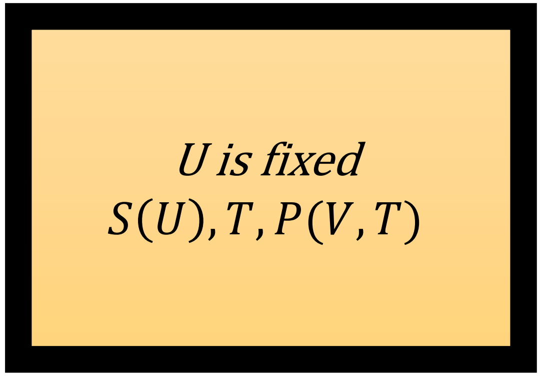
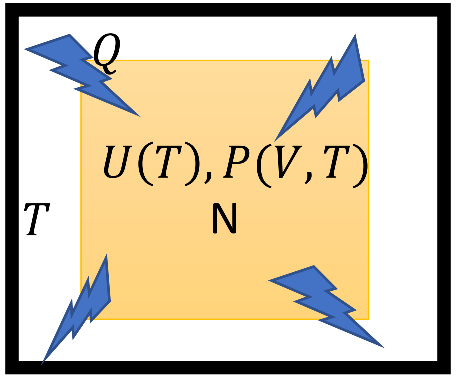
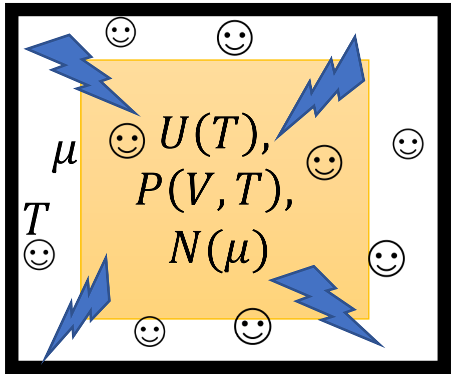






Summary Part 1

19.11.2018

Equilibrium statistical systems

Isolated system at equilibrium

U is fixed
$S(U), T, P(V, T)$

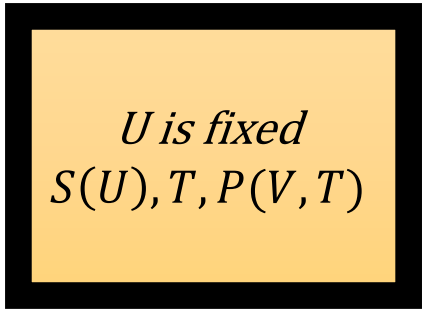
THE EQUIBRIUM STATE IS A MACROSTAE

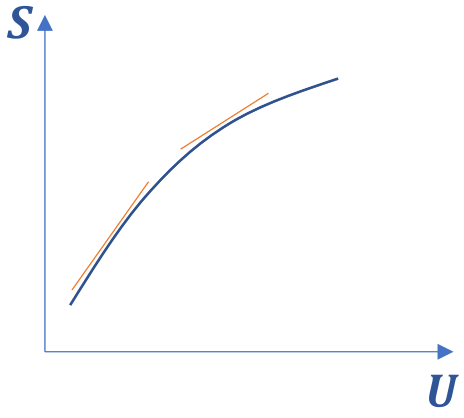
1. WHAT IS THE MULTIPLICITY OF A MACROSTATE?

2. WHAT IS THE ROLE OF ENTROPY?

3. WHAT IS THE CONDITION FOR EQUILIBRIUM?

Isolated system at equilibrium





Multiplicity of a macrostate $\Omega(U,V,N)$ **counts** all **equally-likely** accessible microstates

However, if the particles are **indistinguishable** the total number of accessible microstates is reduced by the number of permutations $\boldsymbol{N}!$

$$\boldsymbol{\Omega(U,V,N)} \to \frac{\boldsymbol{\Omega(U,V,N)}}{\boldsymbol{N}!}$$

Probability that the system is in a *specific microstate*

$$P(s) = \frac{1}{\Omega(U,V,N)}$$

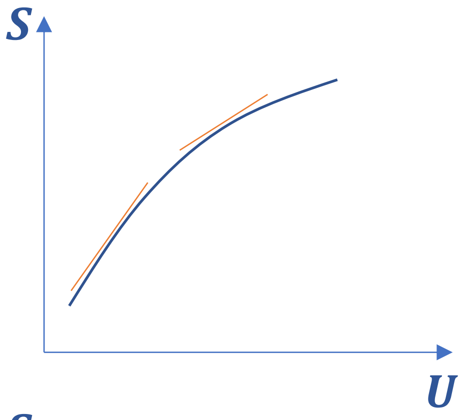
Boltzmann's formula: Entropy of an equilibrium state at fixed U

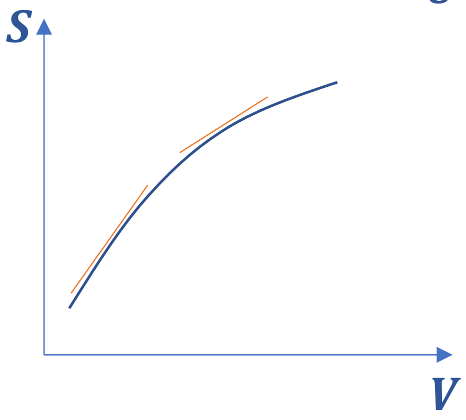
$$S(U,V,N) = k \ln \Omega(U,V,N) \leftrightarrow \mathrm{S(U,V,N)} = -k \sum_s P(s)\, ln\, P(s)$$

Entropy is maximized for an equilibrium state $dS = 0$

THERMODYNAMIC PROPERTIES

U is fixed
$S(U), T, P(V,T)$





Thermodynamic identity for S

$$dS = \frac{1}{T}dU + \frac{P}{T}dV - \frac{\mu}{T}dN$$

Temperature of an equibrium state measures the tendency of the system to give or accept energy

$$T = \left(\frac{\partial S}{\partial U}\right)^{-1}_{V,N}$$

Pressure is the measures the tendency of a system to expand or contract

$$P = T\left(\frac{\partial S}{\partial V}\right)_{U,N} \equiv -\left(\frac{\partial U}{\partial V}\right)_{S,N} \rightarrow P = P(V,T,N)\ equation\ of\ state$$

