

FYS1105 – Formelsamling

Krefter

- Newtons 2. lov: $\mathbf{F} = m\ddot{\mathbf{r}} = \frac{d\mathbf{p}}{dt}$; $\mathbf{p} = m\dot{\mathbf{r}}$
 - * Kartesiske koordinater (3D): $F_x = m\ddot{x}$, $F_y = m\ddot{y}$, $F_z = m\ddot{z}$
 - * Polare koordinater (2D): $F_r = m(\ddot{r} - r\dot{\phi}^2)$, $F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$
- Newtons 2. lov i ikke-inertielle systemer:
 - * System med akselerasjon $\mathbf{A}$: $m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{\text{inertial}}$; $\mathbf{F}_{\text{inertial}} = -m\mathbf{A}$
 - * System som roterer med konstant vinkelhastighet $\boldsymbol{\Omega}$: $m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{\text{cf}}$; $\mathbf{F}_{\text{cor}} = 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}$, $\mathbf{F}_{\text{cf}} = m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}$
- Arbeid utført av en kraft $\mathbf{F}$ langs en kurve C mellom $\mathbf{r}_1$ og $\mathbf{r}_2$: $W(\mathbf{r}_1 \rightarrow \mathbf{r}_2) = \int_C \mathbf{F} \cdot d\mathbf{r}$
- Konservativ kraft $\mathbf{F}$:
 - * $\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$
 - * $U(\mathbf{r}) = -W(\mathbf{r}_0 \rightarrow \mathbf{r})$
- System av N partikler: $\mathbf{F}^{\text{ext}} = \dot{\mathbf{P}}$

Spinn og kraftmoment

- Én partikkel:
 - * $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$
 - * $\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F} = \dot{\boldsymbol{\ell}}$
- System av N partikler:
 - * $\mathbf{L} = \sum_{i=1}^N \boldsymbol{\ell}_i$
 - * Dersom de interne kreftene mellom partiklene er sentrale: $\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}$, der $\boldsymbol{\Gamma}^{\text{ext}}$ er det netto eksterne kraftmomentet på systemet.

Variasjonsregning

Et integral

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] dx,$$

er stasjonært for veien $y(x)$ som oppfyller Euler-Lagrange-likningen:

$$\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0.$$

Lagrange-mekanikk

- Lagrangefunksjonen (N frihetsgrader): $\mathcal{L}[q_1(t), q_2(t), \dots, q_N(t), \dot{q}_1(t), \dot{q}_2(t), \dots, \dot{q}_N(t), t] = T - U$
- Prinsippet om minste virkning: Den fysiske realiserende veien til et system fra punkt 1 til punkt 2 er den som minimerer *virksomheten* $S = \int_{t_1}^{t_2} \mathcal{L} dt$
- Lagrange-likningene: $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$, for $i = 1, 2, \dots, N$

Tolegemeproblemer

For to legemer med posisjon og masse hhv. $\mathbf{r}_1$ og m_1 , og $\mathbf{r}_2$ og m_2 :

- Redusert masse: $\mu = \frac{m_1 m_2}{m_1 + m_2}$
- Massesenter: $\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$
- Relativ posisjon: $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$
- Gravitasjonelt tolegemeproblem:
 - * $\mathbf{r} = \hat{\mathbf{x}} r \cos \phi + \hat{\mathbf{y}} r \sin \phi = r \hat{\mathbf{r}}$
 - * $\mathbf{F} = -\frac{G m_1 m_2}{r^2} \hat{\mathbf{r}} \equiv -\frac{\gamma}{r^2} \hat{\mathbf{r}}, \quad \gamma = G m_1 m_2$
 - * Kepler-bane: $r(\phi) = \frac{c}{1 + \epsilon \cos \phi}$;
 - $c = \frac{\ell^2}{\gamma \mu}$; $\ell = \mu r^2 \dot{\phi}$ er det *bevarte* spinnet i z -retningen
 - Energien til systemet $E = \frac{\gamma^2 \mu}{2 \ell^2} (\epsilon^2 - 1)$ er bevart

Rotasjon av stivt legeme

- $\mathbf{L} = \mathbf{L}(\text{bevegelse av CM}) + \mathbf{L}(\text{bev. relativt CM})$
- $T = T(\text{bevegelse av CM}) + T(\text{bev. relativt CM})$
- $\mathbf{L} = \mathbf{I} \boldsymbol{\omega}$, der $I_{xx} = \sum_{\alpha} m_{\alpha} (y_{\alpha}^2 + z_{\alpha}^2)$, etc., og $I_{xy} = -\sum_{\alpha} m_{\alpha} x_{\alpha} y_{\alpha}$, etc.
- $T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}, \quad T = \frac{1}{2} I_z \omega^2$
- $\dot{\mathbf{L}}(\text{bev. relativt CM}) = \mathbf{\Gamma}^{\text{ext}}(\text{om CM}).$
 $\dot{\mathbf{L}}(\text{bev. av CM}) = \mathbf{R} \times \mathbf{F}^{\text{ext}}$

