Lecture 17

07.11.2018

Degenerate Fermi gases

Reminder: Z – the grand partition function

- Probability, when we can exchange particles:

$$P(s) = \frac{1}{Z} e^{-[E(s) - \mu N(s)]/kT}$$

- Sum of all probabilities equals 1, so we have:

$$Z = \sum_s e^{-[E(s) - \mu N(s)]/kT}$$

$$\text{Gibbs factor} = e^{-[E(s) - \mu_A N_A(s) - \mu_B N_B(s)]/kT}$$

Distribution function - Fermions

- For fermions: n=0 or n=1.
- The grand partition function:

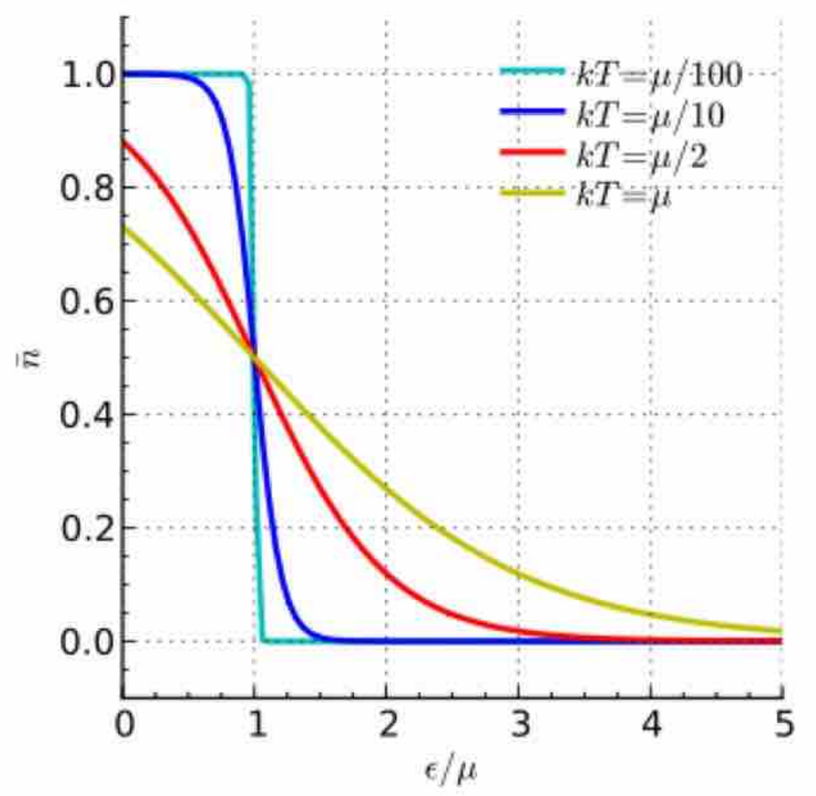
$$Z = 1 + e^{-(\epsilon-\mu)/kT}$$

- Occupancy – probability:

$$\bar{n} = \sum_n n\mathcal{P}(n) = 0 \cdot \mathcal{P}(0) + 1 \cdot \mathcal{P}(1) = \frac{e^{-(\epsilon-\mu)/kT}}{1 + e^{-(\epsilon-\mu)/kT}}$$

- **Fermi-Dirac distribution:**

$$\boxed{\bar{n}_{FD} = \frac{1}{e^{(\epsilon-\mu)/kT} + 1}}$$



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Distribution function - Bosons