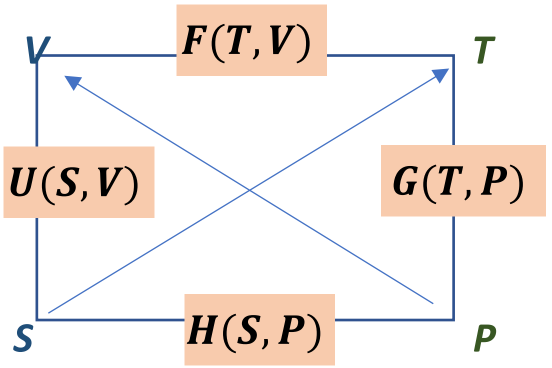
Chemical potential is the measures the tendency of a system to give or take particles

$$\mu = -T\left(\frac{\partial S}{\partial N}\right)_{U,V} \equiv \left(\frac{\partial U}{\partial N}\right)_{S,V}$$

THERMODYNAMIC PROPERTIES

U is fixed

$S(U), T, P(V,T)$



<u>Helmholtz free energy</u>

$$F = U - TS$$

<u>Enthalpy</u>

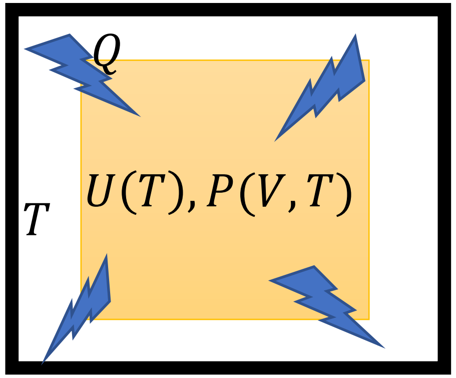
$$H = U + PV$$

<u>Gibbs free energy</u>

$$G = U - TS + PV = N\mu$$

Chemical potential is the energy increase by adding a particle in to the system when the pressure and temperature are constant.

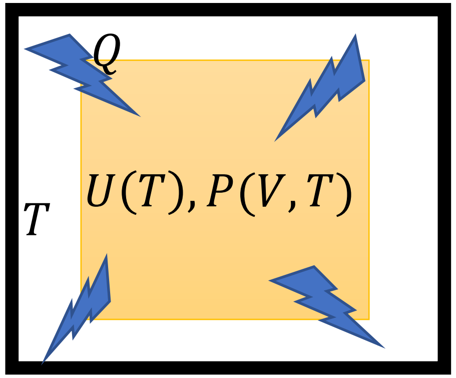
$$\mu = \left(\frac{\partial G}{\partial N}\right)_{T,P}$$

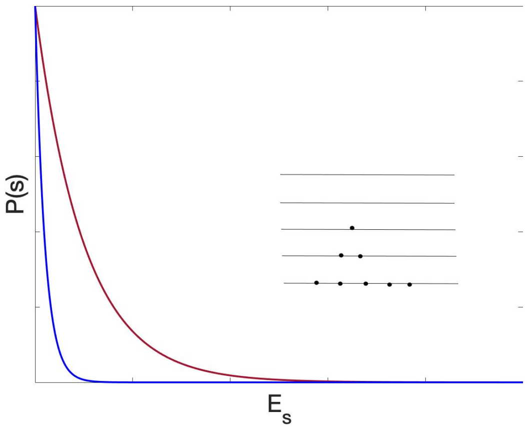


A system in contact with a thermal bath

1. WHAT IS THE PARTITION FUNCTION?
2. WHAT ARE THE FLUCTUATING QUANTITIES?
3. WHAT IS THE EQUILIBRIUM CONDITION?
4. WHAT IS THE ROLE OF ENTROPY?

A system in contact with a thermal bath





System+Thermal bath = isolated system

The probability that the system is in a given microstate is proportional to the probability that the thermal bath is in *any state that accomodate that particular microstate (hence the total number of microstates of the thermal bath corresponding to systems' microstate)*

Probability ratio between two microstates (the system can exchange energy with the thermal bath $\Delta U_R = -\Delta E$*)*

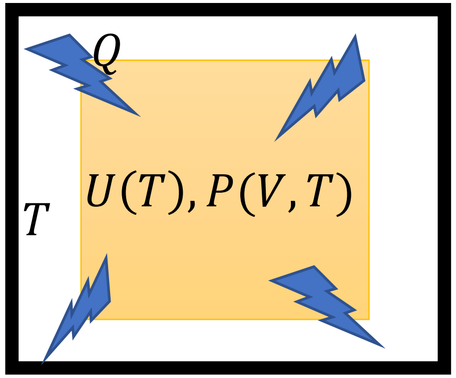
$$\frac{P(s_1)}{P(s_2)} = \frac{\Omega_R(s_1)}{\Omega_R(s_2)} = e^{\frac{[S_R(s_1)-S_R(s_2)]}{k}} = e^{\frac{[U_R(s_1)-U_R(s_2)]}{kT}} = e^{-\frac{[E(s_1)-E(s_2)]}{kT}}$$

Probability of the system in a specific microstate a fixed temperature T

$$P(s) = \frac{1}{Z(T)} e^{-\frac{E_s}{kT}}$$

Boltzmann partition function

$Z(T) = \sum_s e^{-\frac{E_s}{kT}}$ **counts all the accessible microstates weighted by the Boltzmann factor**



A system in contact with a thermal bath

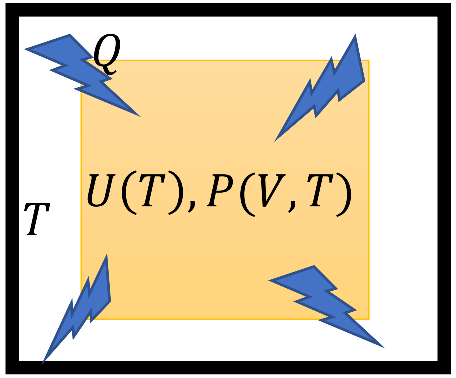
$Z(T) = \sum_s e^{-\frac{E_S}{kT}}$ counts all the accessible microstates weighted by the Boltzmann factor

