# Lecture 18

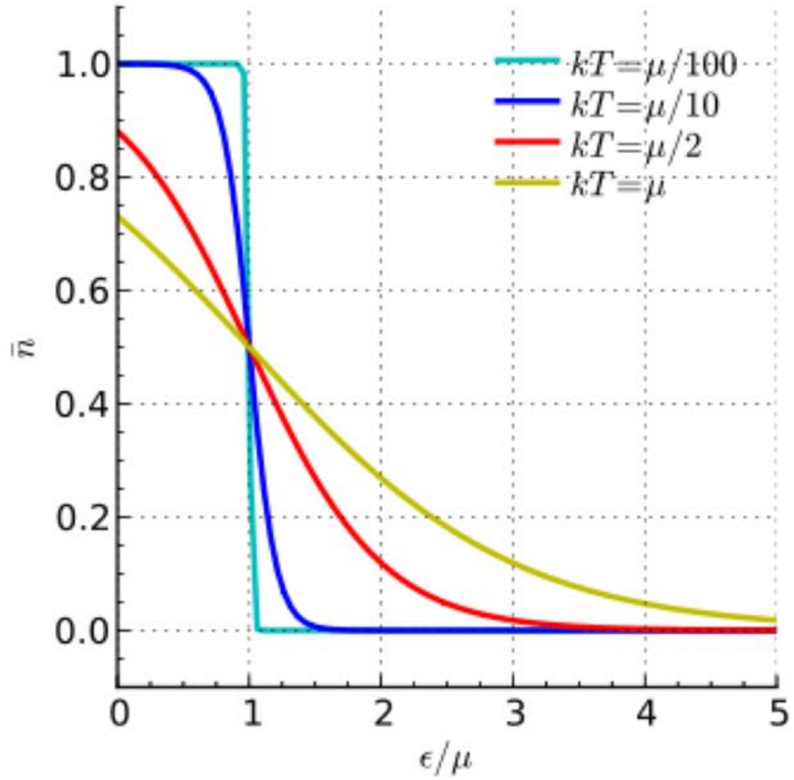
12.11.2018

Black-body radiation

# *Reminder: application of quantum statistics*

**Fermi-Dirac** distribution:

$$\boxed{\bar{n}_{FD} = \frac{1}{e^{(\epsilon-\mu)/kT}+1}}$$

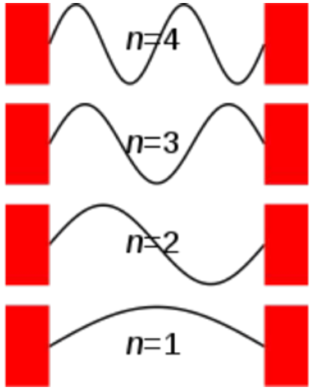



The Boltzmann statistics did not apply. The average volume per particle was much smaller than the quantum volume:

$$\frac{V}{N} \ll v_Q \quad \text{where} \quad v_Q = l_Q^3 = \left(\frac{h}{\sqrt{2\pi mkT}}\right)^3$$

# Reminder: application of quantum statistics

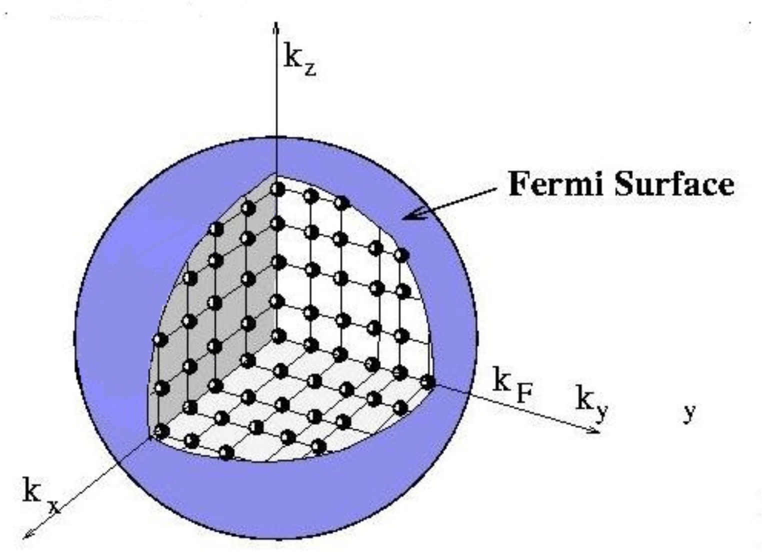
- We applied it to electrons in metal, white dwarfs, neutron stars.
- We found the Fermi energy, pressure, total energy, heat capacity, density of states...




$$\epsilon_F = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3} \qquad \epsilon_F = \frac{h^2 n_{max}^2}{8mL^2}$$

$$U = \frac{3}{5}N\epsilon_F \qquad C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{\pi^2 N k^2 T}{2\epsilon_F}$$

$$P = \frac{2N\epsilon_F}{5V} = \frac{2U}{3V} \qquad g(\epsilon) = \frac{\pi}{2}\left(\frac{8m}{h^2}\right)^{3/2} V\sqrt{2}$$



