FYS1105 – Formelsamling

Krefter

- Newtons 2. lov: $\mathbf{F} = m\ddot{\mathbf{r}} = \frac{d\mathbf{p}}{dt}$; $\mathbf{p} = m\dot{\mathbf{r}}$
  - Kartesiske koordinater (3D): $F_x = m\ddot{x}$, $F_y = m\ddot{y}$, $F_z = m\ddot{z}$
  - Polare koordinater (2D): $F_r = m\left(\ddot{r} - r\dot{\phi}^2\right)$, $F_\phi = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)$
- Newtons 2. lov i ikke-inertielle systemer:
  - System med akselerasjon $\mathbf{A}$: $m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{\text{inertial}}$; $\mathbf{F}_{\text{inertial}} = -m\mathbf{A}$
  - System som roterer med konstant vinkelhastighet $\boldsymbol{\Omega}$: $m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{\text{cf}}$; $\mathbf{F}_{\text{cor}} = 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}$, $\mathbf{F}_{\text{cf}} = m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}$
- Arbeid utført av en kraft $\mathbf{F}$ langs en kurve C mellom $\mathbf{r}_1$ og $\mathbf{r}_2$: $W(\mathbf{r}_1 \to \mathbf{r}_2) = \int_C \mathbf{F} \cdot d\mathbf{r}$
- Konservativ kraft $\mathbf{F}$:
  - $\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla} U(\mathbf{r})$
  - $U(\mathbf{r}) = -W(\mathbf{r}_0 \to \mathbf{r})$
- System av N partikler: $\mathbf{F}^{\text{ext}} = \dot{\mathbf{P}}$

Spinn og kraftmoment

- Én partikkel:
  - $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$
  - $\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F} = \dot{\boldsymbol{\ell}}$
- System av N partikler:
  - $\mathbf{L} = \sum_{i=1}^{N} \boldsymbol{\ell}_i$
  - Dersom de interne kreftene mellom partiklene er sentrale: $\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}$, der $\boldsymbol{\Gamma}^{\text{ext}}$ er det netto eksterne kraftmomentet på systemet.

Variasjonsregning

Et integral

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx,$$

er stasjonært for veien $y(x)$ som oppfyller Euler-Lagrange-likningen:

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0.$$

Lagrange-mekanikk

- Lagrangefunksjonen (N frihetsgrader): $\mathcal{L}[q_1(t), q_2(t), \ldots, q_N(t),\ \dot{q}_1(t), \dot{q}_2(t), \ldots, \dot{q}_N(t), t] = T - U$
- Prinsippet om minste virkning: Den fysisk realiserte veien til et system fra punkt 1 til punkt 2 er den som minimerer *virkningen* $S = \int_{t_1}^{t_2} \mathcal{L} dt$
- Lagrange-likningene: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$, for $i = 1, 2, \ldots, N$

Tolegemeproblemer

For to legemer med posisjon og masse hhv. $\mathbf{r}_1$ og m_1, og $\mathbf{r}_2$ og m_2:

- Redusert masse: $\mu = \frac{m_1 m_2}{m_1 + m_2}$
- Massesenter: $\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$
- Relativ posisjon: $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$
- Gravitasjonelt tolegemeproblem:
  - $\mathbf{r} = \hat{\mathbf{x}} r \cos\phi + \hat{\mathbf{y}} r \sin\phi = r\hat{\mathbf{r}}$
  - $\mathbf{F} = -\frac{G m_1 m_2}{r^2}\hat{\mathbf{r}} \equiv -\frac{\gamma}{r^2}\hat{\mathbf{r}}, \quad \gamma = G m_1 m_2$
  - Kepler-bane: $r(\phi) = \frac{c}{1 + \epsilon\cos\phi}$;
    - $c = \frac{\ell^2}{\gamma\mu}$; $\ell = \mu r^2 \dot{\phi}$ er det *bevarte* spinnet i z-retningen
    - Energien til systemet $E = \frac{\gamma^2 \mu}{2\ell^2}\left(\epsilon^2 - 1\right)$ er bevart

Rotasjon av stivt legeme

- $\mathbf{L} = \mathbf{L}$(bevegelse av CM) $+ \mathbf{L}$(bev. relativt CM)
- $T = T$(bevegelse av CM) $+ T$(bev. relativt CM)
- $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, der $I_{xx} = \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2\right)$, etc., og $I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha$, etc.
- $T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}, \quad T = \frac{1}{2} I_z \omega^2$
- $\dot{\mathbf{L}}$(bev. relativt CM) $= \boldsymbol{\Gamma}^{\text{ext}}$(om CM).
  $\dot{\mathbf{L}}$(bev. av CM) $= \mathbf{R} \times \mathbf{F}^{\text{ext}}$
- Treghetsmoment kule om sentrum, radius a, masse M, uniform massetetthet: $I = \frac{2}{5} M a^2$.
  Treghetsmoment kuleskall om sentrum, masse M, uniform massetetthet over skallet: $I = \frac{2}{3} M a^2$.

