

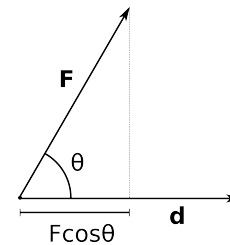
Note 3 - Introduction to Line integrals, Curl and Stoke's Theorem

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1 The line integral of a vector field

The work done by a force \mathbf{F} when a body is following a trajectory \mathcal{C} is equal to the body's change in kinetic energy. This is the statement of the work-energy theorem and it's the main reason why we're interested in the work done by a force. If we have a constant force which is acting on a body moving on a straight line segment \mathbf{d} , then the work done is defined to be



$$W = \mathbf{F} \cdot \mathbf{d} = |\mathbf{F}| |\mathbf{d}| \cos \theta$$

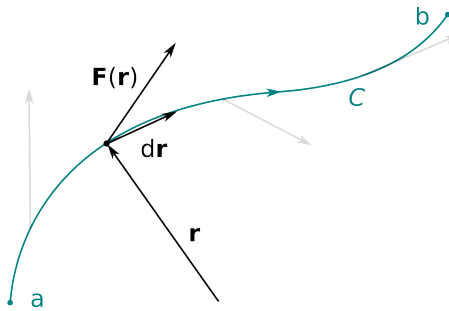
where the dot-product conveniently picks out the component of the force in the direction along \mathbf{d} . If we have a varying force over a more complicated trajectory we divide the trajectory into ever smaller pieces and compute the work done as an integral

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $d\mathbf{r}$ here represents an infinitesimal displacement along the curve \mathcal{C} . This kind of integral is called a line integral over a vector field \mathbf{F} and the calculation of the work done by a force on a body who's trajectory is \mathcal{C} is only one of it's applications. More generally, since the dot product always picks out the component of \mathbf{F} along $d\mathbf{r}$, it can be thought of as a measure of how much \mathbf{F} is going in the direction of \mathcal{C} .

2 Computation

In electromagnetic theory we'll mainly be interested in computing integrals over *closed* curves and later we'll see that such integrals can be evaluated via an important theorem called *Stokes' Theorem*. However this isn't always the easiest method so we'll go through some of the theory of direct computation here. In some easy cases we can use geometric considerations to compute such integrals without doing much calculation, but in the general case we need to parametrize our curve \mathcal{C} by expressing each of our coordinates as a function of some parameter t . Supposed that we have expressed each of our coordinates as a function



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Figure 1: The work from a varying force is found by chopping the trajectory \mathcal{C} in to tiny segments and summing up the contributions from each segment.

of t such that $\mathbf{r} = \mathbf{r}(t)$ for $a \leq t \leq b$, then an infinitesimal segment $d\mathbf{r}$ can be expressed as

$$d\mathbf{r} = \mathbf{r}'(t)dt$$

and the field \mathbf{F} along the path \mathcal{C} can be expressed as $\mathbf{F} = \mathbf{F}(\mathbf{r}(t))$ then this integral will turn out as an integral just over t such that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt.$$

It's generally up to you to chose what parameter to use whether it's an angle, time or maybe one of the coordinates. The line integral will be independent of the parametrization of the curve \mathcal{C} .

2.1 Example

Find the line integral of $\mathbf{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ around a unit circle \mathcal{C} centered at the origin in the xy -plane oriented counter clockwise. A sketch of the field is shown in Figure 2. Geometrically we see that the vectors are always tangent to the curve \mathcal{C} so that $\mathbf{F} \cdot d\mathbf{r} = |\mathbf{F}||d\mathbf{r}| = Fdr$. Now $|\mathbf{F}| = \sqrt{x^2 + y^2} = 1$ such that

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = F \oint_{\mathcal{C}} dr = 2\pi,$$

since $\oint_{\mathcal{C}} dr$ is just the circumference of the circle.

Let's also do it by direct computation. We can parametrize the circle by $\mathbf{r}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$ such that

$$d\mathbf{r} = (-\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}}) dt$$

and

$$\mathbf{F} = -\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}}.$$

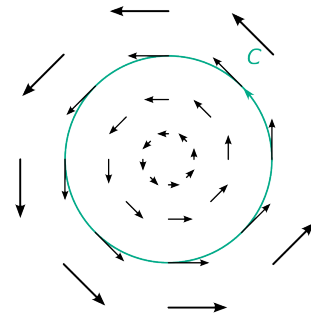


Figure 2: A sketch of the field $\mathbf{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ projected onto the xy -plane.