# *Reminder: distribution functions - comparison*

- For Boltzmann distribution

$$\bar{n}_{Bol} = e^{(\epsilon-\mu)/kT}$$

- Fermi-Dirac distribution

$$\boxed{\bar{n}_{FD} = \frac{1}{e^{(\epsilon-\mu)/kT}+1}}$$

- Bose-Einstein distribution

$$\boxed{\bar{n}_{BE} = \frac{1}{e^{(\epsilon-\mu)/kT}-1}}$$

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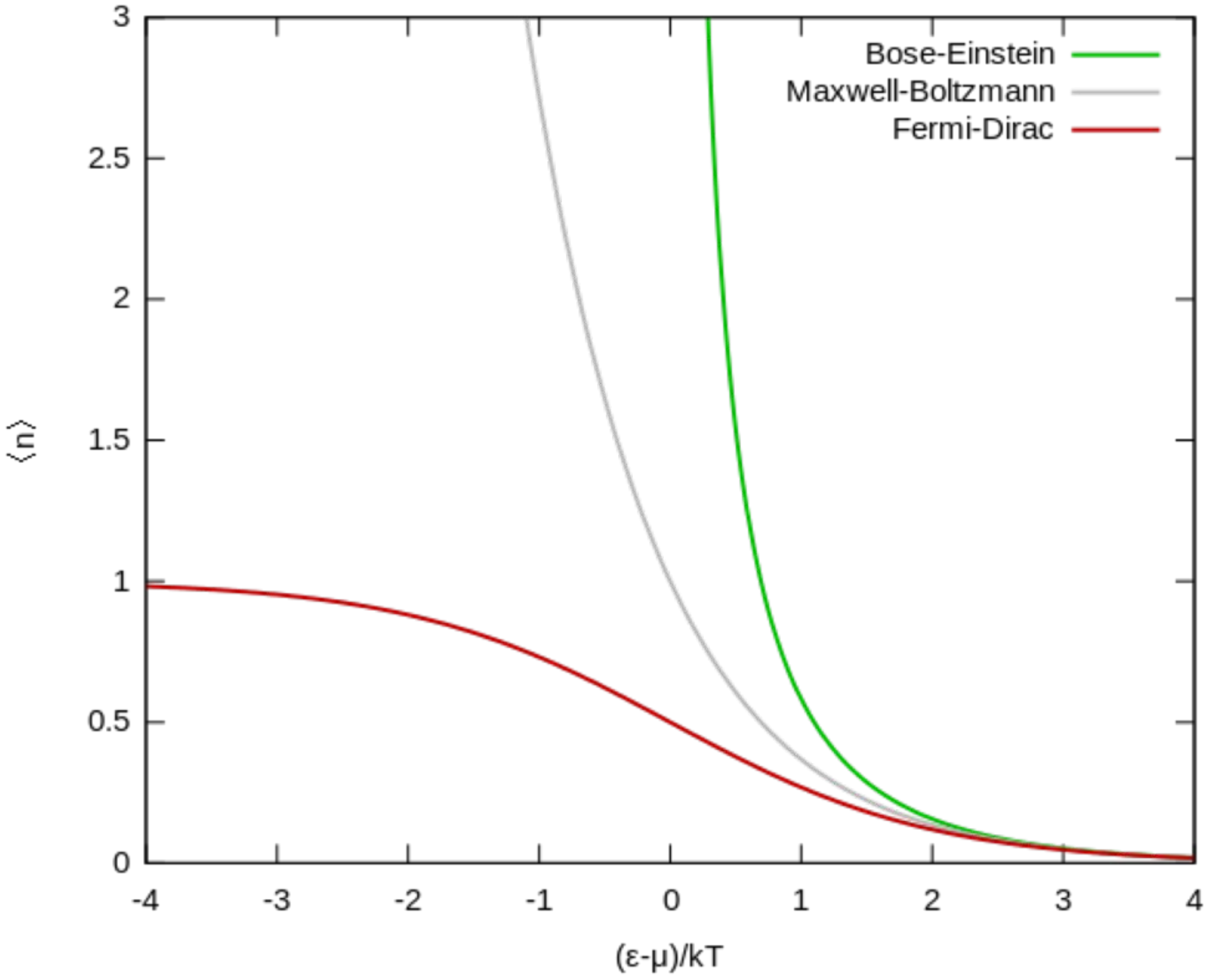


# *Reminder: distribution for bosons*

- For bosons: n=0,1,2 ....
- The grand partition function (μ<ε):

$$Z = 1 + e^{-(\epsilon-\mu)/kT} + e^{-2(\epsilon-\mu)/kT} + e^{-3(\epsilon-\mu)/kT} + \ldots$$

$$= \frac{1}{1 - e^{-(\epsilon-\mu)/kT}}$$

- Occupancy – probability:

$$\bar{n} = \sum_n n\mathcal{P}(n) = 0 \cdot \mathcal{P}(0) + 1 \cdot \mathcal{P}(1) + 2 \cdot \mathcal{P}(2) \ldots$$

$$= \sum_n \frac{n}{Z} e^{-n(\epsilon-\mu)/kT} = -\frac{1}{Z} \sum_s \frac{\partial}{\partial x} e^{-nx}$$

$$= -\frac{1}{Z}\frac{\partial Z}{\partial x}$$

where we used:
$x = (\epsilon - \mu)/kT$

**Bose-Einstein distribution**

$$\boxed{\bar{n}_{BE} = \frac{1}{e^{(\epsilon-\mu)/kT} - 1}}$$

# Let us look at photons - EM radiation

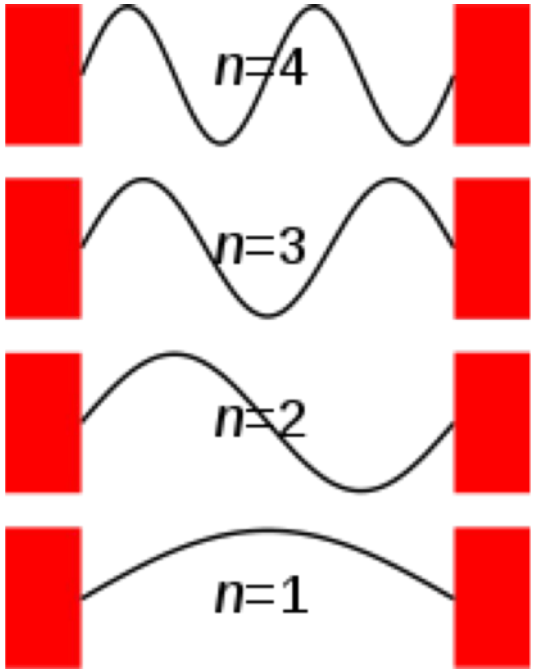
**UV Catastrophe! – radiation in a box.**


- Classical statistical physics: all harmonic oscillator modes have average energies of kT.
- EM field in a metal box – combination of standing waves of different pattern and $f=c/\lambda$. (knots on the box edges).
- Equipartition theorem : energy of each wave is 2 x ½ kT.
- Most of energy will be in a shorter waves – where most of modes are.
- The energy should be proportional to $f^2$.

