Operator for angulærmoment

Klassisk $\vec{L} \equiv \vec{r} \times \vec{p}$ eller som komponenter

$$L_x \equiv y p_z - z p_y \,, \quad L_y \equiv z p_x - x p_z \,, \quad L_z \equiv x p_y - y p_x \qquad \text{syklisk } x \to y,\ y \to z,\ z \to x$$

Vi kjører standardmetoden:

$$\hat{L}_x = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right), \quad \hat{L}_y = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right), \quad \hat{L}_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

Merk at disse ikke kommuterer.

$$\begin{aligned}
[\hat{L}_x, \hat{L}_y] &= [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] \\
&= [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] - \underbrace{[\hat{y}\hat{p}_z, \hat{x}\hat{p}_z]}_{0} - \underbrace{[\hat{z}\hat{p}_y, \hat{z}\hat{p}_x]}_{0} + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z] \\
&= \hat{y}\hat{p}_x [\hat{p}_z, \hat{z}] + \hat{p}_y \hat{x} [\hat{z}, \hat{p}_z] \\
&= -\hat{y}\hat{p}_x [\hat{z}, \hat{p}_z] + \hat{p}_y \hat{x} [\hat{z}, \hat{p}_z] = (-\hat{y}\hat{p}_x + \hat{p}_y \hat{x}) \underbrace{[\hat{z}, \hat{p}_z]}_{i\hbar} \\
&= i\hbar \hat{L}_z
\end{aligned}$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \,, \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

De tilhørende observable er inkompatible, de kan ikke ha samtidige egenfunksjoner og egenverdier.

Kan ikke ha skarpe målinger av mer enn en L_i.

Regneregler

$$[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$$

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$$

$$[\hat{A}, \hat{A}] = 0$$

Operatoren for L^2 kommuterer med komponentene.

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad \left(= \hat{\vec{L}} \cdot \hat{\vec{L}}\right)$$

$$[\hat{L}^2, \hat{L}_x] = \underbrace{[\hat{L}_x^2, \hat{L}_x]}_{0} + [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x]$$

$$= \hat{L}_y^2 \hat{L}_x - \hat{L}_y \hat{L}_x \hat{L}_y + \hat{L}_y \hat{L}_x \hat{L}_y - \hat{L}_x \hat{L}_y^2$$

$$+ \hat{L}_z^2 \hat{L}_x - \hat{L}_z \hat{L}_x \hat{L}_z + \hat{L}_z \hat{L}_x \hat{L}_z - \hat{L}_x \hat{L}_z^2$$

$$= \hat{L}_y [\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x] \hat{L}_y + \hat{L}_z [\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x] \hat{L}_z$$

$$= \hat{L}_y (-i\hbar \hat{L}_z) + (-i\hbar \hat{L}_z) \hat{L}_y + \hat{L}_z i\hbar \hat{L}_y + i\hbar \hat{L}_y \hat{L}_z = 0$$

Det må da finnes samtidige egentilstander ψ slik at

$$\hat{L}^2 \psi = \lambda \psi \quad \text{og} \quad \hat{L}_z \psi = \mu \psi$$

hvor λ og μ er egenverdiene.

Vi skriver angulærmomentoperatorene i sfæriske koordinater

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} = -i\hbar (\vec{r} \times \vec{\nabla}) \quad \text{hvor} \quad \vec{\nabla} \equiv \underbrace{\vec{e}_r \frac{\partial}{\partial r}}_{(\nabla\psi)_r} + \underbrace{\vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta}}_{(\nabla\psi)_\theta} + \underbrace{\vec{e}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi}}_{(\nabla\psi)_\varphi} \qquad \vec{r} = r\vec{e}_r$$

$$\hat{\vec{L}} = -i\hbar \left(r(\vec{e}_r \times \vec{e}_r) \frac{\partial}{\partial r} + (\vec{e}_r \times \vec{e}_\theta) \frac{\partial}{\partial \theta} + \frac{1}{\sin\theta} (\vec{e}_r \times \vec{e}_\varphi) \frac{\partial}{\partial \varphi} \right)$$

$$= -i\hbar \left(\vec{e}_\varphi \frac{\partial}{\partial \theta} - \frac{1}{\sin\theta} \vec{e}_\theta \frac{\partial}{\partial \varphi} \right)$$

Enhetsvektorene kan skrives (Rottmann)

$$\vec{e}_\theta = \cos\theta\cos\varphi\,\vec{e}_x + \cos\theta\sin\varphi\,\vec{e}_y - \sin\theta\,\vec{e}_z$$

$$\vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y$$

Dette gir

$$\hat{L}_x = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\theta\cos\varphi\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cos\theta\sin\varphi\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$ Oppgave 4.21 i Griffiths

Egenfunksjoner til $\hat{L}^2$ og $\hat{L}_z$

$$\hat{L}_z\psi = \mu\psi, \quad \hat{L}_z\psi = -i\hbar\frac{\partial}{\partial\varphi}\psi = \mu\psi \Rightarrow -\hbar^2\frac{\partial^2}{\partial\varphi^2}\psi = \mu^2\psi \Rightarrow \frac{\partial^2}{\partial\varphi^2}\psi = -\frac{\mu^2}{\hbar^2}\psi$$

sammenlign med

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi$$ fra sfæriske harmoniske

så Y er egenfunksjon til $\hat{L}_z$

$$\hat{L}^2\psi = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi = \lambda\psi$$

$$\Rightarrow \left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\varphi^2}\right]\psi = -\frac{\lambda}{\hbar^2}\sin^2\theta\,\psi$$ sammenlign med $\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\varphi^2} = -l(l+1)\sin^2\theta\, Y$

Konklusjon: de sfæriske harmoniske er egenfunksjoner til $\hat{L}^2$ og $\hat{L}_z$.

