LAPLACE'S LIKNING

$$Q \longrightarrow \vec{E} = \sum \frac{Q_i}{4\pi\epsilon_0} \frac{\hat{R}_i}{R_i^2} \rightarrow V(\vec{r})$$



Coulombs lov

$$Q \rightarrow \oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0} \qquad \left(Q = Q_{fri} + Q_b \right)$$

$$\oint_S \vec{D} \cdot d\vec{S} = Q_{fri}; \quad \vec{D} = \epsilon \vec{E} \qquad -\int \vec{P} \cdot d\vec{S}$$



Integral form

$$\oint_S \vec{E}\, d\vec{S} = \frac{Q}{\epsilon_0}$$

$$\int_v \nabla \cdot \vec{E}\, dv = \frac{1}{\epsilon_0} \int_v \rho\, dv$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

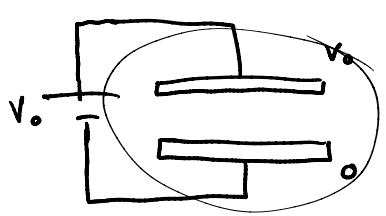
Differensial form

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

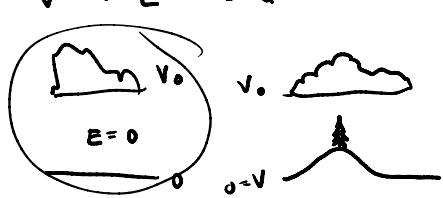
$$\nabla \times \vec{E} = 0$$

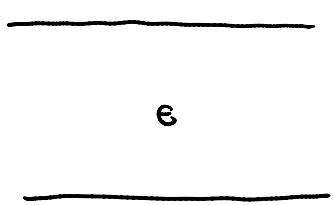


$$\vec{E} \longrightarrow V$$



$$V \rightarrow \vec{E} \rightarrow Q$$





$$\nabla \cdot \vec{D} = \rho_{fri}$$

$$\nabla \cdot (\epsilon \vec{E}) = \rho_{fri}$$

ϵ er uniform

$$\epsilon \nabla \cdot \vec{E} = \rho \Rightarrow \boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon}}$$

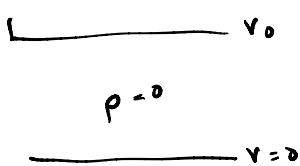
$$\vec{E} = -\nabla V$$

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla V) = \frac{\rho}{\epsilon}$$

$$-\nabla \cdot \nabla V = \frac{\rho}{\epsilon}$$

$$\nabla \cdot \nabla V = -\frac{\rho}{\epsilon}$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon}}$$ Poisson's likning



Spesialtilfelle $\rho = 0$

$$\boxed{\nabla^2 V = 0}$$ Laplace-likningen

TEST

$\nabla^2 V = ?$ a) $V = V_0 x^2$ b) $V = V_0 \cos x$ c) $V = V_{0x}^2 - V_{0y}^2$

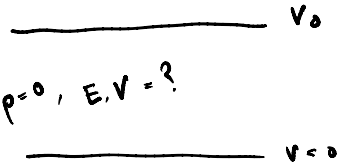
$$\nabla \cdot \nabla V = \nabla \cdot \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

a) $\nabla^2 V = \nabla^2 V_0 x^2 = V_0 \frac{\partial}{\partial x}(2x) = 2V_0$

b) $\nabla^2 V = \nabla^2 (V_0 \cos x) = V_0 \frac{\partial}{\partial x}(-\sin x) = -V_0 \cos x$