- For bosons: n=0,1,2 ....
- The grand partition function ($\mu<\varepsilon$):

$$Z = 1 + e^{-(\epsilon-\mu)/kT} + e^{-2(\epsilon-\mu)/kT} + e^{-3(\epsilon-\mu)/kT} + \ldots$$

$$= \frac{1}{1 - e^{-(\epsilon-\mu)/kT}}$$

- Occupancy – probability:

$$\bar{n} = \sum_n n\mathcal{P}(n) = 0\cdot\mathcal{P}(0) + 1\cdot\mathcal{P}(1) + 2\cdot\mathcal{P}(2)\ldots$$

$$= \sum_n \frac{n}{Z} e^{-n(\epsilon-\mu)/kT} = -\frac{1}{Z}\sum_s \frac{\partial}{\partial x} e^{-nx}$$

$$= -\frac{1}{Z}\frac{\partial Z}{\partial x}$$

where we used:
$x = (\epsilon - \mu)/kT$

Bose-Einstein distribution

$$\boxed{\bar{n}_{BE} = \frac{1}{e^{(\epsilon-\mu)/kT} - 1}}$$

Distribution functions - comparison

- For Boltzmann distribution

$$\bar{n}_{Bol} = e^{(\epsilon-\mu)/kT}$$

- Fermi-Dirac distribution

$$\boxed{\bar{n}_{FD} = \frac{1}{e^{(\epsilon-\mu)/kT}+1}}$$

- **Bose-Einstein distribution**

$$\boxed{\bar{n}_{BE} = \frac{1}{e^{(\epsilon-\mu)/kT}-1}}$$

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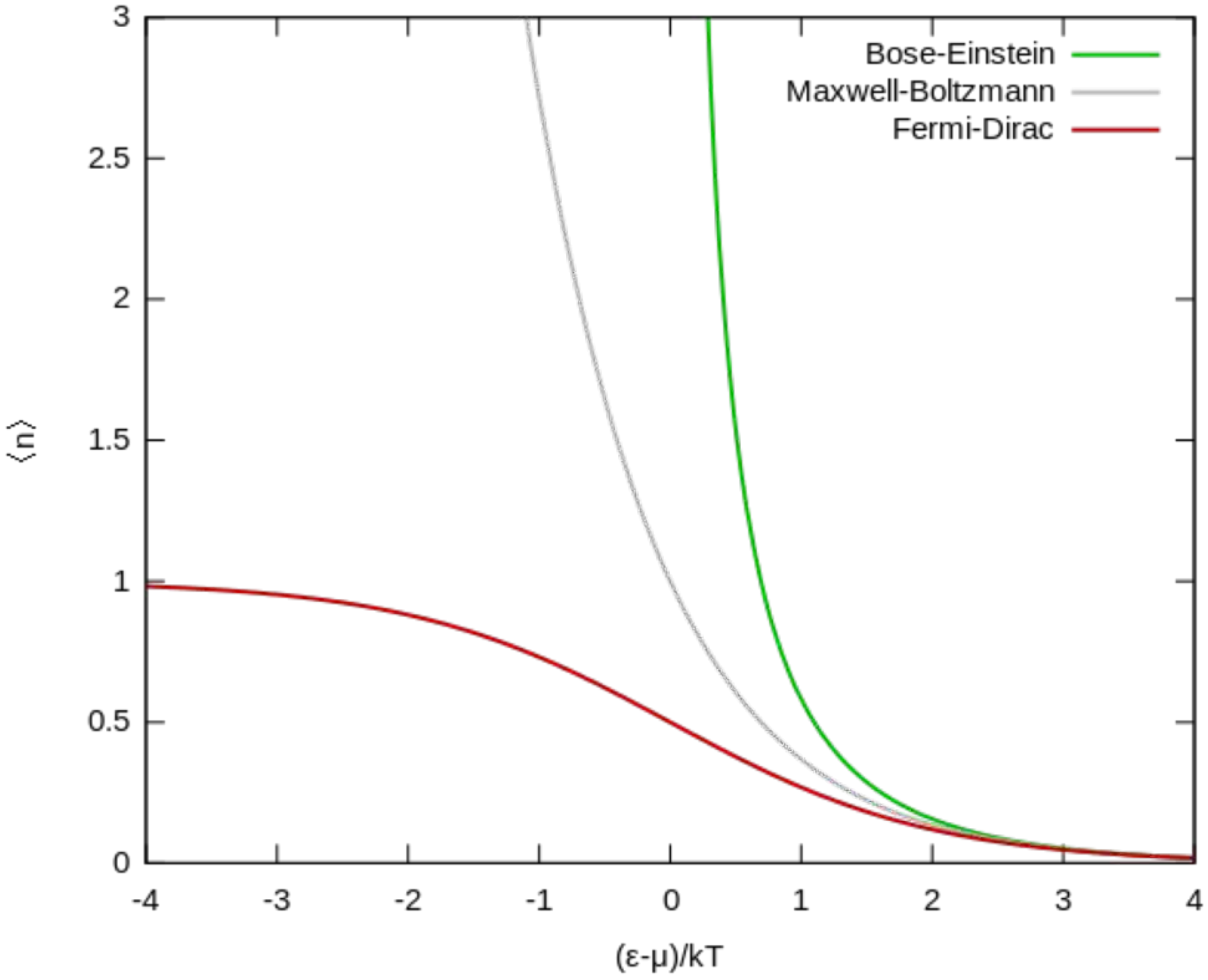