N-***distinguishable, identical and independent*** classical particles

$$Z_N(T,V) = \sum_{\{s_1, s_2 \cdots s_N\}} e^{-\frac{E_N(s_1, \cdots s_N)}{kT}} = Z_1^N(T,V)$$

N-***indistinguishable, identical and independent*** classical particles

$$Z_N(T,V) = \sum_{\{s_1, s_2 \cdots s_N\}} e^{-\frac{E_N(s_1, \cdots s_N)}{kT}} = \frac{1}{N!} Z_1^N(T,V)$$



THERMODYNAMIC PROPERTIES AND AVERAGES

$$Z(\beta) = \sum_s e^{-\beta E_s}, \beta = \frac{1}{kT}$$

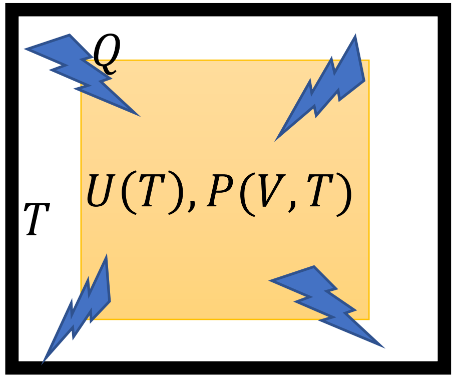
Due to energy exchange with the thermal bath, the energy fluctuations from one microstate to another. Thus, the total energy of an equilibrum macrostate is an average

$$U(T,V,N) = \langle E_s \rangle = \sum_s E_s\, e^{-\beta E_s} = -\frac{1}{Z}\left(\frac{\partial Z}{\partial \beta}\right)_{V,N} \quad = -\left(\frac{\partial \ln Z}{\partial \beta}\right)_{V,N}$$

$$\langle E_s^2 \rangle = \sum_s E_s^2\, e^{-\beta E_s} = \frac{1}{Z}\left(\frac{\partial^2 Z}{\partial \beta^2}\right)_{V,N}$$

How entropy relates to the probability of a microstate

$$S = -k \sum_s P(s) \ln P(s)$$



HELMHOLTZ FREE ENERGY

$$Z(T,V,N) = \sum_s e^{-\beta E_s}, \beta = \frac{1}{kT}$$

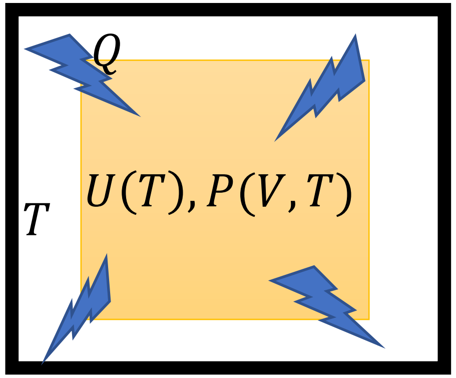
The partition function determines the thermodynamic potential which ***minimized*** *at a given T, V and N.*

Helmholtz free energy

$$\mathbf{F(T,V,N) = -kT \ln \boldsymbol{Z(T,V,N)} \leftrightarrow \boldsymbol{Z = e^{-\beta F}}}$$

Thermodynamic identity

$$\boldsymbol{dF = -S\, dT - PdV + \mu dN}$$



THERMODYNAMIC PROPERTIES

Thermodynamic identity

$$\boldsymbol{dF = -S\,dT - PdV + \mu dN}$$

<u>Entropy</u>

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$$

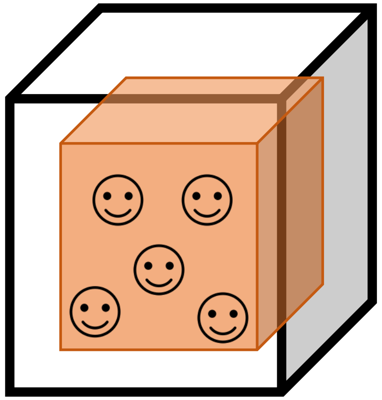
<u>Pressure</u>

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} \rightarrow P = P(V,T,N)\ equation\ of\ state$$

<u>Chemical potential</u>

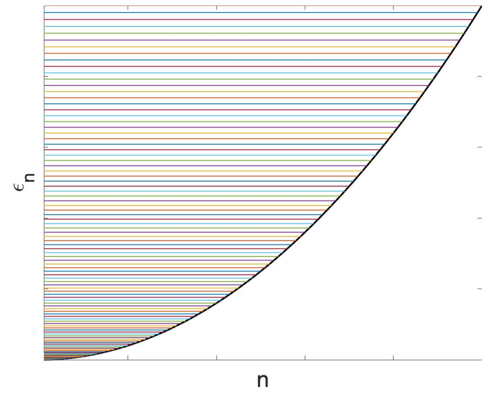
$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$$

Ideal gas in a thermal bath (High T-classical limit)



- Independent and indistinguishable quantum particles
- Quantum state of 1 particle is given by the quantized energy levels and the corresponding wavefunction (the energy is associated with a wavefunction rather then the particle itself!)

$$\epsilon_n = \frac{\vec{p}\cdot\vec{p}}{2m} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2), \qquad \boldsymbol{n_x, n_y, n_z = 0, 1, 2, \cdots}$$



- One-particle partition function (3D)

$$Z_1(T,V) = \sum_{n_x}\sum_{n_y}\sum_{n_z} e^{-\beta\frac{h^2}{8mL^2}(n_x^2+n_y^2+n_z^2)},$$

$$Z_1(T,V) = \left(\sum_n e^{-\beta\frac{h^2}{8mL^2}n^2}\right)^3 \approx_{\substack{n\gg 1\\ high\,T}} \frac{1}{2}\int_{-\infty}^{\infty} dn\ e^{-\beta\frac{h^2}{8mL^2}n^2} = \frac{V}{\Lambda^3(T)}, \qquad \boldsymbol{\Lambda(T) = \sqrt{\frac{h}{2\pi mkT}}} \textbf{ (quantum length)}$$