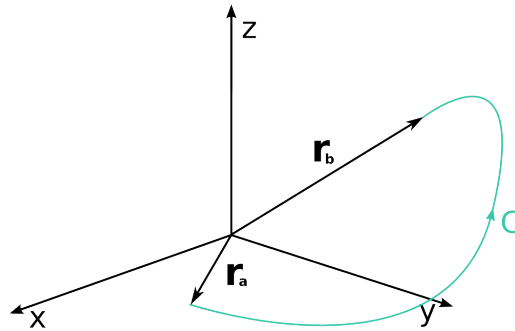


Figure 3

You might want to check that this parametrization indeed satisfies the equation for a unit circle, $x^2 + y^2 = 1$. We get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi.$$

2.2 Example

Let us now find line integral of $\mathbf{F} = \hat{\mathbf{r}}/r^2$ around a unit circle centered at the origin \mathcal{C} going counter clockwise. This is a very important field for our purposes since it has the same mathematical form as the field from a point charge. To accomplish this in Cartesian coordinates we need to convert by using the relations $r^2 = x^2 + y^2 + z^2$ and

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + z^2}}$$

such that

$$\mathbf{F} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

Now this field is only dependent of r so we might as well orient our coordinate axes such that our curve \mathcal{C} lies in the xy -plane. We can then use the same parametrization as in the previous example. This gives

$$\mathbf{F} = \frac{\cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}}}{(\cos^2 t + \sin^2 t)^{\frac{3}{2}}} = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}}$$

such that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos t \sin t + \sin t \cos t) dt = 0.$$

In this case it is however easier and more natural to use a spherical polar coordinate system. In this system the field along the circle \mathcal{C} where $r = 1$ is just $\mathbf{F} = \hat{\mathbf{r}}$ and by letting circle reside in the xy -plane, such that we don't have to worry about ϕ , a parametrization is obtained by letting $t = \theta$ for $0 \leq \theta \leq 2\pi$. Furthermore since r is constant, dr is zero

$$d\mathbf{r} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} = 0\hat{\mathbf{r}} + d\theta\hat{\boldsymbol{\theta}}$$

such that

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 0d\theta = 0$$

In fact it turns out that the line integral over this field is zero for *any* closed curve since

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{r_a}^{r_b} \frac{1}{r^2} \hat{\mathbf{r}} \cdot (dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}) = \int_{r_a}^{r_b} \frac{1}{r^2} dr = -\frac{1}{r} \Big|_{r_a}^{r_b} = \left(\frac{1}{r_b} - \frac{1}{r_a} \right) = 0$$

when $r_a = r_b$. Note the importance of this result. It says that the work around any closed loop in an electric field from a point charge is zero. We'll get back to it later.

3 Curl of a vector field

We earlier found that we can think of the line integral along a curve \mathcal{C} of a vector field \mathbf{F} as measuring how much \mathbf{F} is going in the same direction of \mathcal{C} . If we have a closed loop like the circle in the above examples this means that we can then think of the line integral as measuring how much the field \mathbf{F} goes in a circle \mathcal{C} , or rather how much \mathbf{F} is *circulating* around \mathcal{C} ¹.

Suppose now that we wanted a measure how much the field \mathbf{F} is 'circulating' at a *point* \mathbf{r} (figure 4). One way to do this is to take a closed curve \mathcal{C} with \mathbf{r} as its center and evaluate the integral $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ with the curve \mathcal{C} getting smaller and smaller. However since $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \rightarrow 0$ when the curve shrinks to zero we divide by the area A enclosed by the curve and define the *curl* at a point \mathbf{r} and in the direction of $\hat{\mathbf{n}}$ to have the value

$$\text{curl } \mathbf{F} \cdot \hat{\mathbf{n}} = \lim_{A \rightarrow 0} \frac{\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}}{|A|} \quad (1)$$

where $\hat{\mathbf{n}}$ is the unit normal to the planar area enclosed by \mathcal{C} . We get the curl in any direction by varying the direction of $\hat{\mathbf{n}}$ (figure 4). This is then our measure of *circulation* or *rotation* of a vector field \mathbf{F} around a point \mathbf{r} . If we think of \mathbf{F} as the velocity field of a fluid and place a paddle wheel at a point with curl, it will rotate. Because of the directionality and magnitude the curl of \mathbf{F} is itself a vector field.

