δ- funksjon

δ-funksjon er ikke en funksjon (fordeling)

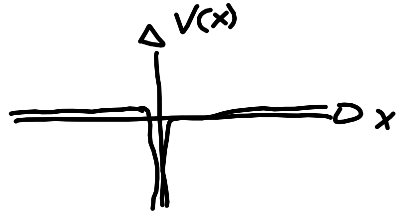
$$\delta(x) = \begin{cases} \infty & \text{når } x = 0 \\ 0 & \text{ellers} \end{cases} \quad \text{slik at} \quad \int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx = 1 \quad \text{for } \forall \varepsilon > 0$$

Kan representeres som grensen til en funksjon

$$\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-x_0)^2}{2\sigma^2}} = \delta(x-x_0)$$

Vi har at $f(x)\delta(x-a) = f(a)\delta(x-a)$ på en slik måte at

$$\int_{a-\varepsilon}^{a+\varepsilon} f(x)\delta(x-a)\,dx = f(a)$$



Vi skal se på potensialet $V(x) = -\alpha\,\delta(x)$

En uendelig dyp og smal brønn.

Spredningstilstander ($E>0$)

TUSL $\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi$ der $k=\frac{\sqrt{2mE}}{\hbar}$

for $x\neq 0$

Har løsninger $\psi(x) = Ae^{ikx} + Be^{-ikx}$ for $x<0$

$\psi(x) = Fe^{ikx} + Ge^{-ikx}$ for $x>0$

Vi ser på en innkommende bølge fra venstre, $G=0$.

Grensebetingelsen for „lining" i $x=0$

ψ kontinuerlig $\Rightarrow$ $\boxed{A+B=F}$ $(*)$

ψ' diskontinuerlig. Hva nå? Vi kan finne hvor diskontinuerlig den er ved å bruke TUSL.

$$\Delta\psi' \equiv \lim_{\varepsilon\to 0}\left(\frac{d\psi}{dx}\Big|_{x=\varepsilon} - \frac{d\psi}{dx}\Big|_{x=-\varepsilon}\right) = \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\frac{d^2\psi}{dx^2}dx$$

Utnytter TUSL $-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2\psi}{dx^2}dx + \int_{-\varepsilon}^{\varepsilon}V\psi\,dx = \int_{-\varepsilon}^{\varepsilon}E\psi\,dx$

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$$\underline{\Delta\psi'} = \lim_{\varepsilon\to 0}\left(\frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon}V(x)\psi(x)\,dx\right) = \frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}-\alpha\,\delta(x)\psi(x)\,dx$$

$$= -\frac{2m\alpha}{\hbar^2}\lim_{\varepsilon\to 0}\psi(0) = \underline{-\frac{2m\alpha}{\hbar^2}\psi(0)} \qquad (+)$$

For vår løsning blir

$$\Delta\psi' \equiv \lim_{\varepsilon\to 0}\left(\frac{d\psi}{dx}\Big|_{x=\varepsilon} - \frac{d\psi}{dx}\Big|_{x=-\varepsilon}\right) = \lim_{\varepsilon\to 0}\left(ikFe^{ik\varepsilon} - ikAe^{-ik\varepsilon} + ikBe^{+ik\varepsilon}\right)$$

$$= ikF - ikA + ikB \qquad (++)$$

Setter $(+) = (++)$

$$\boxed{-\frac{2m\alpha}{\hbar}(A+B) = ikF - ikA + ikB} \qquad (**)$$

Løser (*) og (**) for B og F

$$B = \frac{i\beta}{1-i\beta} A \quad \text{og} \quad \underline{F = \frac{1}{1-i\beta} A} \quad \text{hvor} \quad \underline{\beta = \frac{m\alpha}{\hbar^2 k}}$$

Da blir

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{|1-i\beta|^2} = \frac{1}{1+\beta^2}$$

$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}$$

(sjekk at $R + T = 1$)

$$\beta^2 = \frac{m^2\alpha^2}{k^2\hbar^4} = \frac{m^{\cancel{2}}\alpha^2}{\frac{2\cancel{m}E}{\cancel{\hbar^2}}\hbar^{\cancel{4}2}} = \frac{m\alpha^2}{2\hbar^2 E}$$

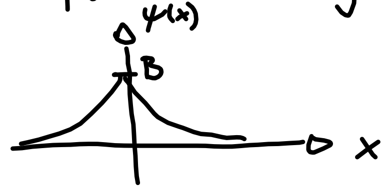
$$T = \frac{1}{1+\frac{m\alpha^2}{2\hbar^2 E}}, \quad T \xrightarrow{E\to\infty} 1$$

Bundne tilstander $(E<0)$

TUSL $\quad \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = \lambda^2\psi, \quad \lambda = \frac{\sqrt{-2mE}}{\hbar}$

$$\psi(x) = Ae^{-\lambda x} + Be^{\lambda x} \qquad x<0 \quad \text{må ha } A=0 \text{ for normering}$$

$$\psi(x) = F e^{-\kappa x} + G e^{\kappa x} \quad \text{for } x>0 \;, \quad G=0 \text{ pga. normering.}$$



$$\psi(x) = \begin{cases} Be^{\kappa x} & x<0 \\ Be^{-\kappa x} & x>0 \end{cases}$$

Bruker grensebetingelser

ψ kontinuerlig $\Rightarrow$ $B=F$

ψ' diskontinuerlig $\Rightarrow$ $\Delta\psi' = -\frac{2m\alpha}{\hbar^2}\psi(0) = \underline{-\frac{2m\alpha}{\hbar^2} B}$

$$\Delta\psi' = \lim_{\varepsilon\to 0}\left(\left.\frac{d\psi}{dx}\right|_{x=\varepsilon} - \left.\frac{d\psi}{dx}\right|_{x=-\varepsilon}\right)$$

$$= \lim_{\varepsilon\to 0}\left(-\kappa B e^{-\kappa\varepsilon} - \kappa B e^{\kappa\varepsilon}\right)$$

$$= -\kappa(B+B) = \underline{-2\kappa B}$$

$$\Rightarrow -\frac{2m\alpha}{\hbar^2}\cancel{B} = -2\kappa\cancel{B}$$

$$\Rightarrow \kappa = \frac{m\alpha}{\hbar^2}$$

$$\Rightarrow \underline{E} = -\frac{\kappa^2\hbar^2}{2m} = -\frac{m^{\cancel{2}}\alpha^2}{\hbar^{\cancel{4}2}}\cdot\frac{\cancel{\hbar^2}}{2\cancel{m}} = \underline{-\frac{m\alpha^2}{2\hbar^2}}$$

Kun <u>en eneste bunden tilstand</u>!

Finner B ved å normere løsning: Oblig 7 oppgave 1a.