- N-particle partition function (3D)

$$Z_N(T,V) = \frac{\mathbf{1}}{\boldsymbol{N}!}\left(\frac{V}{\Lambda^3(T)}\right)^N$$

Maxwell-Bolzmann distribution

- Probability that the particle in a state with velocity vector

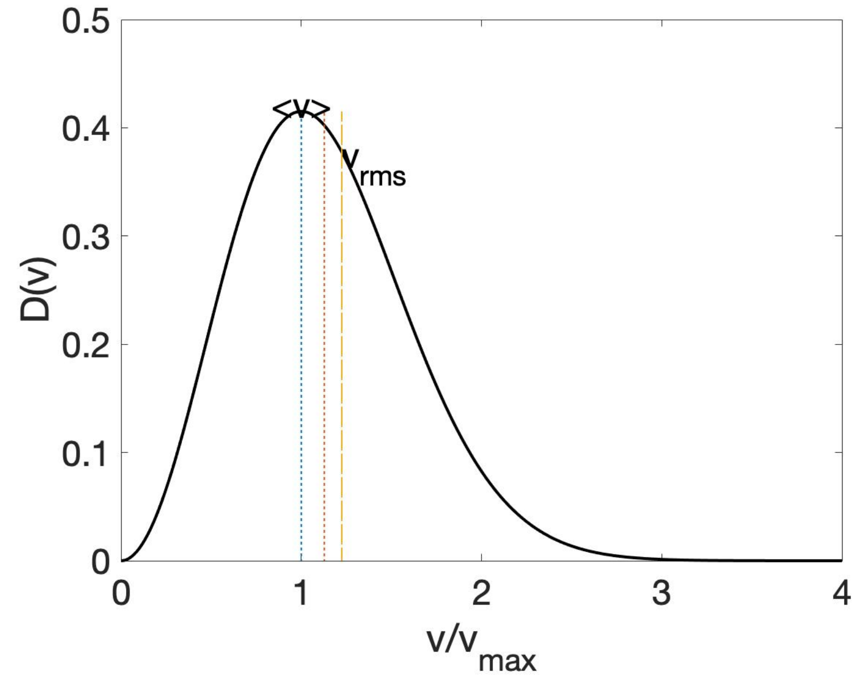
$$P_{3D}(\vec{V}) \sim e^{-\beta m \frac{\vec{V}\cdot\vec{V}}{2}}$$

- Probability that a particles has a *speed* between v and $v + dv$ $(v = |\vec{V}|)$

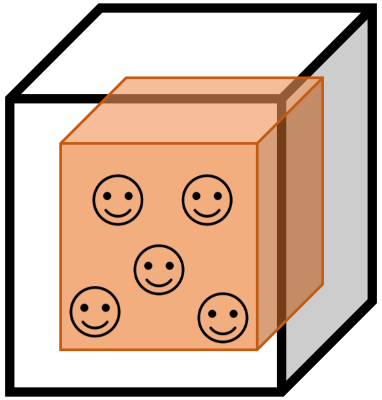
$$\boldsymbol{D^{(3D)}(v)dv} \sim P_{3D}(\vec{V})dV_x dV_y dV_z = e^{-\beta m \frac{v^2}{2}} 4\pi v^2 dv$$

$$\boldsymbol{\int_0^\infty dv\ D^{(3D)}(v) = 1}$$

$$\boldsymbol{D^{(3D)}(v) = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{m}{2kT}v^2}}$$



N-free particles in a thermal bath



- N-particle partition function

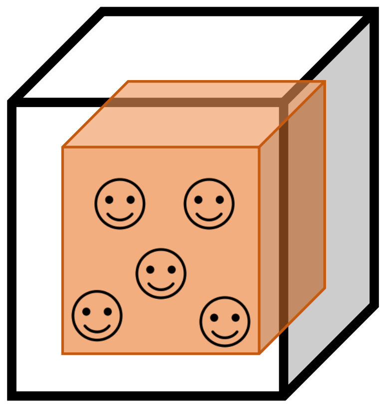
$$Z_N(T,V) = \frac{Z_1^N}{N!} = \frac{1}{N!}\left(\frac{V}{\Lambda^3(T)}\right)^N$$

- Helmholtz free energy

$$F_N(T,V) = -kT\, ln\, Z_N(T,V) = -NkT\left[ln\left(\frac{Z_1}{N}\right) - 1\right]$$

$$F_N(T,V) = -NkT\left[ln\left(\frac{V}{N\Lambda^3(T)}\right) - 1\right]$$

N-free particles in a thermal bath



- N-particle partition function

$$Z_N(T,V) = \frac{Z_1^N}{N!} = \frac{1}{N!}\left(\frac{V}{\Lambda^3(T)}\right)^N, \qquad \Lambda(T) = \sqrt{\frac{h}{2\pi mkT}}$$

- Energy energy

$$U = -\frac{\partial}{\partial\beta} ln\, Z_N(T,V) = 3N\frac{d}{d\beta}\, ln\, \Lambda(\beta) = \frac{3\mathrm{N}}{2}\mathrm{kT}$$

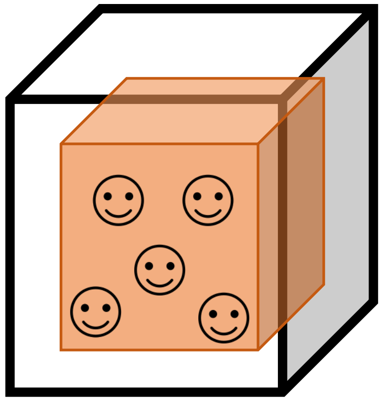
- Entropy

$$S = \frac{U-F}{T} = \frac{3\mathrm{N}k}{2} + Nk + Nk\left[ln\left(\frac{V}{N\Lambda^3(T)}\right)\right] = Nk\left[ln\left(\frac{V}{N\Lambda^3(T)}\right) + \frac{5}{2}\right]$$