Let us look closer at the Fermi - gas

- We consider the gas of fermions at a very low temperature.
- What do we mean by low temperature? – The Boltzmann statistics does not apply. The average volume per particle is much smaller than the quantum volume:

$$\frac{V}{N} \ll v_Q \qquad \text{where} \qquad v_Q = l_Q^3 = \left(\frac{h}{\sqrt{2\pi m k T}}\right)^3$$

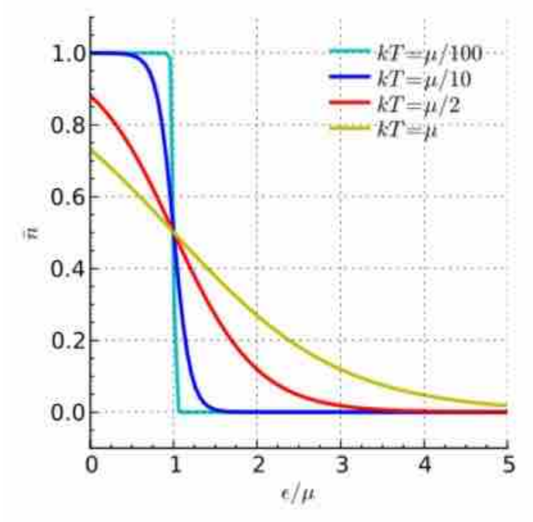
Let us look closer at the Fermi - gas

- In metal V/N is ca. $(0.2\ \mathrm{nm})^3$, while for an electron in room temperature $v_Q=(4.3\ \mathrm{nm})^3$. Boltzmann statistics does not work!
- Temperature is too low for the Boltzmann statistics to apply: T => 0.
- We neglect other interactions with lattice, atoms, etc.
- Electrons in a metal can be treated as ideal gas. But their concetration far exceeds concentration of particles in a conventional gas.
- Electrons are fermions – hence Fermi-Dirac statistics apply: number of particles ocuppying state s (energy ε_s) is given by:

$$\boxed{\bar{n}_{FD,s} = \frac{1}{e^{(\epsilon_s-\mu)/kT}+1}}$$

Fermi – gas at T=0

$$\boxed{\bar{n}_{FD,s} = \frac{1}{e^{(\epsilon_s - \mu)/kT} + 1}}$$

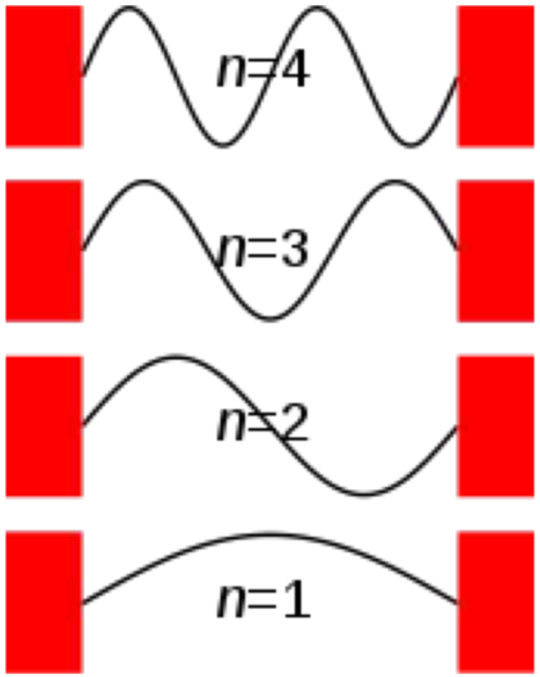


- A Step-Function.
- Here μ is called the Fermi energy: $\epsilon_F = \mu(T = 0)$

$$\mu = \left(\frac{\partial U}{\partial N}\right)_{S,V}$$

- Degenerate gas: all states below the Fermi energy occupied; all states above are free.
- The Fermi energy determined by the total number of electrons in a given volume. Intensive quantity...
- How to find the Fermi energy,, total energy, and the pressure of such an electron gas?

Fermi energy of electron gas



Particle in a box

- Assume electrons to be free particles in a box of $V=L^3$. (no interactions with crystal lattice, ions, etc.).
- Treat them as particle in a box! Wavefunctions are sine waves depending on a level n:

$$\lambda_n = \frac{2L}{n}$$

- Momenum is (for each dimension):
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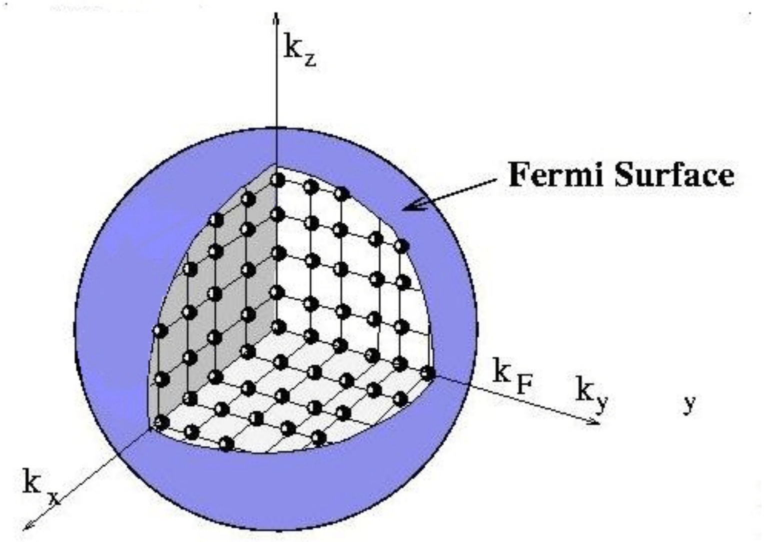
$$p = \frac{h}{\lambda_n} = \frac{hn}{2L}$$

Fermi energy of electron gas



- The energy is then given by:

$$\epsilon = \frac{\|\vec{p}\|^2}{2m} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2)$$

- We are thus filling a part of a sphere in *n*-space. Each node has TWO states (because of different spin).
- Lower energy – lower indices.
- Maximum energy related to radius of the Fermi surface.
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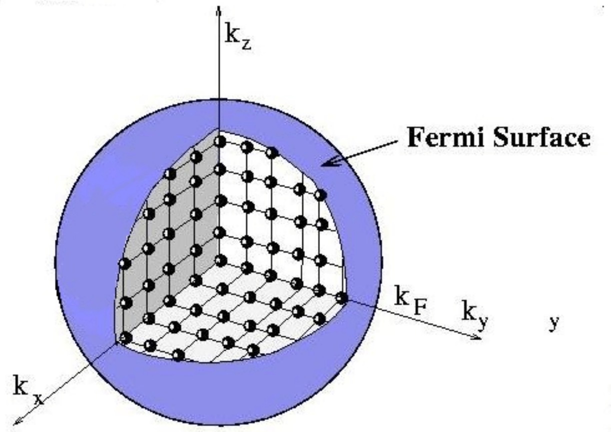
$$\epsilon_F = \frac{h^2 n_{max}^2}{8mL^2}$$