# UV Catastrophe

1D box of size *a,* standing waves:

$$a = n\frac{\lambda}{2} \rightarrow \lambda = 2an, f = \frac{nc}{2a} \qquad n \in I$$




This corresponds to 1 state, but EM waves have 2 polarizations – two degrees of freedom

Density of states (for each n we have *c/2a*):

$$n(f)df = \frac{2}{c/2a} = 2\frac{2a}{c}$$

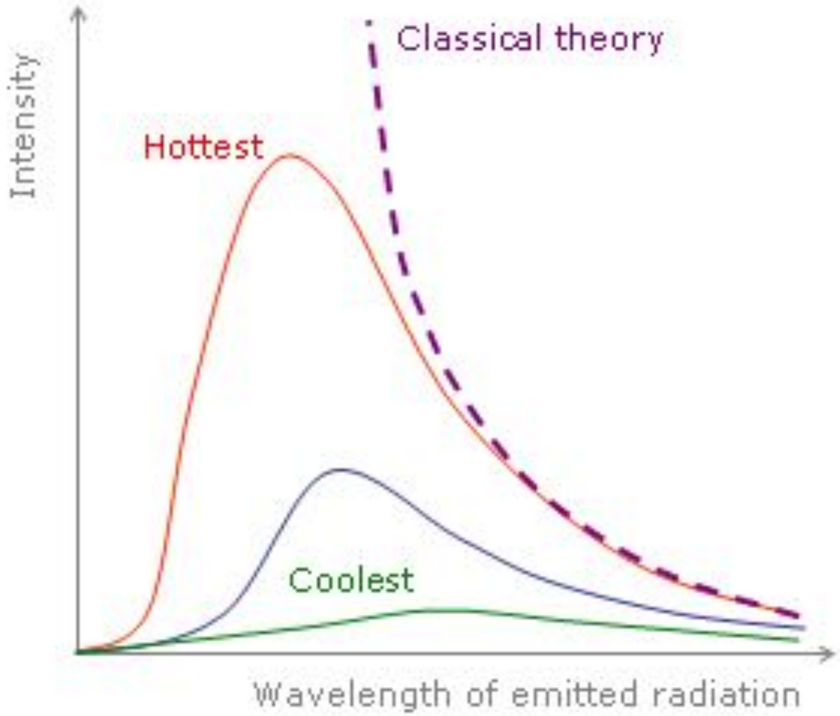
# UV Catastrophe

In 3D sphere

$$n(f)df = \left(\frac{2a}{c}\right)^3 \frac{1}{8} 4\pi f^2 2df = \frac{8\pi a^3}{c^3} f^2 df$$

This leads to the Rayleight-Jeans law for black-body emmissivity per volume unit, where each wave carries energy of *kT*.

$$\rho(f)df = \frac{8\pi kT}{c^3} f^2 df$$




And this **does not agree** with observations at short wavelengths...

# The Planck Distribution

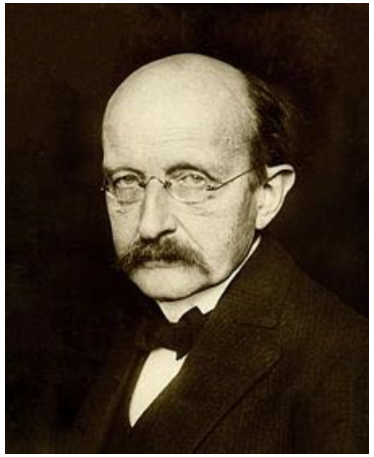
*Planck*: for a black-body, the mean energy of EM standing waves is a function of frequency.

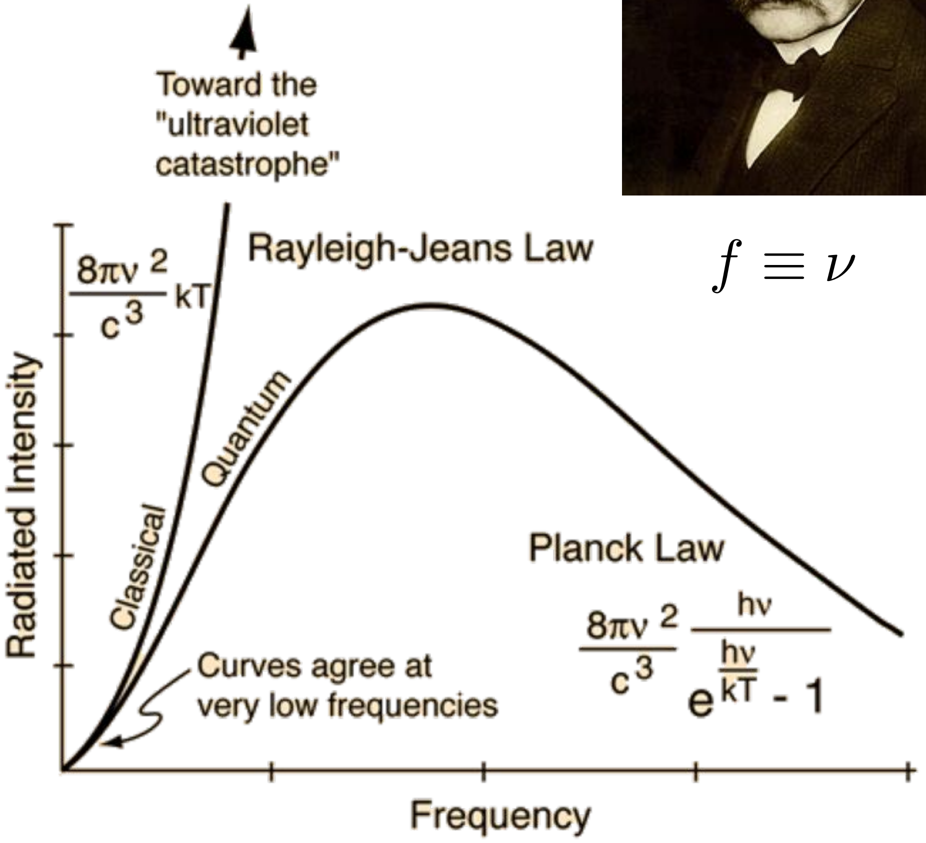
This contradicts equipartition theorem (because here the mean energy does not depend on *f*).