- **Equation of state**

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{kT}{V}$$

N-free particles in a thermal bath

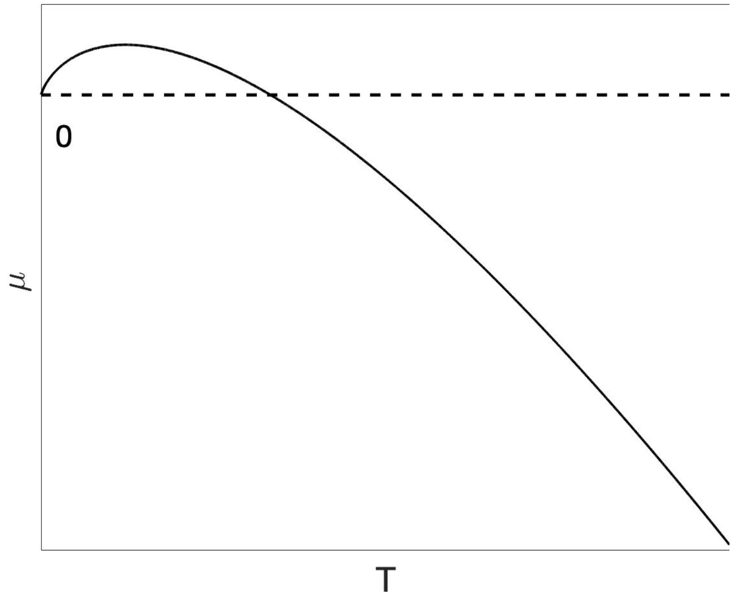


- Helmholtz free energy

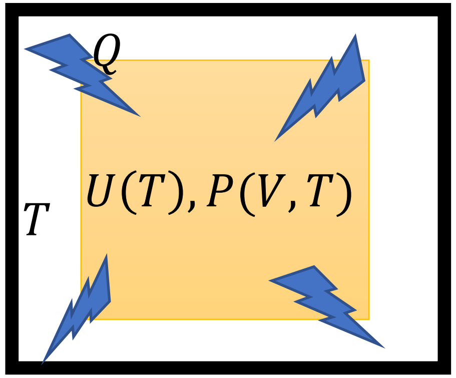
$$F_N(T,V) = -NkT\left[ln\left(\frac{V}{N\Lambda^3(T)}\right) - 1\right]$$

- **Chemical potential**

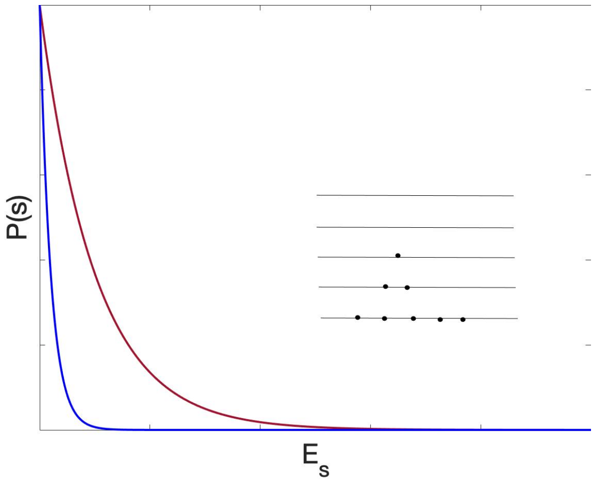
$$\mu(T,V) = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -kT\,ln\left(\frac{V}{N\Lambda^3(T)}\right)$$



$$Z_1 = \frac{V}{\Lambda^3(T)} = Ne^{-\beta\mu}$$



Ideal gas in contact with a thermal bath

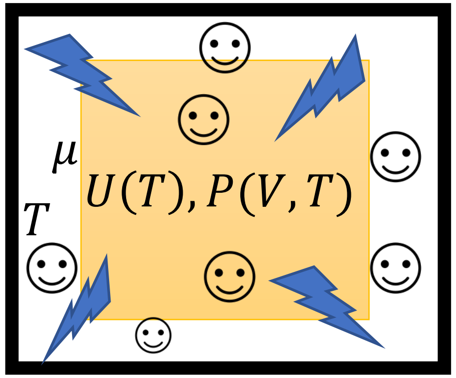


Probability of one particle to be in a in a specific energy state a fixed temperature T

$$P(s) = \frac{1}{Z_1(T)} e^{-\beta E_s}$$

Boltzmann distribution for the average number of particles (occupation number) in a given energy state

$$\langle N_s \rangle = NP(s) = e^{-\beta(E_s - \mu)}$$

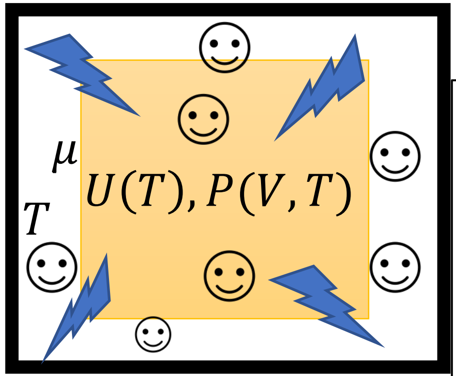


A system in contact with a thermal and particle reservoir

WHAT IS THE GIBBS FACTOR?

WHAT QUANTITIES FLUCTUATES?

WHAT IS THE EQUILIBRIUM CONDITION?