Fermi energy of electron gas



- How many states are available? Twice (due to spin) the volume of the Fermi surface (for positive n).

$$N = 2 \cdot \frac{1}{8} \cdot \frac{4}{3}\pi n_{max}^3 = \frac{\pi n^3}{3}$$

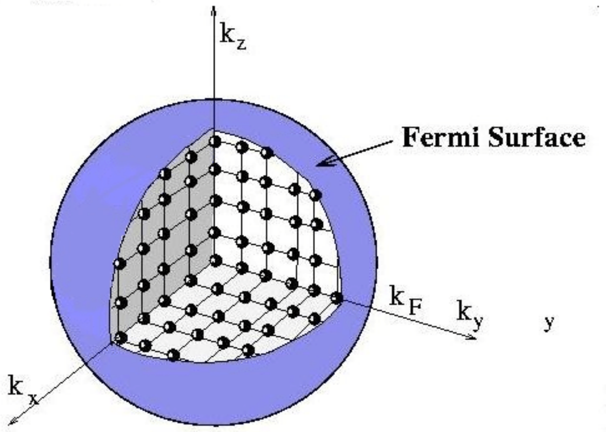
$$\epsilon_F = \frac{h^2 n_{max}^2}{8mL^2}$$

- So the Fermi energy is:

$$\epsilon_F = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3}$$

- But it depends in fact only on the electron density!
- It is the highest energy of all electrons. Average energy is lower (more than half of ε_F).

Total energy of Fermi gas



- We need to sum over all energies

$$U = 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\epsilon(\vec{n})$$

- By switching to integrals and evaluating of the 1/8 sphere:

$$\epsilon_F = \frac{h^2 n_{max}^2}{8mL^2}$$

$$U = 2\int_0^{n_{max}} dn \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi n^2 \sin\theta \epsilon(n)$$

- We obtain:

$$U = \frac{3}{5}N\epsilon_F$$

- So can use the Fermi energy as indicator of the applicability of Fermi-Dirac statistics:

$$\boxed{kT \ll \epsilon_F}$$

Fermi temperature

$$T_F = \epsilon_F/k$$

What about the pressure?

- We can use the well known formula (see thermodynamic identity)

$$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$$

- And obtain the degeneracy pressure!

$$P = -\frac{\partial}{\partial V}\left[\frac{3}{5}N\frac{h^2}{8m}\left(\frac{3N}{\pi}\right)^{2/3}V^{-2/3}\right] = \frac{2N\epsilon_F}{5V} = \frac{2}{3}\frac{U}{V}$$

- Which increases with reducing volume.

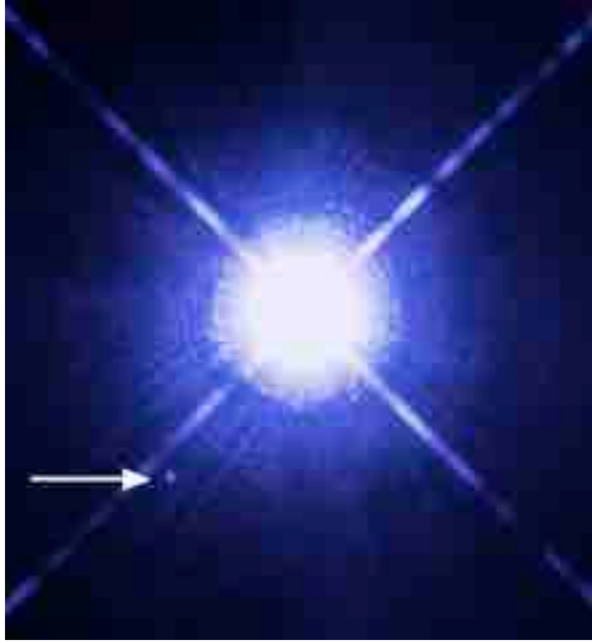
- Degeneracy pressure keeps the matter from collapsing. It folows from the exclusion principle.