The allowed energies are quantized and $\Delta\varepsilon$ depends on the frequency.

$$\Delta\epsilon = hf$$

This makes sense: low frequencies $\Delta\varepsilon$ is small and almost continuum and $\varepsilon$~kT.






$$f \equiv \nu$$

$$E_n = 0, hf, 2hf, \ldots$$

h = 6,63 $10^{-34}$ J s

# The Planck Distribution (2)

The partition function for a single oscillator is then:

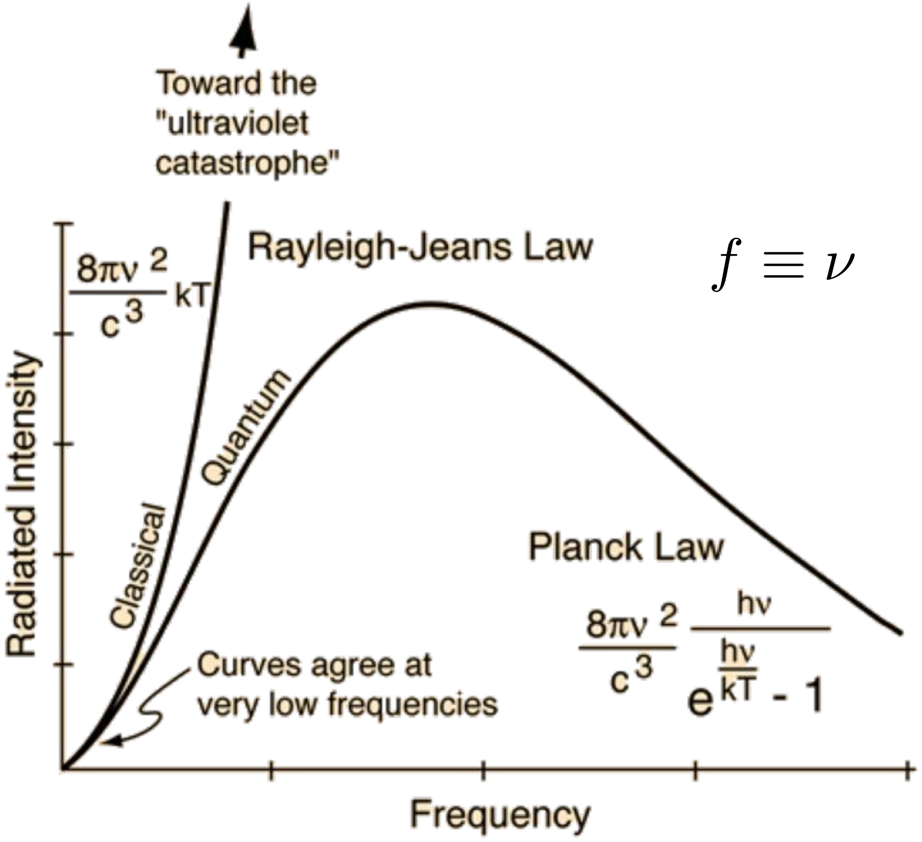
$$Z = 1 + e^{-\beta hf} + e^{-2\beta hf} + \ldots$$
$$= \frac{1}{1 - e^{-\beta hf}}$$

Average energy:

$$\bar{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{hf}{e^{\beta hf} - 1}$$

So if we have energy units of hf, then the average number of units of energy in the oscillator:

$$\bar{n}_P = \frac{1}{e^{\beta hf} - 1}$$




$$f \equiv \nu$$

$$E_n = 0, hf, 2hf, \ldots$$

h = 6,63 $10^{-34}$ J s

# The Planck Distribution (3)

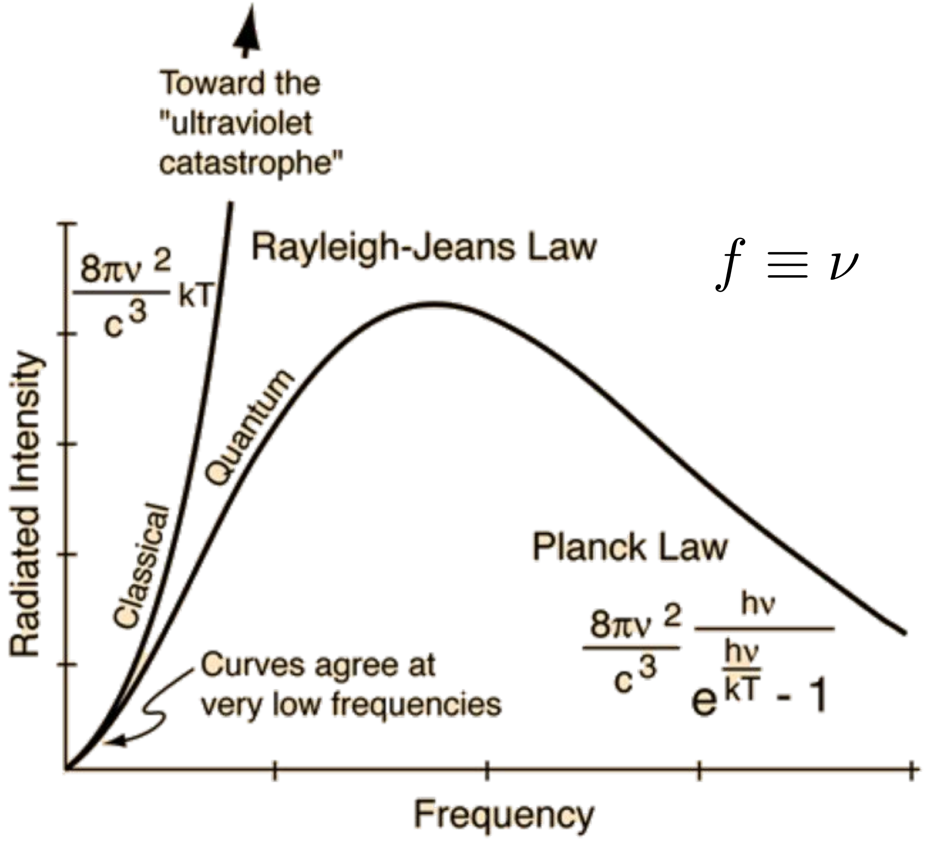
Consequently **Planck Law** for the radiated intensity of black-body is:

$$\rho(f)df = \frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/kT}-1}df$$

Short wavelengths are surpressed exponentially.

Ultraviolet catastrophe does not occur.

The energies asre quantized.



# Photons

The units of energy (hf in Planck distribution) can be though of photons.

Photons are bosons so they follow Bose-Einstein distritribution:

$$\boxed{\bar{n}_{BE} = \frac{1}{e^{(\epsilon-\mu)/kT} - 1}}$$

Energy for photons is $\varepsilon = hf$. But what is $\mu$ for photons?

When comparing with Plancks distribution we should have $\mu$=0.

Does it make sense?

$$\bar{n}_P = \frac{1}{e^{\beta hf} - 1}$$

# Photons – $\mu$

**The chemical potential of photons is zero.**

Photons are created/destroyed in any quantity – their number is not conserved.

Helmholtz free energy *F,* must be minimum when *T,V* are fixed. But *N* is not constrained – if it changes a bit, *F* should not change:

$$\left(\frac{\partial F}{\partial N}\right)_{T,V} = 0$$

$$\left(\frac{\partial F}{\partial N}\right)_{T,V} = \mu$$

On the other hand if a photon is absorbed/created by electron, the equilibrium condition for such a reaction is equivalent to :

$$e \longleftrightarrow e + \gamma$$

$$\mu_e = \mu_e + \mu_\gamma$$

The Bose-Einstein distribution is equivalent to the Planck distribution.

# Photons – total energy

$$\bar{n}_P = \frac{1}{e^{\beta h f} - 1}$$

Distribution tells us how many photons there are in a given mode. But how many photons / or how much energy are there in total in a box?

**Energy**:

Allowed wavelengths and momenta:

$$\lambda = \frac{2L}{n}; \quad p = \frac{hn}{2L}$$

$$\epsilon = pc = \frac{hcn}{2L}$$

Photon energies are pc not $p^2/2m$ (relativistic particles).

$$\epsilon = \sqrt{p_x^2 + p_y^2 + p_z^2} = \frac{hc}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

In 3D we need to consider all directions:

$$\epsilon = \frac{hc}{2L}\|\vec{n}\|$$

# Photons – total energy (2)

The average energy in the mode is e times occupancy of that mode.

$$\bar{\epsilon} = \epsilon n_P(\epsilon)$$

Total energy is from summing over all modes.

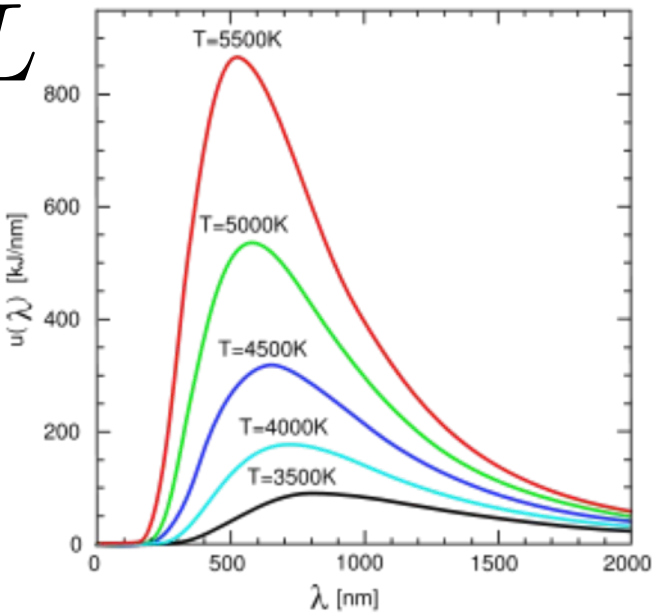
$$U = 2 \sum_{n_x, n_y, n_z} \epsilon \bar{n}_P(\epsilon) = \sum_{n_x, n_y, n_z} \frac{hcn}{L} \frac{1}{e^{hcn/2LkT} - 1}$$

And if the number of modes is large, we can go over to integrals (and spherical coordinates.

$$U = \int_0^{\infty} dn \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi n^2 \sin\theta \frac{hcn}{L} \frac{1}{e^{hcn/2LkT} - 1}$$

Note that now we integrate to infinity...