A system in contact with a thermal and particle reservoir

The system can exchange energy and particles with a reservoir and it is in equilibrium at a fixed T and chemical potential $\boldsymbol{\mu}$

Probability of the system being in a given microstate is proportional to the probability that the reservois is in *any state that accomodate that particular microstate (hence the total number of microstates of the thermal bath corresponding to a given system's microstate)*

Probability ratio between two microstates (the system can exchange energy $\Delta \mathrm{U_R} = -\Delta \mathrm{E}$*, and particles* $\Delta \mathrm{N_R} = -\Delta \mathrm{N}$*)*

$$\frac{P(s_1)}{P(s_2)} = \frac{\Omega_R(s_1)}{\Omega_R(s_2)} = e^{\frac{[S_R(s_1)-S_R(s_2)]}{k}} = e^{\beta \Delta U_R} e^{-\beta\mu\Delta N_R} = e^{-\beta\Delta E}\, e^{\,\beta\mu\Delta N}$$

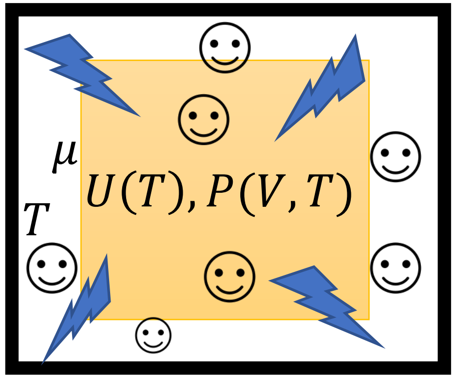
Probability of the system in a specific microstate a fixed T and $\boldsymbol{\mu}$

$$\boldsymbol{P(s) = \frac{1}{\Xi(T,\mu)} e^{-\,\beta(E_s-\mu\, N_s)}}$$

Grand Partition function

$\Xi(T,\mu) = \sum_s e^{-\,\beta(E_s-\mu\, N_s)}$ counts all the accessible microstates weighted by the Gibbs factor

What is the microstate s?



Non-interacting particle system in contact with a thermal and particle reservoir

Each identical particle can occupy discrete energy states ϵ_j, $j = 1, 2, \cdots$ is the state number

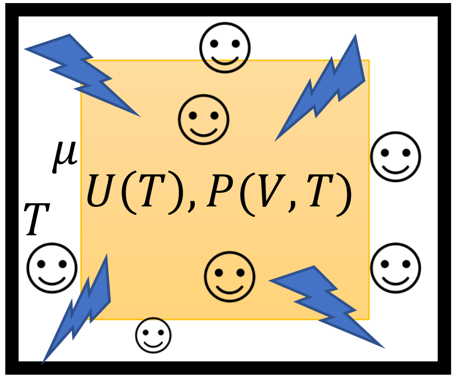
For N identical particles, we can have N_j number of particles (occupation number) in the energy state ϵ_j

The energy of a specific microstate with $N_s = \sum_j N_j$ particles is $E_s = \sum_j N_j \epsilon_j$

$\sum_s$ ≡sum over all particles number N_s and over all the partitions of particles N_s in the quantum states with total energy E_s

$$\Xi(T,\mu) = \sum_{N_s} \sum_{\substack{\{N_j\} \\ \sum_j N_j = N_s}} e^{-\beta(E_s - \mu N_s)} = \sum_{\{N_j\}} e^{-\beta \sum_j N_j(\epsilon_j - \mu)}$$

$$\Xi(T,\mu) = \left(\sum_{N_1} e^{-\beta N_1(\epsilon_1 - \mu)}\right) \cdot \left(\sum_{N_2} e^{-\beta N_2(\epsilon_2 - \mu)}\right) \cdot\cdot \left(\sum_{N_3} e^{-\beta N_3(\epsilon_3 - \mu)}\right) \cdots$$



Occupation number of a state

Probability of the system in a specific microstate a fixed T and $\boldsymbol{\mu}$

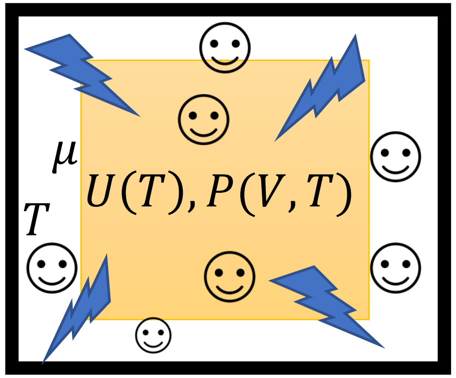
$$P(s) = \frac{1}{\Xi(T,\mu)} e^{-\beta(E_s - \mu N_s)} = \frac{e^{-\beta N_1(\epsilon_1-\mu)} \cdot e^{-\beta N_2(\epsilon_2-\mu)} \cdot e^{-\beta N_3(\epsilon_3-\mu)} \ldots}{\left(\sum_{N_1} e^{-\beta N_1(\epsilon_1-\mu)}\right) \cdot \left(\sum_{N_2} e^{-\beta N_2(\epsilon_2-\mu)}\right) \cdot\cdot \left(\sum_{N_3} e^{-\beta N_3(\epsilon_3-\mu)}\right) \ldots}$$

$$\boldsymbol{P(s) = \frac{e^{-\beta N_1(\epsilon_1-\mu)}}{\left(\sum_{N_1} e^{-\beta N_1(\epsilon_1-\mu)}\right)} \cdot \frac{e^{-\beta N_2(\epsilon_2-\mu)}}{\left(\sum_{N_2} e^{-\beta N_2(\epsilon_2-\mu)}\right)} \cdot \frac{e^{-\beta N_3(\epsilon_3-\mu)}}{\left(\sum_{N_3} e^{-\beta N_3(\epsilon_3-\mu)}\right)} \cdots}$$

$$\boldsymbol{P(s) = P(N_1) \cdot P(N_2) \cdot P(N_3) \cdots}$$

Probability for the occupation number $\boldsymbol{N}$ of the given state at fixed T and $\boldsymbol{\mu}$

$$\boldsymbol{P(N) = \frac{e^{-\beta N(\epsilon-\mu)}}{\left(\sum_N e^{-\beta N(\epsilon-\mu)}\right)}}$$