White dwarf star

- White (degenrate) dwarf, - a burn out star, stellar core remnant composed mostly of electron-degenerate matter.
- Very dense object: mass is comparable to that of the Sun, while its volume is comparable to that of Earth.
- Faint luminosity comes from the emission of stored thermal energy.
- No fusion. Thermal energy can not counteract the gravitaional collapse – but the degenerate electron pressure can when the average distance between electrons is comparable to the Broglie wavelength!
- What is the physics of such a star (e.g., Sirus B)?

Sirius A and *Sirius B*

White dwarf star



- Total energy of the star: E=K+U.
- Let us assume uniform mass distribution, then potential energy:

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

- Let us assume electrons to be highly degenerate, and nonrelativistic, the total kinetic energy is:

$$E_k = \frac{3}{5}N\epsilon_F = \frac{3}{5}N \cdot \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{2/3}$$

- If 1 electron corresponds to 1 proton + 1 neutron: $N=M/2m_p$.

$$E_k = \frac{3h^2}{40m_e}\left(\frac{M}{2m_p}\right)^{5/3}\left(\frac{9}{4\pi^2 R^3}\right)^{2/3} = A\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2}$$

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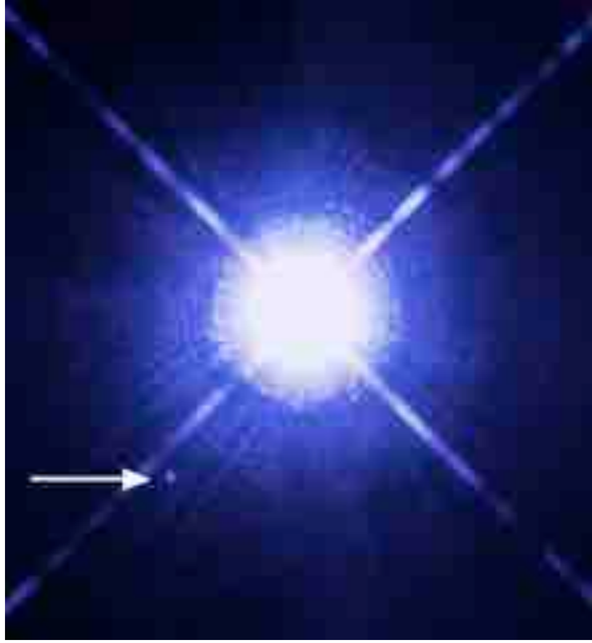
White dwarf star



- By finding the minimum of the total energy, we can find the equilibrium radius:
- Dwarf star with a larger mass has a smaller equilibrium radius! Higher mass gives larger gravitational attraction.

$$R = 0.029 \frac{h^2}{G m_e m_p^{5/3}} \frac{1}{M^{1/3}}$$

- If we choose the mass of the Sun, the radius of the dwarf star would be 7200 km. A bit more than radius of the Earth. The density would be M/V, giving 1.3x 10^9 kg/m^3 (1.3 million times the density of water).
- Effectively we can use T=0, since Fermi temperature is 2.3 x 10^9 K.
- If the mass of dwarf star is > 3 mass of the sun, it can be relativistic and unstable – its radius will tend to go to zero, and Carbon-oxygen dwarf can reignite at >1.4 mass of the sun and 1a supernova explosion can happen. It can end up into neutron star or black hole... Stable dwarf will turn into red and black dwarf – after radiating its energy.
- Nowadays the critical mass is calculated as 1.4 mass of the Sun.