# Let us change variables: $\epsilon = hcn/2L$




And express energy per unit volume:

$$\frac{U}{V} = \int_0^\infty \frac{8\pi\epsilon^3}{(hc)^3} \frac{1}{e^{\epsilon/kT} - 1} d\epsilon$$

This gives us a spectrum (energy density per unit photon energy):

$$u(\epsilon) = \frac{8\pi}{(hc)^3} \frac{\epsilon^3}{e^{\epsilon/kT} - 1}$$

Which we also derived earlier today:

$$\rho(f)df = \frac{8\pi f^2}{c^3} \frac{hf}{e^{hf/kT} - 1} df$$

Peak of the spectrum: $\varepsilon_{max}=2.82kT$.

Temperature determines the spectrum (Wiens displacement law).

# Total energy

Can be relatively easily found from:

$$\frac{U}{V} = \int_0^\infty \frac{8\pi\epsilon^3}{(hc)^3} \frac{1}{e^{\epsilon/kT} - 1} d\epsilon$$

We just need to integrate it (non-trival matter), which gives total energy density:

$$\frac{U}{V} = \frac{8\pi^5 (kT)^4}{15(hc)^3}$$

The average energy of photon ~ *kT*. Total energy ~*NkT*. Total energy is thus ~*C VkT*, where *C* is a constant (because *N* ~ *V*).

# Entropy

As in the case of fermions we can find heat capacity it from:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

Which this time gives:

$$C_V = 4\frac{8\pi^5 k^4 V}{15(hc)^3}T^3$$

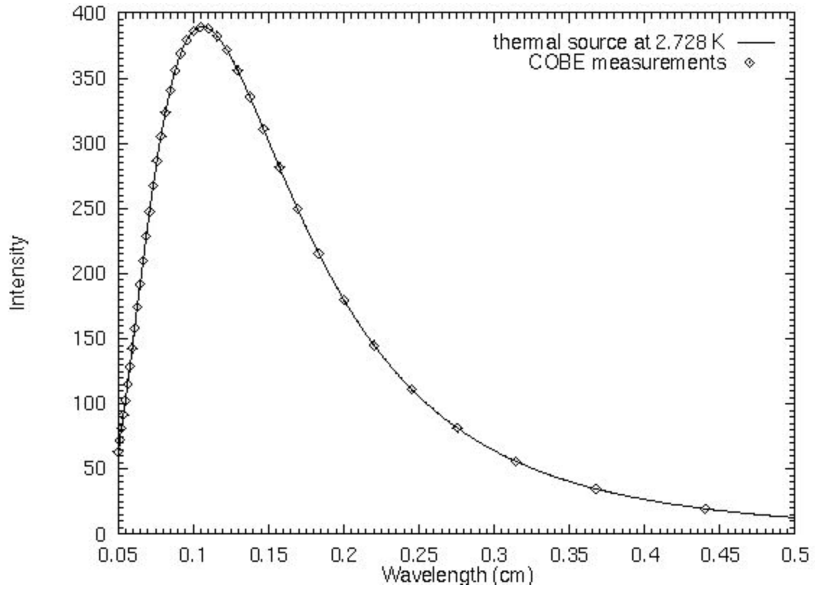
We can also find absolute entrophy by integrating $C_V$ over all temperatures:

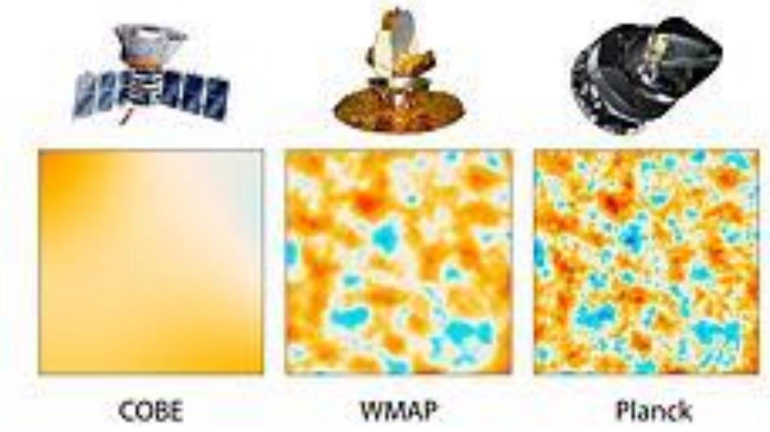
$$S(T) = \int_0^T \frac{C_V(T')}{T'}dT' = \frac{32\pi^5}{45}V\left(\frac{kT}{hc}\right)^3 k$$

It can be shown that it is proportional to the total number of photons.

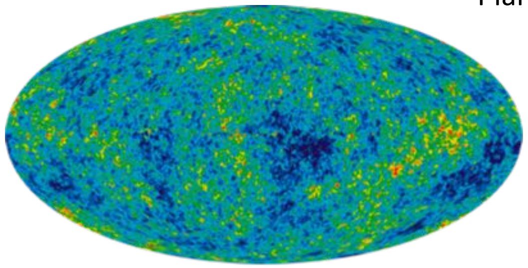
# Cosmic Background Radiation

- Cosmic microwave background radiation – leftover from the early state of the universe.
- The universe was then filled with ionised gas interacting with EM radiation – temperature was then 3000 K.
- Expansion of the universe Doppler shifted the wavelenght.
- Currently – observed «perfect thermal spectrum» at T= 2.7260±0.0013 K.
- Photons energy have a peak at $\varepsilon$=2.82 $kT$=6.6 x $10^{-4}$ eV (Wiens law) – far infrared (mm wavelengts) – best observed from space.
- Total energy: 0.26 MeV/$m^3$. – very small.