Non-interacting FERMIONS in contact with a thermal and particle reservoir

The occupation number for each quantum state is $\boldsymbol{N = 0, 1}$

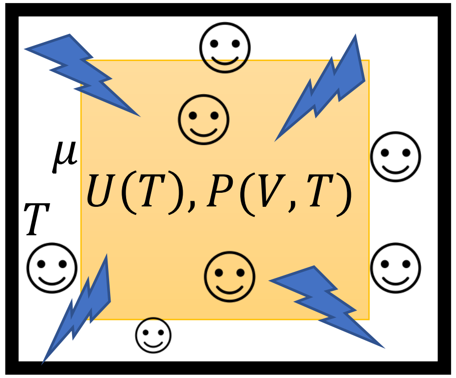
Probability for the occupation number N of the given energy state a fixed T and $\boldsymbol{\mu}$

$$P(N) = \frac{e^{-\beta N(\epsilon-\mu)}}{1 + e^{-\beta(\epsilon-\mu)}}$$

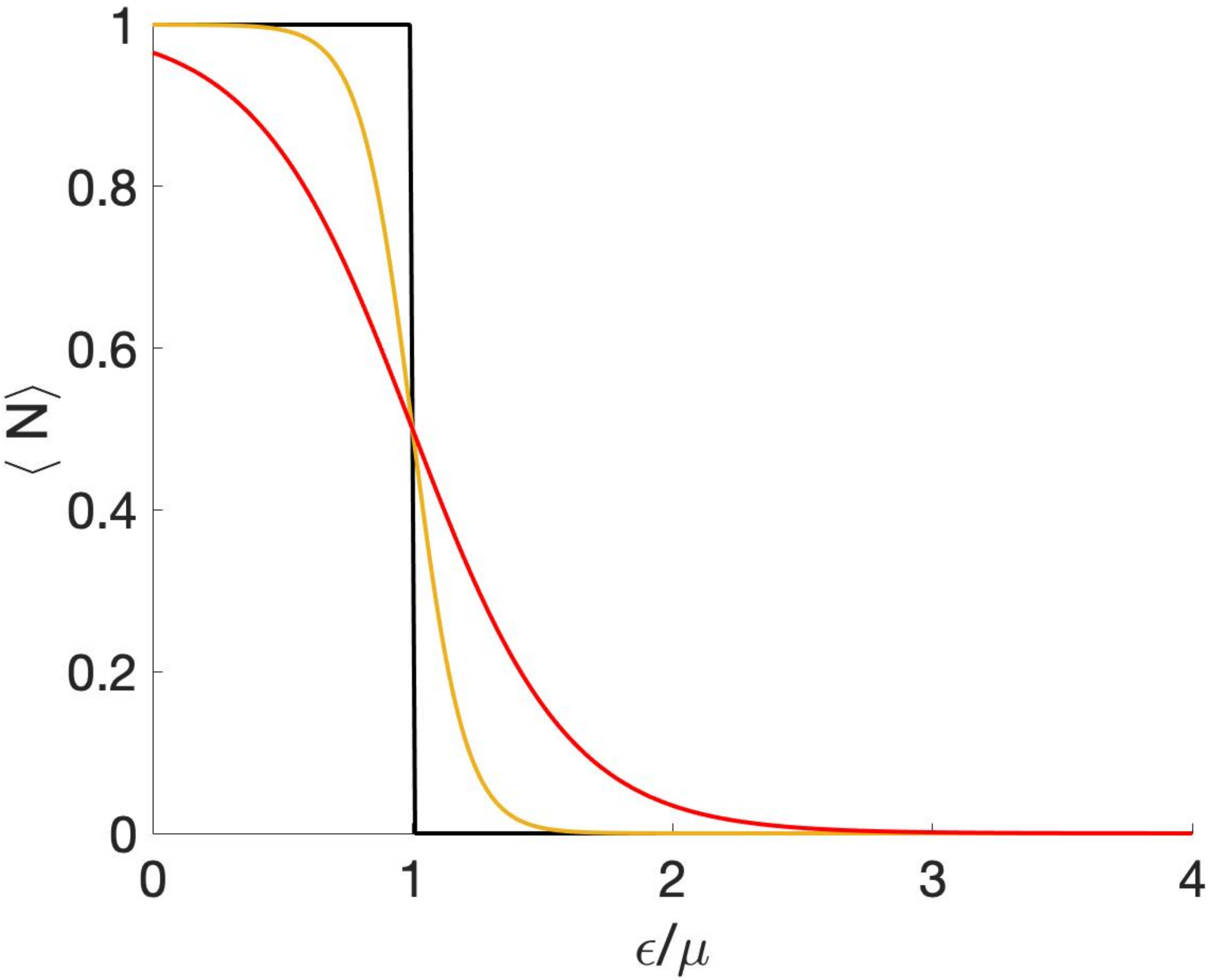
Average occupation number $\langle N \rangle$ of the given energy state ϵ a fixed T and $\boldsymbol{\mu}$

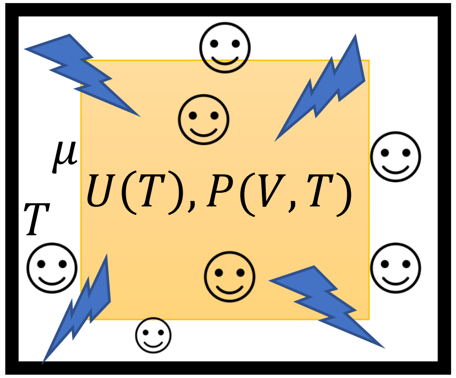
FERMI-DIRAC distribution

$$\langle N \rangle(\epsilon) = \sum_{N=0}^{1} NP(N) = \frac{e^{-\beta(\epsilon-\mu)}}{1 + e^{-\beta(\epsilon-\mu)}} \rightarrow \langle N \rangle(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}$$



Fermi Dirac distribution





Non-interacting BOSONS in contact with a thermal and particle reservoir

The occupation number for each state is $\boldsymbol{N = 0, 1, 2 \cdots}$

$\sum_{N=0}^{\infty} e^{-\beta N(\epsilon-\mu)} = \frac{1}{1-e^{-\beta(\epsilon-\mu)}}, \quad for\ \mu < \epsilon\ \ (for\ every\ \epsilon!)$