Neutron star

- Can form if the dwarf star is too heavy to stabilise (mass > 3 mass of the Sun).
- Extreme pressures: electrons combine with protons to form neutrons.
- White-dwarf is effectively transformed into a gas of neutrons. The mean separation between the neutrons gets comparable with their de Broglie wavelength.
- Degeneracy pressure of the neutrons can halt the collapse of the star

Crab Nebula with Crab pulsar

Pulsars, 1967, discovered by Jocelyn Bell and Antony Hewish

Neutron star

- We can follow similar procedure as for the dwarf star by letting:

$$m_p \to m_p/2$$

$$m_e \to m_p$$

- The radius of such a star is:

$$R = 0.000011 R_S \left(\frac{M_S}{M}\right)^{1/3}$$

- So if the star had the mass of the sun, its radius would be about 10 km!

- Rotating neutron star is called pulsar.

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Crab Nebula with Crab pulsar

Pulsars, 1967, discovered by Jocelyn Bell and Antony Hewish

Neutron star

- When relativistic effects are taken into account, it is found that there is a critical mass above which a neutron star cannot be maintained against gravity.

- The critical mass, which is known as the *Oppenheimer-Volkoff limit:*

$$M = 6.9 M_S$$

- But even at lower mass ($M = 2\ M_S$), neutron star can collapse into the black hole.

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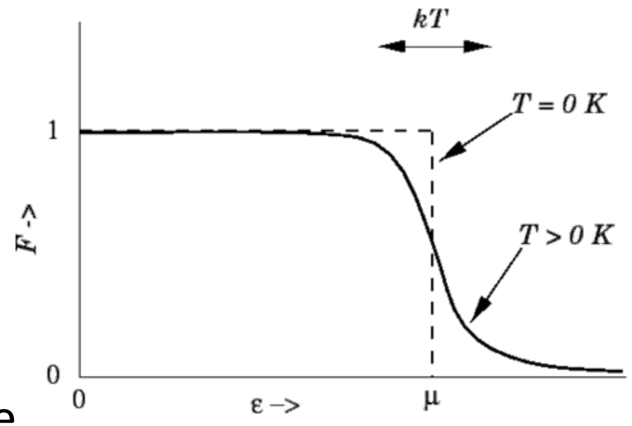
Crab Nebula with Crab pulsar

Pulsars, 1967, discovered by Jocelyn Bell and Antony Hewish

Fermi gas at T>0

$$\boxed{\bar{n}_{FD} = \frac{1}{e^{(\epsilon-\mu)/kT}+1}}$$



- Most of fermions DO NOT have the thermal energy of ca. kT – most of the states are already occupied!
- Only electrons close to Fermi energy can jump on unoccupied states above the Fermi energy.
- The number of states affectedby the increase in T is proportional to number of affected electrons (NkT) and energy acquired by electron (kT).

$$Extra\ energy \propto NkT \times kT$$

- Dimensionless analysis gives the proportiaonality constant:

$$U = \frac{3}{5}N\epsilon_F + \frac{\pi^2}{4}N\frac{(kT)^2}{\epsilon_F}$$

- And we can now calculate head capacity – since we have temperature...

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{\pi^2 N k^2 T}{2\epsilon_F}$$

The density of states

- We can rewrite, by changing variables, the energy integral for Fermi gas at zero temperature:

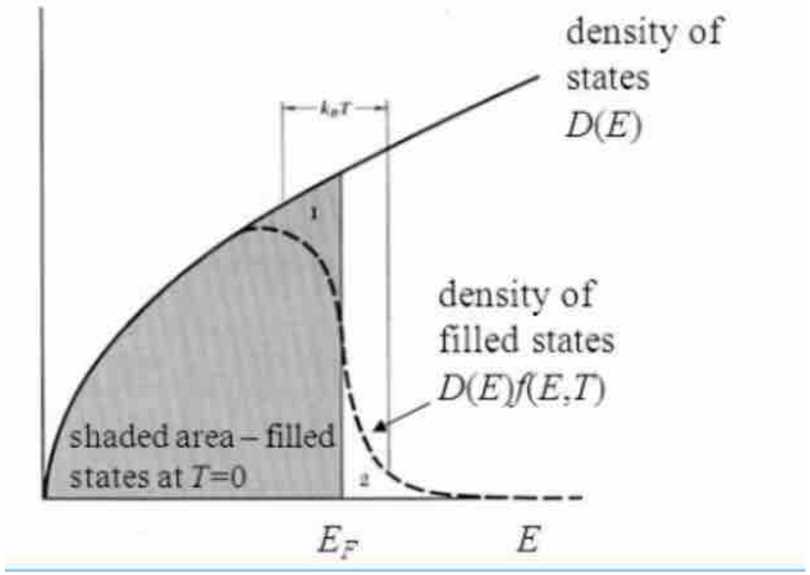
$$U = \int_0^{\epsilon_F} \epsilon \left[\frac{\pi}{2} \left(\frac{8mL^2}{h^2} \right)^{3/2} \sqrt{2} \right] d\epsilon$$