Cosmic Background Explorer (NASA, 1989-93)
Wilkinson Microwave Anisotropy Probe (NASA, 2001-10)
Planck (ESA, 2009-13)



# Stefan-Boltzmann law

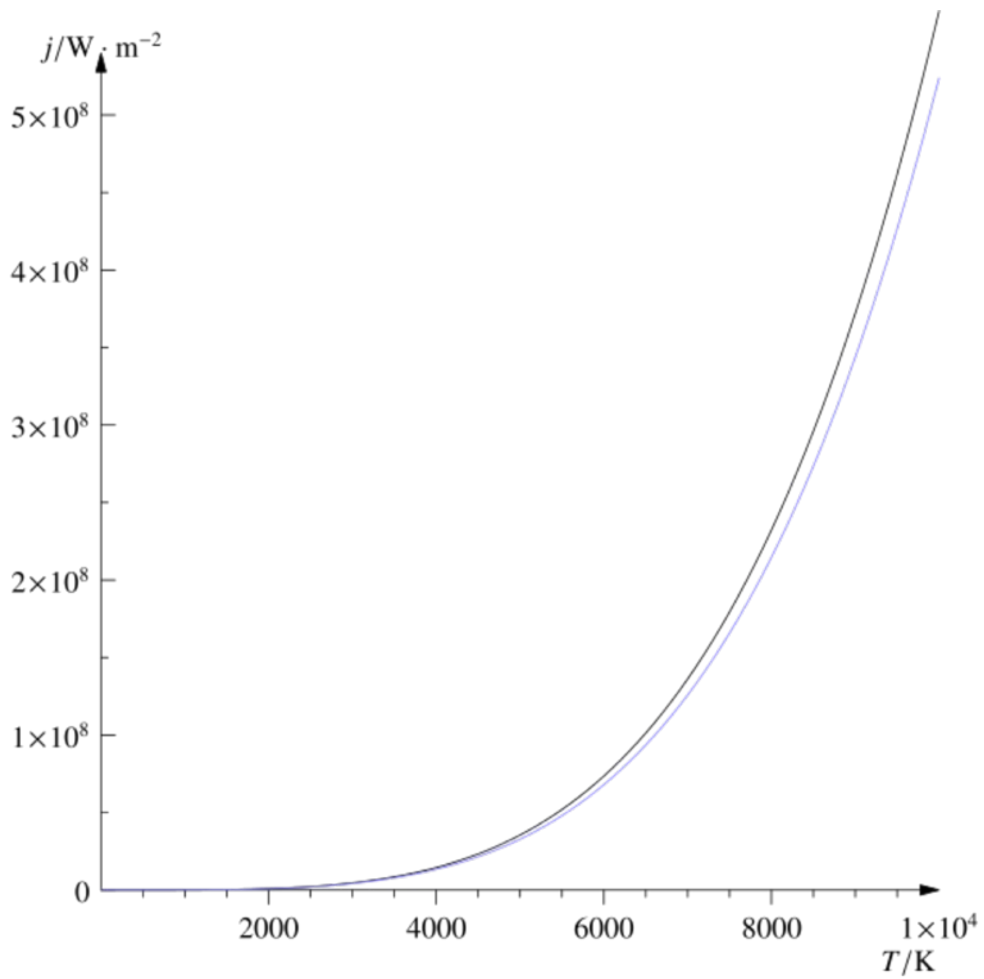
Tells us about the power emitted per unit area:

$$P_{esc} = \sigma T^4$$

where Stefan-Boltzmann constant is:

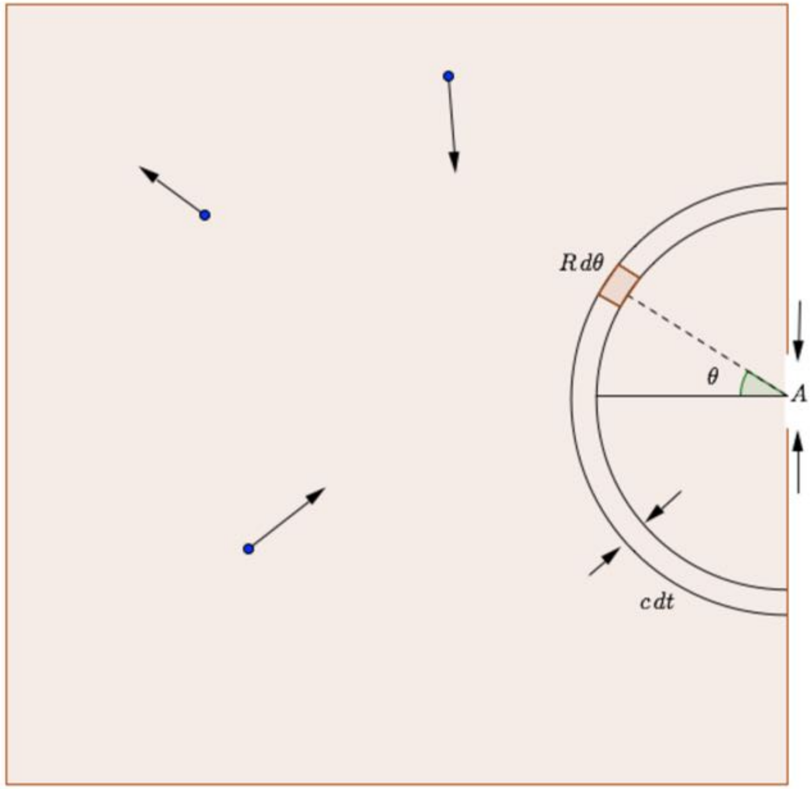
$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3} = 5.670373 \times 10^{-8}\,\mathrm{W\,m^{-2}K^{-4}}$$

It can be easily determined by considering radiation escaping a box.



# Stefan-Boltzmann law (2)

Let us consider an opening of radius A.




