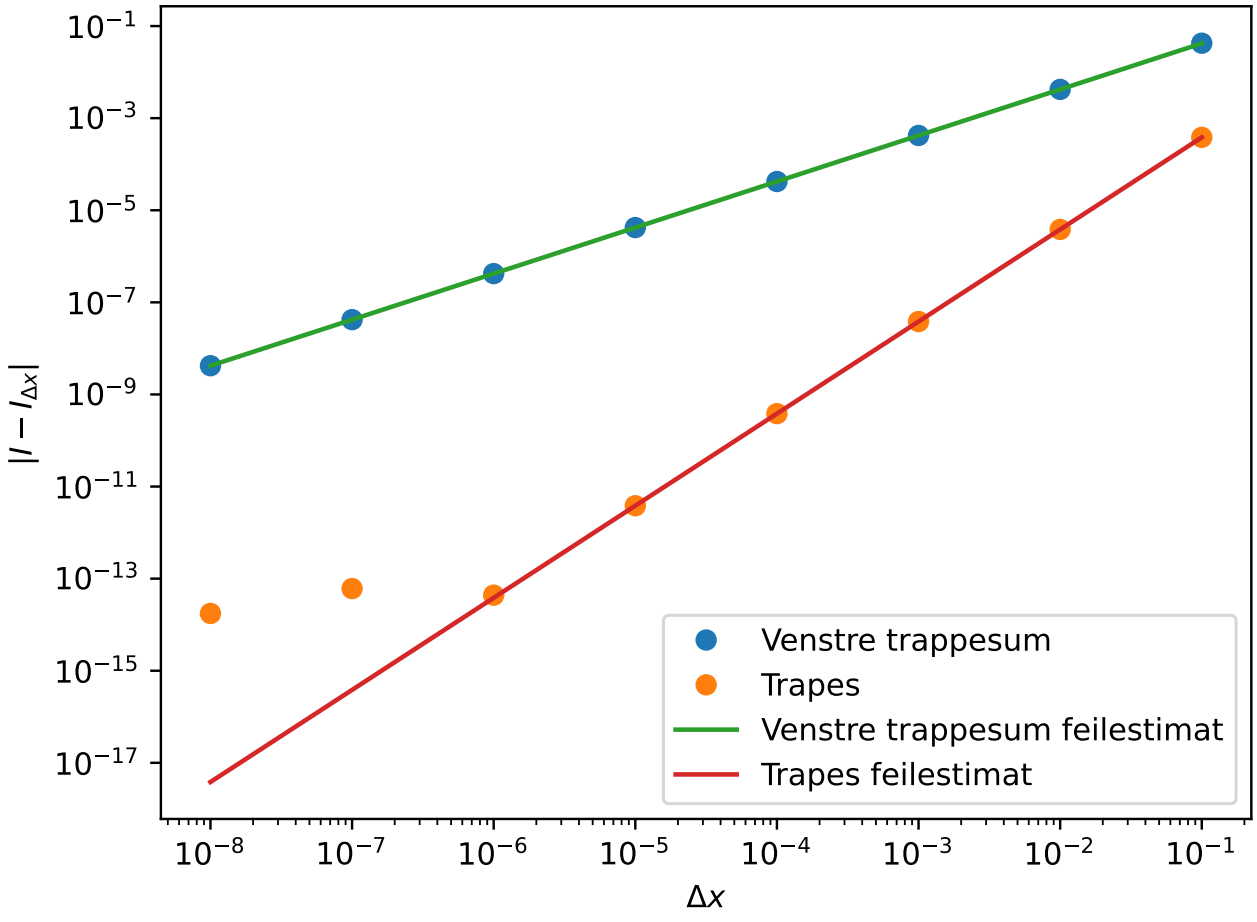
$$= \sum_{i=0}^{n-1} f(x_i)\Delta x + \frac{1}{2}\Delta x[f(b) - f(a)].$$

$$I = \int_a^b f(x)dx \qquad I_{\Delta x} = \sum_{i=0}^{n-1} \frac{1}{2}[f(x_i) + f(x_{i+1}]\Delta x$$

Feilestimat (se forelesningsnotater)

$$E = I - I_{\Delta x} = -\frac{1}{12}\Delta x^2[f'(b) - f'(a)]$$

Feilen blir proporsjonal med Δx^2


10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^{-13}
10^{-15}
10^{-17}
$|I - I_{\Delta x}|$
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
Δx
Venstre trappesum
Trapes
Venstre trappesum feilestimat
Trapes feilestimat