Oscillatorer

- Enkel harmonisk oscillator: $\ddot{x} = -\omega^2 x$
- Bevegelseslikning n koblede oscillatorer: $\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}$
  - $\mathbf{q}$: Kolonnevektor for de n generaliserte koordinatene, $\mathbf{q}^T = (q_1, \ldots, q_n)$, der $\mathbf{q} = 0$ ved et stabilt likevektspunkt
  - $\mathbf{M}$: $n \times n$ generalisert "masse"-matrise; $T = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j \dot{q}_k$
  - $\mathbf{K}$: $n \times n$ generalisert "fjærkonstant"-matrise; $U = \frac{1}{2}\sum_{j,k} K_{jk} q_j q_k$
- Normalmodus: $\mathbf{q}(t) = \mathrm{Re}\left(\mathbf{a} e^{i\omega t}\right)$, der $\mathbf{a}$ er en konstant kolonnevektor.

Hamilton-mekanikk

- Hamiltonfunksjonen (N frihetsgrader): $\mathcal{H}(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{N} p_i \dot{q}_i - \mathcal{L}$, hvor $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$
- Hamilton-likningene: $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$, $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$, for $i = 1, 2, \ldots, N$

Kontinuumsmekanikk

- Bølgeligningen 1D: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, med løsning $u(x,t) = f(x - ct) + g(x + ct)$, der f og g er vilkårlige funksjoner.
- Bølgeligningen 3D: $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$
- Def. spenningstensor $\mathbf{\Sigma}$: $\mathbf{F}(d\mathbf{A}) = \mathbf{\Sigma} d\mathbf{A}$. Spenningstensoren er symmetrisk.
- Def. deformasjonstensor $\mathbf{E}$ med elementer ϵ_{ij}: $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right)$.
  Dekomposisjon: $\mathbf{E} = e\mathbf{1} + \mathbf{E}'$, der $e = \frac{1}{3}\operatorname{tr}\mathbf{E}$.
- Hookes lov, generalisert: $\mathbf{\Sigma} = \alpha e \mathbf{1} + \beta \mathbf{E}'$
- Hookes lov fjær: $\mathbf{F} = -k\mathbf{r} \Leftrightarrow U = \frac{1}{2}k\mathbf{r}^2$
- Elastisitetsmoduler: $dp = -\mathrm{BM} \cdot \frac{dV}{V}$, $\quad \frac{dF}{A} = \mathrm{YM} \cdot \frac{dl}{l}$, $\quad \frac{F}{A} = \mathrm{SM} \cdot \frac{dy}{dx}$
- Newtons viskositetslov: skjærspenning $= \mu \cdot$ hastighetsgradient, $\mu =$ viskositet.
- Materialderivert: $\frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla \xi$
- Strømtetthet: $\mathbf{J} = \varrho \mathbf{v}$
- Strøm (i masse/tidsenhet): $I = \int_S \mathbf{J} \cdot d\mathbf{S}$
- Kontinuitetsligningen: $\nabla \cdot (\varrho \mathbf{v}) = -\frac{\partial \varrho}{\partial t}$
- Bevegelsesligning, ikke-viskøs væske: $\varrho \frac{d\mathbf{v}}{dt} = \varrho \mathbf{g} - \nabla p$
- Bernoullis teorem (ikke-viskøs, inkompressibel væske): $\frac{d}{dt}\left(\frac{1}{2}\varrho v^2 + \varrho g z + p\right) = 0$