Oscillatorer

- Enkel harmonisk oscillator: $\ddot{x} = -\omega^2 x$
- Bevegelseslikning n koblede oscillatorer: $\mathbf{M} \ddot{\mathbf{q}} = -\mathbf{K} \mathbf{q}$
 - * $\mathbf{q}$: Kolonnevektor for de n generaliserte koordinatene, $\mathbf{q}^T = (q_1, \dots, q_n)$, der $\mathbf{q} = 0$ ved et stabilt likevektspunkt
 - * $\mathbf{M}$: $n \times n$ generalisert "masse"-matrise; $T = \frac{1}{2} \sum_{j,k} M_{jk} \dot{q}_j \dot{q}_k$
 - * $\mathbf{K}$: $n \times n$ generalisert "fjærkonstant"-matrise; $U = \frac{1}{2} \sum_{j,k} K_{jk} q_j q_k$
- Normalmodus: $\mathbf{q}(t) = \text{Re}(\mathbf{a} e^{i\omega t})$, der $\mathbf{a}$ er en konstant kolonnevektor.

Hamilton-mekanikk

- Hamiltonfunksjonen (N frihetsgrader): $\mathcal{H}(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^N p_i \dot{q}_i - \mathcal{L}$, hvor $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$
- Hamilton-likningene: $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$, $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$, for $i = 1, 2, \dots, N$

Kontinuumsmekanikk

- Bølgeligningen 1D: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, med løsning $u(x, t) = f(x - ct) + g(x + ct)$, der f og g er vilkårlige funksjoner.
- Bølgeligningen 3D: $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$
- Def. spenningstensor $\mathbf{\Sigma}$: $\mathbf{F}(d\mathbf{A}) = \mathbf{\Sigma} d\mathbf{A}$. Spenningstensoren er symmetrisk.
- Def. deformasjonstensor $\mathbf{E}$ med elementer ϵ_{ij} : $\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} \right)$.
Dekomposisjon: $\mathbf{E} = e\mathbf{1} + \mathbf{E}'$, der $e = \frac{1}{3} \text{tr} \mathbf{E}$.
- Hookes lov, generalisert: $\mathbf{\Sigma} = \alpha e \mathbf{1} + \beta \mathbf{E}'$
- Hookes lov fjær: $\mathbf{F} = -k\mathbf{r} \Leftrightarrow U = \frac{1}{2} k \mathbf{r}^2$
- Elastisitetsmoduler: $dp = -\text{BM} \cdot \frac{dV}{V}$, $\frac{dF}{A} = \text{YM} \cdot \frac{dl}{l}$, $\frac{F}{A} = \text{SM} \cdot \frac{dy}{dx}$
- Newtons viskositetslov: skjærspenning $= \mu \cdot$ hastighetsgradient, $\mu =$ viskositet.
- Materialderivert: $\frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla \xi$
- Strømtetthet: $\mathbf{J} = \rho \mathbf{v}$
- Strøm (i masse/tidsenhet): $I = \int_S \mathbf{J} \cdot d\mathbf{S}$
- Kontinuitetsligningen: $\nabla \cdot (\rho \mathbf{v}) = -\frac{\partial \rho}{\partial t}$
- Bevegelsesligning, ikke-viskøs væske: $\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \nabla p$
- Bernoullis teorem (ikke-viskøs, inkompressibel væske): $\frac{d}{dt} \left(\frac{1}{2} \rho v^2 + \rho g z + p \right) = 0$

Vektorformler

$$\nabla^2 \mathbf{A} = (\nabla^2 A_x) \hat{\mathbf{x}} + (\nabla^2 A_y) \hat{\mathbf{y}} + (\nabla^2 A_z) \hat{\mathbf{z}}$$

Differensielle vektoridentiteter:

$$\begin{aligned} \hat{\mathbf{x}} \cdot \nabla U &= \frac{\partial U}{\partial x} \quad (x \text{ vilkårlig akse}) \\ \nabla(U + W) &= \nabla U + \nabla W \\ \nabla(UW) &= U \nabla W + W \nabla U \\ \nabla f(U) &= f'(U) \nabla U \\ \nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} \\ &\quad + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \\ \nabla \cdot (\mathbf{A} + \mathbf{B}) &= \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \\ \nabla \cdot (U \mathbf{A}) &= U \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla U \\ \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \\ \nabla \times (\mathbf{A} + \mathbf{B}) &= \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \\ \nabla \times (U \mathbf{A}) &= (\nabla U) \times \mathbf{A} + U \nabla \times \mathbf{A} \\ \nabla \cdot (\nabla \times \mathbf{A}) &= 0 \\ \nabla \cdot (\nabla U) &= \nabla^2 U \\ \nabla \times (\nabla U) &= 0 \\ \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \end{aligned}$$

