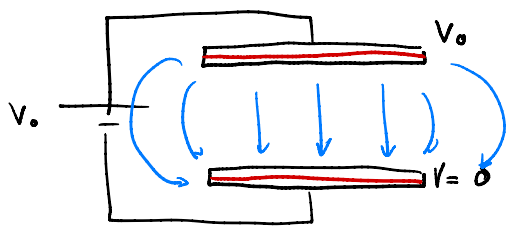
Laplace - likning

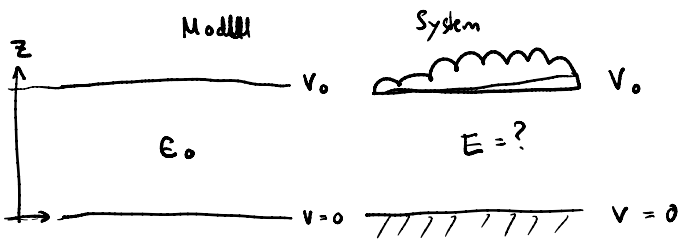
Sett på Q_i, $\rho(\vec{r})$ $\longrightarrow$ $\vec{E}$, V

Gitt Q: finner $\vec{E}$ eller V



Finn $\vec{E}$?

Ved å ta utgangspunkt i Gauss lov



$$\left.\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{E} &= -\nabla V \end{aligned}\right\} \quad \nabla \cdot (-\nabla V) = \frac{\rho}{\epsilon_0} = -\nabla^2 V = -\nabla \cdot \nabla V = \frac{\rho}{\epsilon_0}$$

Hvis $\rho = 0$ i et område $\Rightarrow \nabla^2 V = 0$

TEST Hva er $\nabla^2 V$: $V = V(x, y, z)$

$$\nabla V = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) V(x,y,z) = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$\nabla \cdot \nabla V = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplace likning

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Poissons likning

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho(x,y,z)}{\epsilon_0}$$

TEST

$$V(x,y,z) = V_0 x^2$$

Mulig løsning av Laplace, Poisson

```
x = np.linspace(-2,2,L)
y = np.linspace(-2,2,L)
rx,ry = np.meshgrid(x,y,indexing="ij")
```

Laplace likning

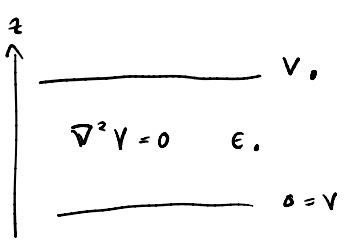
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 2V_0$$

Poisson

$$\nabla^2 V = 2V_0 = -\frac{\rho}{\epsilon_0}$$

Bevegelseslikning

$$m\frac{d^2x}{dt^2} = F(x); \quad x(t_0), \; v(t_0)$$



Initialverdi-problem

Grenseverdi-problem

Randverdi

Grenseverdier

Dirichlet grense-bet. $V(0)$ $V(x) = ?$ $V(a)$



von Neumann grense bet. $\left.\frac{\partial V}{\partial x}\right|_{x=0}$ $\left.\frac{\partial V}{\partial x}\right|_{x=a}$

TEST $V(x) = V_0 \frac{x}{L}$

Hva er grensebet ; $x = 0$ og $x = L$?

Dirichlet $V(0) = V_0 \frac{0}{L} = 0$; $V(L) = V_0 \frac{L}{L} = V_0$

von Neumann $V'(0) = \frac{V_0}{L}$ $V'(L) = \frac{V_0}{L}$

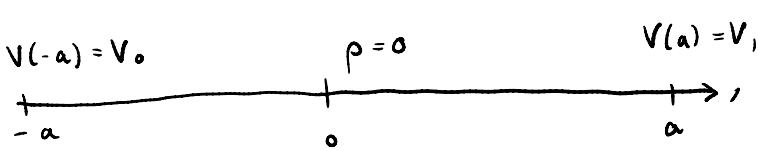
$x(t) = v_0 t$; $x(0) = 0$, $x'(0) = v_0$