De har egenverdier $\lambda = \hbar^2 \ell(\ell+1)$ og $\mu = \hbar m$.

Når vi løste H-atomet så fant vi også ψ s.l.h. at

$$\hat{H}\psi = E\psi$$

(Er $[\hat{H}, \hat{L}^2] = 0$ og $[\hat{H}, \hat{L}_z] = 0$? Ja, for $V(r)$. Oppgave 4.19 i Griffiths)

Egenverdier for $\hat{L}^2$ og $\hat{L}_z$

Vi finner egenverdier med stigeoperatorer. Vi definerer

$$\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y$$

$$[\hat{L}_z, \hat{L}_\pm] = [\hat{L}_z, \hat{L}_x] \pm i[\hat{L}_z, \hat{L}_y] = i\hbar\hat{L}_y \pm i(-i\hbar\hat{L}_x) = \hbar(i\hat{L}_y \pm \hat{L}_x) = \pm\hbar\hat{L}_\pm$$

$$[\hat{L}^2, \hat{L}_\pm] = [\hat{L}^2, \hat{L}_x] \pm i[\hat{L}^2, \hat{L}_y] = 0$$

$$\hat{L}_\pm\hat{L}_\mp = (\hat{L}_x \pm i\hat{L}_y)(\hat{L}_x \mp i\hat{L}_y) = \hat{L}_x^2 + \hat{L}_y^2 \pm i\hat{L}_y\hat{L}_x \mp i\hat{L}_x\hat{L}_y$$

$$= \hat{L}^2 - \hat{L}_z^2 \mp i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x) = \hat{L}^2 - \hat{L}_z^2 \mp i(i\hbar\hat{L}_z) = \hat{L}^2 - \hat{L}_z^2 \pm \hbar\hat{L}_z$$

som gir $\hat{L}^2 = \hat{L}_\pm\hat{L}_\mp + \hat{L}_z^2 \mp \hbar\hat{L}_z$

Anta at ψ er en egentilstand til $\hat{L}^2$ og $\hat{L}_z$. Da er også $\hat{L}_\pm \psi$ det.

Bevis: $\underline{\hat{L}^2 \hat{L}_\pm \psi} = \hat{L}_\pm \hat{L}^2 \psi = \underline{\lambda \hat{L}_\pm \psi}$

altså er $\hat{L}_\pm \psi$ er en egentilstand til $\hat{L}^2$ med samme egenverdi λ som ψ.

$\hat{L}_z \hat{L}_\pm \psi = \hat{L}_\pm \hat{L}_z \psi \pm \hbar \hat{L}_\pm \psi = \mu \hat{L}_\pm \psi \pm \hbar \hat{L}_\pm \psi = (\mu \pm \hbar) \hat{L}_\pm \psi$

altså er $\hat{L}_\pm \psi$ en egentilstand til $\hat{L}_z$ med egenverdi: $\mu \pm \hbar$.

Det må finnes en høyeste egenverdi for $\hat{L}_z$, μ_t, og en egentilstand ψ_t slik at $\hat{L}_+ \psi_t = 0$, hvis ikke vil $\mu^2 > \lambda$.

$\hat{L}^2 \psi_t = (\hat{L}_- \hat{L}_+ + \hat{L}_z^2 + \hbar L_z)\psi_t = (\mu_t^2 + \hbar \mu_t)\psi_t$ slik at $\underline{\lambda = \mu_t(\mu_t + \hbar)}$ $(*)$

Det må finnes en laveste egenverdi for $\hat{L}_z$, μ_b, og en egentilstand ψ_b slik at $\hat{L}_- \psi_b = 0$.

$\hat{L}^2 \psi_b = (\hat{L}_+ \hat{L}_- + \hat{L}_z^2 - \hbar \hat{L}_z)\psi_b = (\mu_b^2 - \hbar \mu_b)\psi_b$ slik at $\lambda = \mu_b(\mu_b - \hbar)$ $(**)$

For at $(*) = (**)$ så må $\mu_b = \mu_t + \hbar$, som er absurd, eller $\underline{\mu_b = -\mu_t}$.

Det vil si at μ går fra $-\mu_t$ til μ_t i steg på $\hbar$.

Om vi lar $\mu_t = l\hbar$ så er $\lambda = l\hbar(l\hbar + \hbar) = \hbar^2 l(l+1)$ og μ går i N trinn fra $-l\hbar$ til $l\hbar$, det vil si: $l\hbar = -l\hbar + N\hbar \Rightarrow 2l\cancel{\hbar} = N\cancel{\hbar} \Rightarrow l = \frac{N}{2}$

Egentilstandene til $\hat{L}^2$ og $\hat{L}_z$ er karakterisert av egenverdiene l og m

$$\hat{L}^2 \psi_{lm} = \hbar^2 l(l+1)\psi_{lm} \quad \text{og} \quad \hat{L}_z \psi_{lm} = \hbar m \psi_{lm}$$

hvor $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ og $m = -l, -l+1, \ldots, 0, \ldots, l-1, l$