Integralidentiteter:

$$\begin{aligned} \int_V \nabla U dV &= \oint_S U d\mathbf{S} \\ \int_V \nabla \cdot \mathbf{A} dV &= \oint_S \mathbf{A} \cdot d\mathbf{S} \quad (\text{Divergensteoremet}) \\ \int_V \nabla \times \mathbf{A} dV &= \oint_S d\mathbf{S} \times \mathbf{A} \\ \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} &= \oint_C \mathbf{A} \cdot d\mathbf{l} \quad (\text{Stokes' teorem}) \end{aligned}$$

Kartesisk koordinatsystem:

$$\begin{aligned} \nabla U &= \frac{\partial U}{\partial x} \hat{\mathbf{x}} + \frac{\partial U}{\partial y} \hat{\mathbf{y}} + \frac{\partial U}{\partial z} \hat{\mathbf{z}} \\ \nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ \nabla \times \mathbf{A} &= \hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \\ &\quad + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ \nabla^2 U &= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \end{aligned}$$

$$\int_{\mathbb{R}^3} f(\mathbf{r}) dV = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) dx dy dz$$

$$\int_S f(\mathbf{r}) dS = \begin{cases} \iint f(x, y, z) dy dz & \text{hvis } x \text{ er konstant p\aa } S \\ \iint f(x, y, z) dx dz & \text{hvis } y \text{ er konstant p\aa } S \\ \iint f(x, y, z) dx dy & \text{hvis } z \text{ er konstant p\aa } S \end{cases}$$

$$d\mathbf{r} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$$

Sylindrisk koordinatsystem:

$$\begin{aligned} \nabla U &= \frac{\partial U}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial U}{\partial \phi} \hat{\phi} + \frac{\partial U}{\partial z} \hat{\mathbf{z}} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r} \frac{\partial(r A_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\ \nabla \times \mathbf{A} &= \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \\ &\quad + \hat{\phi} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{\mathbf{z}} \left(\frac{\partial(r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right) \end{aligned}$$

$$\nabla^2 U = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$\int_{\mathbb{R}^3} f(\mathbf{r}) dV = \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} f(r, \phi, z) r dr d\phi dz$$

$$\begin{aligned} \int_S f(\mathbf{r}) dS &= \begin{cases} \iint f(r, \phi, z) r d\phi dz & \text{hvis } r \text{ er konstant p\aa } S \\ \iint f(r, \phi, z) dr dz & \text{hvis } \phi \text{ er konstant p\aa } S \\ \iint f(r, \phi, z) r dr d\phi & \text{hvis } z \text{ er konstant p\aa } S \end{cases} \end{aligned}$$

$$d\mathbf{r} = dr \hat{\mathbf{r}} + r d\phi \hat{\phi} + dz \hat{\mathbf{z}}$$

Sfærisk koordinatsystem:

$$\begin{aligned} \nabla U &= \frac{\partial U}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \hat{\phi} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} \\ &\quad + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta A_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \end{aligned}$$

$$\begin{aligned}\nabla \times \mathbf{A} &= \frac{\hat{\mathbf{r}}}{r \sin \theta} \left(\frac{\partial(\sin \theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right) \\ &+ \frac{\hat{\boldsymbol{\theta}}}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial(r A_\phi)}{\partial r} \right) \\ &+ \frac{\hat{\boldsymbol{\phi}}}{r} \left(\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right)\end{aligned}$$

$$\begin{aligned}\nabla^2 U &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) \\ &+ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) \\ &+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2}\end{aligned}$$

$$\int_{\mathbb{R}^3} f(\mathbf{r}) dV = \int_0^{2\pi} \int_0^\pi \int_0^\infty f(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

$$\begin{aligned}\int_S f(\mathbf{r}) dS \\ = \begin{cases} \iint f(r, \theta, \phi) r^2 \sin \theta d\theta d\phi & \text{hvis } r \text{ er konstant p\aa } S \\ \iint f(r, \theta, \phi) r \sin \theta dr d\phi & \text{hvis } \theta \text{ er konstant p\aa } S \\ \iint f(r, \theta, \phi) r dr d\theta & \text{hvis } \phi \text{ er konstant p\aa } S \end{cases}\end{aligned}$$

$$d\mathbf{r} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}$$