c) $\nabla^2 V = \nabla^2 (V_0 x^2 - V_0 y^2) = 2V_0 - 2V_0 = 0$

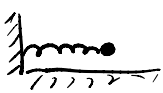
Løsning av Laplace likning

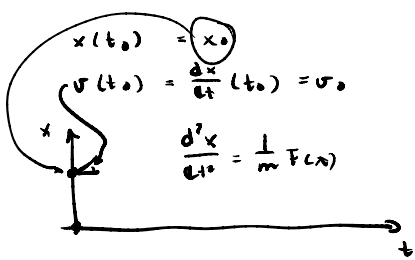


GRENSEBETINGELSER

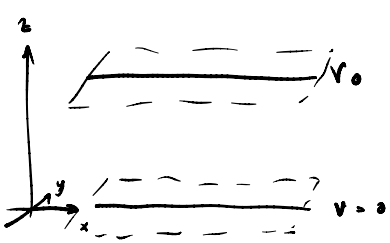
Newtons 2dre lov

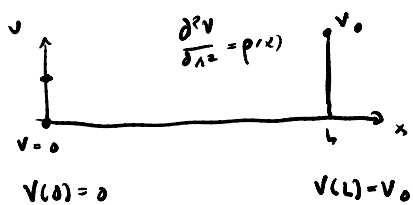
$$m \frac{d^2x}{dt^2} = F(x) = -kx$$





Poissons likning

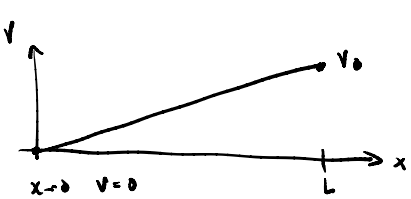




$V(0) = 0$ $\qquad$ $V(L) = V_0$

TEST $\qquad V(x) = V_0 (x/L)$

Grense-bet. $x = 0$ og $x = L$?

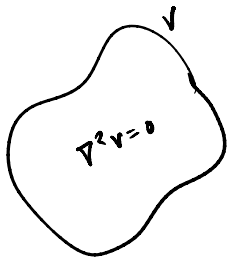


EKSISTENS & UNIKHETS-TEOREN

$\nabla^2 V = 0$, $\nabla^2 V = -\rho/\epsilon$

(1) Det finnes en løsning

(2) Løsningen er unik gitt grensebetingelser.



Coulombs lov
Gauss lov

$Q_i \rightarrow \vec{E} \rightarrow V$

$\nabla \cdot \vec{E} = \rho/\epsilon_0$

Laplace likning

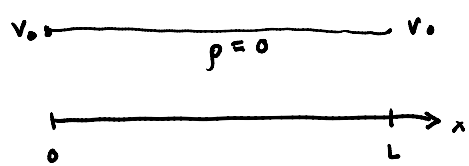
$V \rightarrow \vec{E} \rightarrow Q_i$

$\vec{E} = -\nabla V \Rightarrow \nabla^2 V = -\rho/\epsilon_0$

$\nabla^2 V = 0$, $\rho = 0$

TEST

1d



Finn $V(x)$, $0 < x < L$

$$\nabla^2 V = -\rho/\epsilon_0 \overset{\rho=0}{=} 0 \overset{1d}{=} \frac{\partial^2 V}{\partial x^2} = 0$$

$$\frac{d^2V}{dx^2} = 0, \quad V(0) = V_0, \quad V(L) = V_0$$

$$\frac{dV}{dx} = C \quad \Rightarrow \quad V(x) = Cx + B$$

$$V(0) = B = V_0 \quad \Rightarrow \quad V(x) = Cx + V_0$$

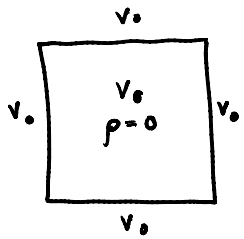
$$V(L) = V_0 = C \cdot L + V_0 \Rightarrow C = 0, \quad V(x) = V_0$$

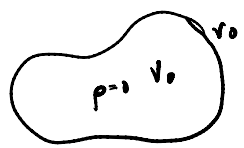
TEST (2d)

$V(x,y) = ?$

$$\nabla^2 V = 0$$

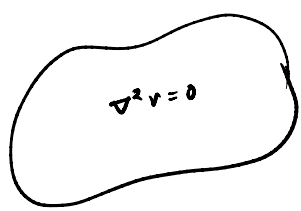
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$





TEOREM

For Laplace likninys løsninger er ekstremalverdiene på randen.