- This integral includes number of single states per unit energy, which is **the density of states:**

$$g(\epsilon) = \frac{\pi}{2} \left(\frac{8m}{h^2} \right)^{3/2} V\sqrt{2}$$

- Density of states is proportional to volume and does not depend on N. It is proportional to the $\varepsilon^{1/2}$.

- We can estamate the number of states between two energies by integrating the density of states.



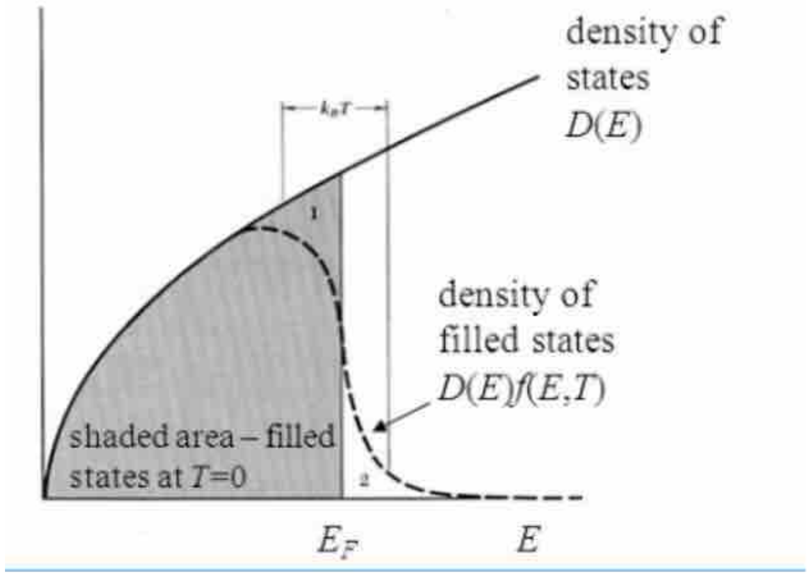
The density of states

- Once we find density of states with quantum mechanics, we follow the analysis using thermal physics.
- For zero temperature:

$$N = \int_0^{\epsilon_F} g(\epsilon) d\epsilon$$

- For non-zero temperature, we need to write explicitly the Fermi-Dirac distribution funciton:

$$\begin{aligned} N &= \int_0^{\infty} g(\epsilon) \bar{n}_{FD}(\epsilon) d\epsilon \\ &= \int_0^{\infty} g(\epsilon) \frac{1}{e^{(\epsilon-\mu)/kT}+1} d\epsilon \end{aligned}$$



Corresponding energy

- Can be found by letting ε into integral:

$$U = \int_0^\infty \epsilon g(\epsilon) \bar{n}_{FD}(\epsilon) d\epsilon$$
$$= \int_0^\infty \epsilon g(\epsilon) \frac{1}{e^{(\epsilon-\mu)/kT}+1} d\epsilon$$

- For nonzero temperatures, chemical potenetial is slightly shifted as compared to zero temperature and different from ε_F.

- This is because of statistics an density of states being larger to the right so we could increase the number of electrons artificially.

- We want to avoid «creating electrons» in the statistics just by increasing temperature.