Teoremer

Eksistens : Finnes en Løsning $V(x,y,z)$ } Gitt Lekning og grensebetingelser

Unikehet : Finnes kun en løsning. }

TEST



Finn $V(x)$ på intervallet $-a \leq x \leq a$

Vi vet $\nabla^2 V = 0$; $\frac{d^2V}{dx^2} = 0$; $V(x) = Ax + B$

$V(-a) = V_0 = A(-a) + B = V_0$

$V(a) = V_1 = A(a) + B = V_1$

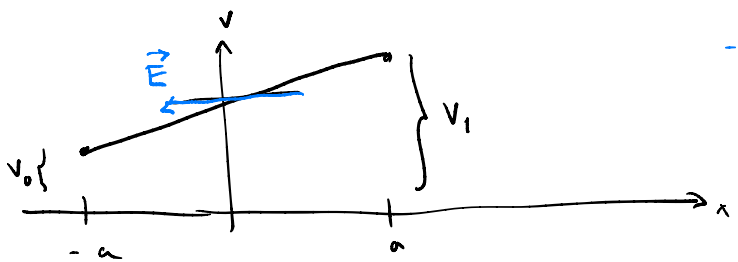
$+$ $\div$

$B + B = V_0 + V_1$

$B = \frac{1}{2}(V_0 + V_1)$

$-2Aa = V_0 - V_1$

$A = \frac{V_1 - V_0}{2a}$

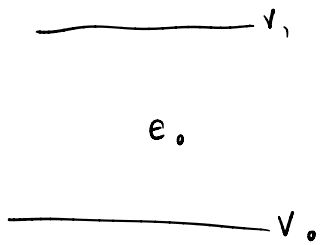


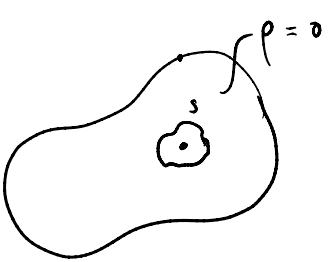
V må tilfredsstille: $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

Hvis $\rho = 0$ $\nabla^2 V = 0$

(I 2d: $V(x,y) = V_0 e^{-y} \sin x$)

For Laplace likning ($\nabla^2 V = 0$) må ekstremalpunkter ligge på grensene.

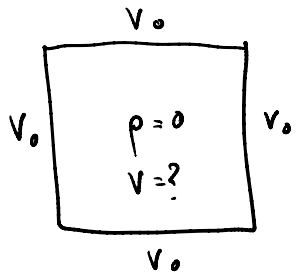




TEST

$V = ?$





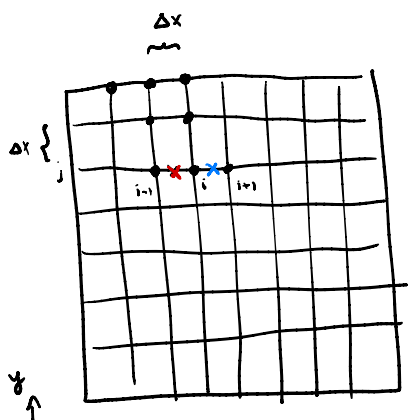
Numensk løsning av $\nabla^2 V = 0$

Vi velger en diskretisering

$$V(x_{ij}, y_{ij}) = V_{ij}$$

$$x_{ij} = x_0 + i\Delta x$$

$$y_{ij} = y_0 + j\Delta x$$



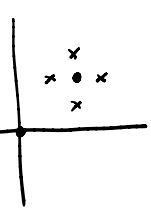
$$\times \quad \frac{\partial V}{\partial x}\left(x_{ij} - \frac{\Delta x}{2}, y_{ij}\right) \approx \frac{V(x_{ij}, y_{ij}) - V(x_{ij} - \Delta x, y_{ij})}{\Delta x}$$

$$\times \quad \frac{\partial V}{\partial x}\left(x_{ij} + \frac{\Delta x}{2}, y_{ij}\right) \approx \frac{V(x_{ij} + \Delta x, y_{ij}) - V(x_{ij}, y_{ij})}{\Delta x}$$

$$\frac{\partial^2 V}{\partial x^2}(x_{ij}, y_{ij}) \approx \frac{\overbrace{V(x_{ij} + \Delta x, y_{ij})}^{V_{i+1,j}} - V(x_{ij}, y_{ij}) - \left(V(x_{ij}, y_{ij}) - V(x_{ij} - \Delta x, y_{ij})\right)}{\Delta x^2}$$

(V_{ij}, V_{ij}, $V_{i-1,j}$)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \approx \frac{1}{\Delta x^2}\left[V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{ij}\right]$$

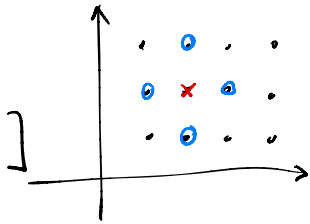


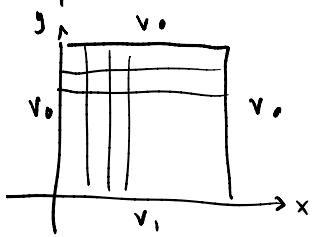
$= 0$ (Laplace likning)

$$V_{ij} = \frac{1}{4}\left[V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}\right]$$

Jacobi-iterasjoner:

$$V_{i,j}^{(t+1)} = \frac{1}{4}\left[V_{i+1,j}^{(t)} + \cdots\right.$$





Numpy-implementasjon

Matrise $V[i,j]$

$b[i,j]$ / tall
\ 'nan'