$$\bar{E} = -\nabla V$$



$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

$$V = V_0 x^2 - V_0 y^2$$

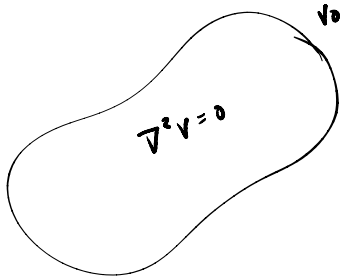
$V = V_0$

$V = V_0 x^2 - V_0 y^2$

$V = V_0 e^{-y} \sin x$

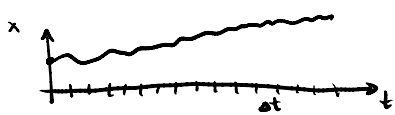
$V = V_0 x$

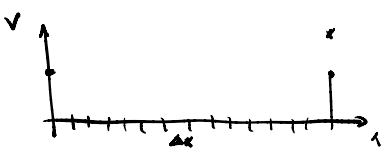
$V = V_0 y$



Laplace lukning : 2d

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

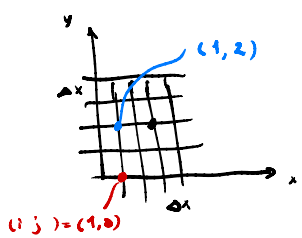




$V(x_{i,j}, y_{i,j}) = V_{i,j}$

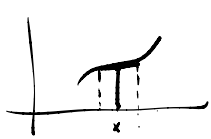
$x_{i,j} = x_0 + i\Delta x$

$y_{i,j} = y_0 + j\Delta x$



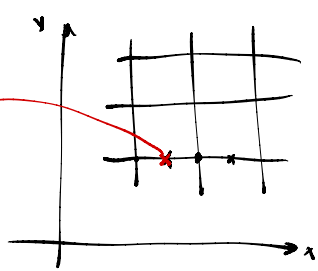
$$\frac{\partial^2 V}{\partial x^2} \simeq ?$$

$$\frac{\partial f}{\partial x} \simeq \frac{f(x + \frac{\Delta x}{2}) - f(x - \frac{\Delta x}{2})}{\Delta x}$$



$$\frac{\partial V}{\partial x}(x - \frac{\Delta x}{2}, y)$$

$$\simeq \frac{V(x,y) - V(x - \Delta x, y)}{\Delta x}$$



$$\frac{\partial V}{\partial x}\left(x+\frac{\Delta x}{2}, y\right) \simeq \frac{V(x+\Delta x, y) - V(x, y)}{\Delta x}$$

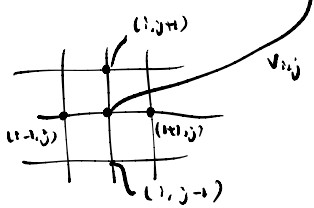
$$\frac{\partial^2 V}{\partial x^2}(x, y) \simeq \frac{\dfrac{V(x+\Delta x, y) - V(x, y)}{\Delta x} - \dfrac{V(x, y) - V(x-\Delta x, y)}{\Delta x}}{\Delta x}$$

$$\frac{\partial^2 V}{\partial x^2}(x, y) \simeq \frac{V(x+\Delta x, y) + V(x-\Delta x, y) - 2V(x, y)}{\Delta x^2}$$

$$\frac{\partial^2 V}{\partial y^2}(x, y) \simeq \frac{V(x, y+\Delta x) + V(x, y-\Delta y) - 2V(x, y)}{\Delta x^2}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \simeq \frac{1}{\Delta x^2}\Big[\overset{V_{i+1,j}}{V(x+\Delta x, y)} + \overset{V_{i-1,j}}{V(x-\Delta x, y)} + \overset{V_{i,j+1}}{V(x, y+\Delta x)} + \overset{V_{i,j-1}}{V(x, y-\Delta x)} - 4\overset{V_{i,j}}{V(x, y)}\Big]$$

$= 0$



$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = 0$$

$$V_{i,j} = \frac{1}{4}\left(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}\right)$$