Vektorformler

Differensielle vektoridentiteter:

$$
\begin{aligned}
\hat{\mathbf{x}} \cdot \nabla U &= \frac{\partial U}{\partial x} \ (x \text{ vilkårlig akse}) \\
\nabla(U+W) &= \nabla U + \nabla W \\
\nabla(UW) &= U\nabla W + W\nabla U \\
\nabla f(U) &= f'(U)\nabla U \\
\nabla(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \\
&\quad + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \\
\nabla\cdot(\mathbf{A}+\mathbf{B}) &= \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B} \\
\nabla\cdot(U\mathbf{A}) &= U\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla U \\
\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B} \\
\nabla\times(\mathbf{A}+\mathbf{B}) &= \nabla\times\mathbf{A} + \nabla\times\mathbf{B} \\
\nabla\times(U\mathbf{A}) &= (\nabla U)\times\mathbf{A} + U\nabla\times\mathbf{A} \\
\nabla\cdot(\nabla\times\mathbf{A}) &= 0 \\
\nabla\cdot(\nabla U) &= \nabla^2 U \\
\nabla\times(\nabla U) &= 0 \\
\nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}
$$

Integralidentiteter:

$$\int_V \nabla U dV = \oint_S U d\mathbf{S}$$

$$\int_V \nabla\cdot\mathbf{A} dV = \oint_S \mathbf{A}\cdot d\mathbf{S} \quad \text{(Divergensteoremet)}$$

$$\int_V \nabla\times\mathbf{A} dV = \oint_S d\mathbf{S}\times\mathbf{A}$$

$$\int_S \nabla\times\mathbf{A}\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{l} \quad \text{(Stokes' teorem)}$$

Kartesisk koordinatsystem:

$$\nabla U = \frac{\partial U}{\partial x}\hat{\mathbf{x}} + \frac{\partial U}{\partial y}\hat{\mathbf{y}} + \frac{\partial U}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$
\begin{aligned}
\nabla\times\mathbf{A} &= \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \\
&\quad + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)
\end{aligned}
$$

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

$$\nabla^2\mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$$

$$\int_{\mathbb{R}^3} f(\mathbf{r})dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)dxdydz$$

$$\int_S f(\mathbf{r})dS = \begin{cases} \iint f(x,y,z)dydz & \text{hvis } x \text{ er konstant på } S \\ \iint f(x,y,z)dxdz & \text{hvis } y \text{ er konstant på } S \\ \iint f(x,y,z)dxdy & \text{hvis } z \text{ er konstant på } S \end{cases}$$

$$d\mathbf{r} = dx\ \hat{\mathbf{x}} + dy\ \hat{\mathbf{y}} + dz\ \hat{\mathbf{z}}$$

Sylindrisk koordinatsystem:

$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial U}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$
\begin{aligned}
\nabla\times\mathbf{A} &= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) \\
&\quad + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \frac{\hat{\mathbf{z}}}{r}\left(\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)
\end{aligned}
$$

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$\int_{\mathbb{R}^3} f(\mathbf{r})dV = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} f(r,\phi,z)rdrd\phi dz$$

$$
\begin{aligned}
&\int_S f(\mathbf{r})dS \\
&= \begin{cases} \iint f(r,\phi,z)rd\phi dz & \text{hvis } r \text{ er konstant på } S \\ \iint f(r,\phi,z)drdz & \text{hvis } \phi \text{ er konstant på } S \\ \iint f(r,\phi,z)rdrd\phi & \text{hvis } z \text{ er konstant på } S \end{cases}
\end{aligned}
$$

$$d\mathbf{r} = dr\ \hat{\mathbf{r}} + rd\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}}$$

Sfærisk koordinatsystem:

$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$
\begin{aligned}
\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \mathbf{A} = & \frac{\hat{\mathbf{r}}}{r \sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right) \\
& + \frac{\hat{\boldsymbol{\theta}}}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right) \\
& + \frac{\hat{\boldsymbol{\phi}}}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla^2 U = & \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) \\
& + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) \\
& + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}
\end{aligned}
$$

$$
\int_{\mathbb{R}^3} f(\mathbf{r})dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f(r,\theta,\phi)r^2\sin\theta dr d\theta d\phi
$$

$$
\begin{aligned}
& \int_S f(\mathbf{r})dS \\
& = \begin{cases} \iint f(r,\theta,\phi)r^2\sin\theta d\theta d\phi & \text{hvis } r \text{ er konstant på } S \\ \iint f(r,\theta,\phi)r\sin\theta dr d\phi & \text{hvis } \theta \text{ er konstant på } S \\ \iint f(r,\theta,\phi)r dr d\theta & \text{hvis } \phi \text{ er konstant på } S \end{cases}
\end{aligned}
$$

$$
d\mathbf{r} = dr\ \hat{\mathbf{r}} + rd\theta\ \hat{\boldsymbol{\theta}} + r\sin\theta d\phi\ \hat{\boldsymbol{\phi}}
$$