4 Computation of Curl

If you found the definition of curl in equation 1 complicated, do not worry. The point was that the curl had something to do with 'circulation' or 'rotation' of a vector field and is related to closed loop integrals. If you have this intuition

¹In fluid dynamics where \mathbf{F} often represents the velocity in a fluid, the integral around such a closed loop is actually called the *circulation* around \mathcal{C} .

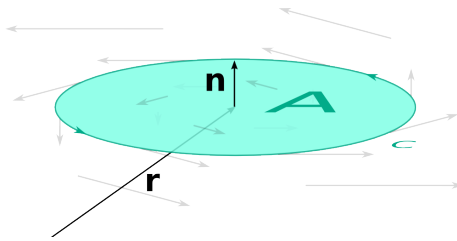


Figure 4: The definition of curl is related to a line integral over a closed loop.

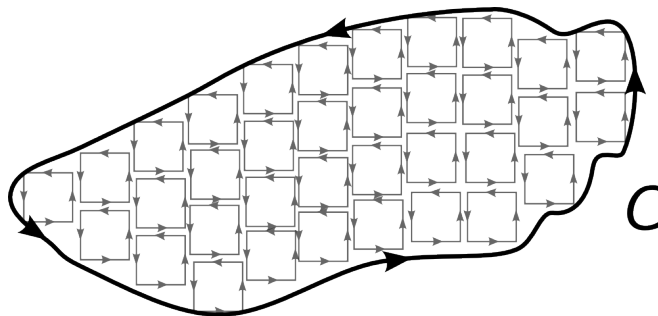


Figure 5: The curl in the field at neighbouring points cancel each other.

you'll be fine. This definition (equation 1) is not very handy for computing either. Luckily it can be shown that in Cartesian coordinates

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{k}} \quad (2)$$

And many just take this as the definition of curl. As complicated as this formula might seem, this is an improvement since this is actually a cross product with the del-operator $\nabla = \frac{\partial}{\partial x} \hat{\mathbf{i}} + \frac{\partial}{\partial y} \hat{\mathbf{j}} + \frac{\partial}{\partial z} \hat{\mathbf{k}}$ which can conveniently be expressed as

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

5 Stokes's Theorem

Looking back at the definition of curl in equation 1 it shouldn't be too surprising that there is a connection between the line integral of a closed loop and the curl of a vector field. This connection is encapsulated in *Stokes Theorem* which states that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{A}, \quad (3)$$

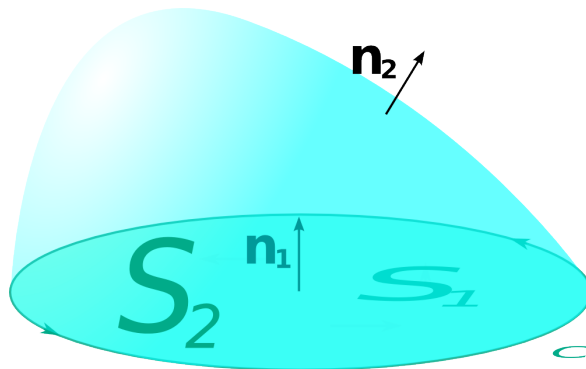


Figure 6: Stokes' Theorem can be used for any surface \mathcal{S} whose boundary is the curve \mathcal{C} .

where \mathcal{S} is *any* surface bounded by the closed curve \mathcal{C} and $d\mathbf{A} = \hat{\mathbf{n}}dA$ is the infinitesimal area vector perpendicular at any point to \mathcal{S} (figure 6). The orientation of $\hat{\mathbf{n}}$ and \mathcal{C} is governed by the right hand rule². In words this theorem states that the line integral of a vector field \mathbf{F} around a closed loop is equal to the surface integral of the curl of \mathbf{F} on any surface bounded by the curve \mathcal{C} .

Some intuition on why Stokes' theorem is true can be gained by considering the diagram in Figure 5. From the definition of curl we saw that it is really nothing but line integrals of small loops. If we cover an entire surface with the curve \mathcal{C} as it's boundary with many such small loops we see that neighboring contributions will have a tendency to cancel each other so that if we sum their contributions we get nothing but the entire integral around \mathcal{C} . This is the essence of the theorem and the idea behind a formal proof.

5.1 Example

Consider the field of Example 2.1 $\mathbf{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$. Let's compute the same line integral around the unit circle \mathcal{C} in the xy-plane, but now by using Stokes' Theorem. Even though any surface enclosed by \mathcal{C} would do, we chose the easiest one, namely the portion of the xy-plane enclosed by \mathcal{C} . First we find the curl of \mathbf{F} (equation 2).