The volume colored in $Rd\theta$ is: $Rd\theta \times R\sin\theta d\phi \times cdt$

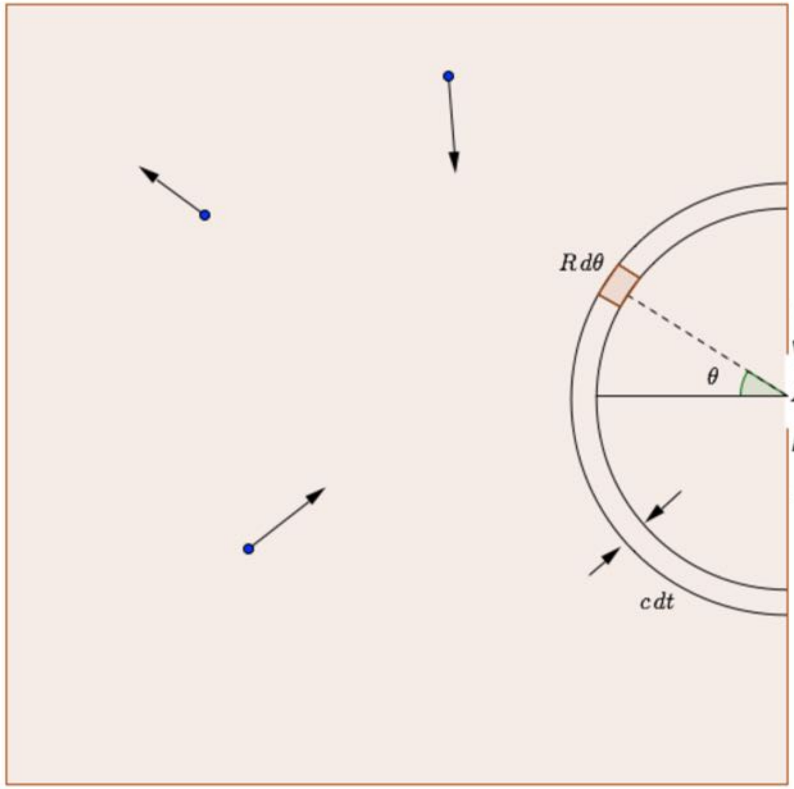
Energy density is: $\frac{U}{V} = \frac{8\pi^5 (k_{\mathrm{B}}T)^4}{15(hc)^3},$

And the total energy in $Rd\theta$: $\frac{U}{V} cdtR^2 \sin\theta d\theta d\phi$

Probability of escaping: $\frac{Acos\theta}{4\pi R^2}$

So the total energy escaping: $\frac{Acos\theta}{4\pi}\frac{U}{V} cdt\sin\theta d\theta d\phi$

# Stefan-Boltzmann law (3)




Finally we need to integrate

$$\frac{A\cos\theta}{4\pi}\frac{U}{V}cdt\sin\theta d\theta d\phi$$

over all angles, and obtain:

$$E_{tot} = \frac{A}{4}\frac{U}{V}cdt$$

This depends on the area *A*, so by dividing we obtain power per unit area:

$$P_{esc} = \frac{c}{4}\frac{U}{V}c$$

To get the Stefan Boltzman law we need to instert *U/V*:

$$\frac{U}{V} = \frac{8\pi^5(k_{\mathrm{B}}T)^4}{15(hc)^3},$$

Applicable to the black-body – perfect emitter / absorber.

But also can be appiled to a body with emissivity/albedo *e*

$$P_{esc} = \sigma e T^4$$

# Examples

**The Sun**

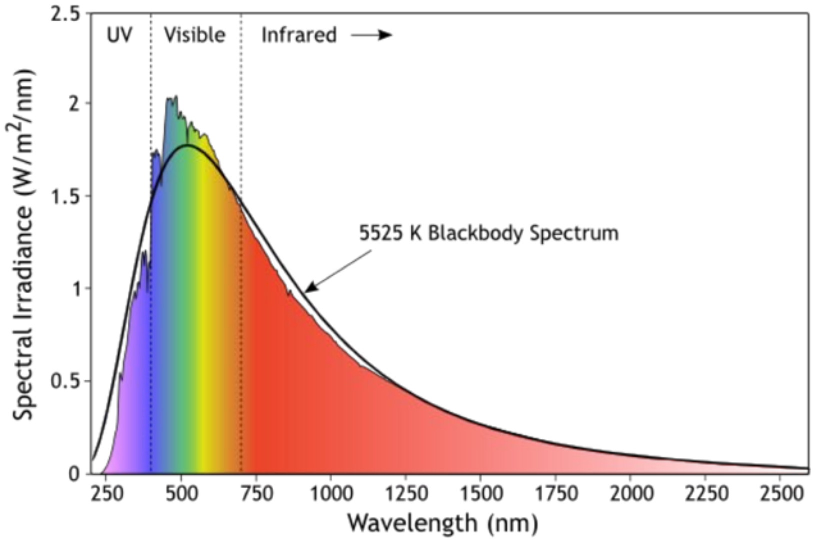
Solar radiation received by the earth $P_E$=1370 W/m$^2$ (solar constant) gives luminosity of $L$=3.9 x $10^{26}$ W.

Here we used that $P_S=4\pi R^2 P_E$, and that distance to the Sun is 150 Mkm, $A$ surface of the sun: 6.1 x $10^{18}$m$^2$.

Solar surface temperature:

$$T = \left(\frac{L}{\sigma A}\right)^{1/4} = \left(\frac{3\times 10^{26}}{\sigma 6.1\times 10^{18}}\right)^{1/4} = 5800K$$

Thus the maximum of the spectrum is at (Wienn's law):
2.82 $kT$ = 1.41 eV (infrared).



# Examples

**The Earth**

Solar energy absorbed by Earth: $P_E\,\pi R^2$

Energy emitted by Earth: $4\pi R^2\sigma T^4$

Energy absorbed = energy emitted:

$$T = \left(\frac{1370\mathrm{W/m^2}}{4\cdot 5.67\times 6.1\times 10^{-8}\mathrm{W/m^2}}\right)^{1/4} = 279K$$

With the clouds – albedo included: *T*=255 K.

Finally when we add reflection of emitted energy: *T*=303 K (greenhouse effect)