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{k}}.$$

The unit vector normal to the xy-plane is $\hat{\mathbf{k}}$ so that $d\mathbf{A} = \hat{\mathbf{k}}dA$. Now applying Stokes Theorem

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}} \nabla \times \mathbf{F} \cdot d\mathbf{A} = \int_{\mathcal{S}} 2\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}dA = 2 \int_{\mathcal{S}} dA = 2\pi$$

where $\int_{\mathcal{S}} dA$ is the area of the unit disk \mathcal{S} .

²If your index finger points in the direction of \mathcal{C} and your ring finger points towards the surface \mathcal{S} then your thumb will indicate the direction of $\hat{\mathbf{n}}$. See Figure 6.

5.2 Example

The radial field of Example 2.2 $\mathbf{F} = \hat{\mathbf{r}}/r^2$ has no obvious rotation so we should expect that the curl is zero. You could check this by computing $\nabla \times \mathbf{F}$ directly but it actually follows from Stokes Theorem. We did show in Example 2.2 that $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for *any* closed curve \mathcal{C} . By Stokes' it then follows that

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}} \nabla \times \mathbf{F} \cdot d\mathbf{A} = 0$$

for *any* surface \mathcal{S} , but this can only be true if

$$\nabla \times \mathbf{F} = 0$$

6 Fields with $\nabla \times \mathbf{F} = 0$

When there is no curl in the field it follows from Stokes Theorem that for *any* closed curve \mathcal{C}

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0.$$

This result is in itself important. For a force field it means that the work done, and thus the change in kinetic energy, is the same for *any* closed loop. Exploring this a little further, let's now imagine that the curve \mathcal{C} consists of two curves \mathcal{C}_1 and \mathcal{C}_2 connecting two points \mathcal{P}_1 and \mathcal{P}_2 as in Figure 7. This means that

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r},$$

such that when a vector field has zero curl

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

for *any* two curves \mathcal{C}_1 and \mathcal{C}_2 . This is called path independence. Considering a force field again, this means that the work done in taking any path between two points will necessarily be the same. You might have encountered such fields before in mechanics. They are called conservative because they conserve energy. As it turns out such field can always be written as the gradient of some scalar function ϕ called the potential such that $\mathbf{F} = \nabla\phi$. I'll leave it up to you to show that $\nabla \times \nabla\phi = 0$.

7 Electromagnetism and Curl

Like for divergence the concept of curl is also important in electromagnetic theory. We showed in an example that $\nabla \times F = 0$ for a $\frac{\hat{\mathbf{r}}}{r^2}$ field and since every electrostatic field is a superposition of fields on this form this result carries over to electric field \mathbf{E} .

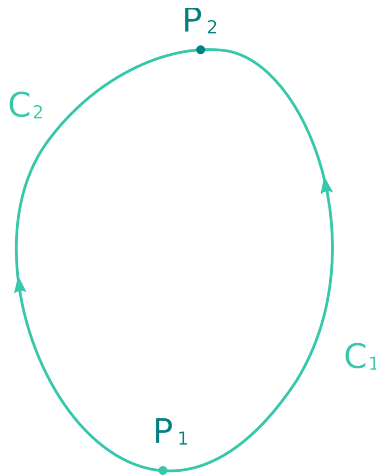


Figure 7: Irrotational fields are path independent, meaning that any line integral of the field connecting two points give exactly the same value.

$$\nabla \times \mathbf{E} = 0$$

so that by Stokes Theorem

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = 0$$

for any closed curve \mathcal{C} . So *electrostatic* fields have are irrotational and therefore conservative. However, we'll see that when we have changing magnetic fields in space, this induces curl in the electric field. This law is called *Faraday's law*

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

and is the basis for how dynamo's work and therefore for our entire electricity based society. We'll also meet *Ampere's law* which states that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

where \mathbf{J} is the current density at the point in question. This law is telling us that curl, or rotation, in the magnetic field are created by moving charges (currents) or changing electric fields. Notice the the fact that changing electric field induces curl in the electric field and vice versa, so that if there are no magnetic fields to begin with we can actually induce one by a varying electric field. Now the magnetic field will itself be changing and in turn induce an electric field. Repeat the argument and you'll find that this process is self sustaining and as we'll discover this is the secret behind the nature of electromagnetic waves.