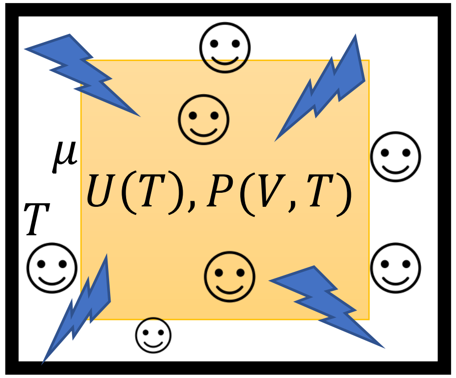
Probability for the occupation number N of the given energy state a fixed T and $\boldsymbol{\mu}$

$$P(N) = \left(1 - e^{-\beta(\epsilon-\mu)}\right) e^{-\beta N(\epsilon-\mu)}$$

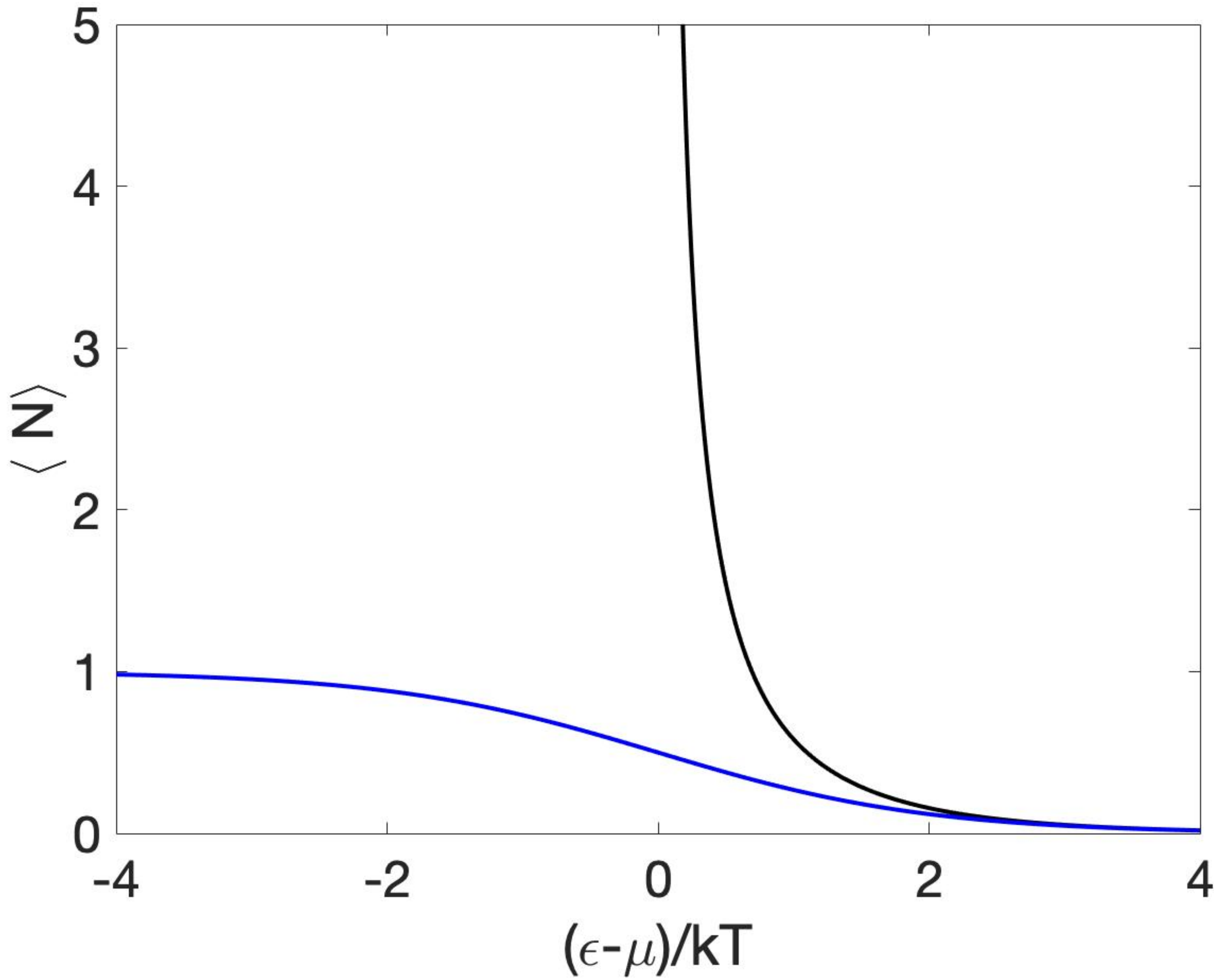
Average occupation number $\langle N \rangle$ of the given energy state ϵ a fixed T and $\boldsymbol{\mu}$

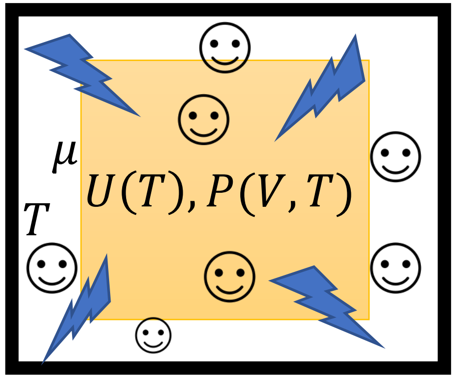
BOSE-EINSTEIN distribution

$$\langle N \rangle(\epsilon) = \sum_{N=0}^{\infty} N P(N) = \left(1 - e^{-\beta(\epsilon-\mu)}\right) \sum_{N=0}^{\infty} N\, e^{-\beta N(\epsilon-\mu)} \rightarrow \langle N \rangle(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}$$



Bose Einstein distribution





Classical limit

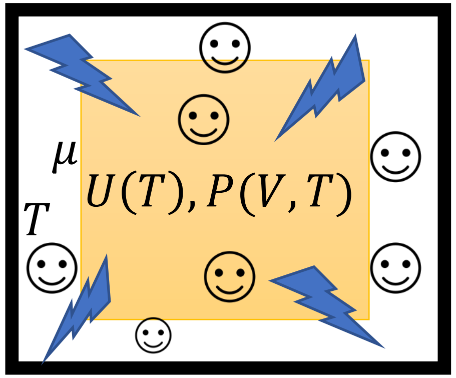
QUANTUM distribution for the average occupation number of an energy state

$$\langle N \rangle(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1}$$

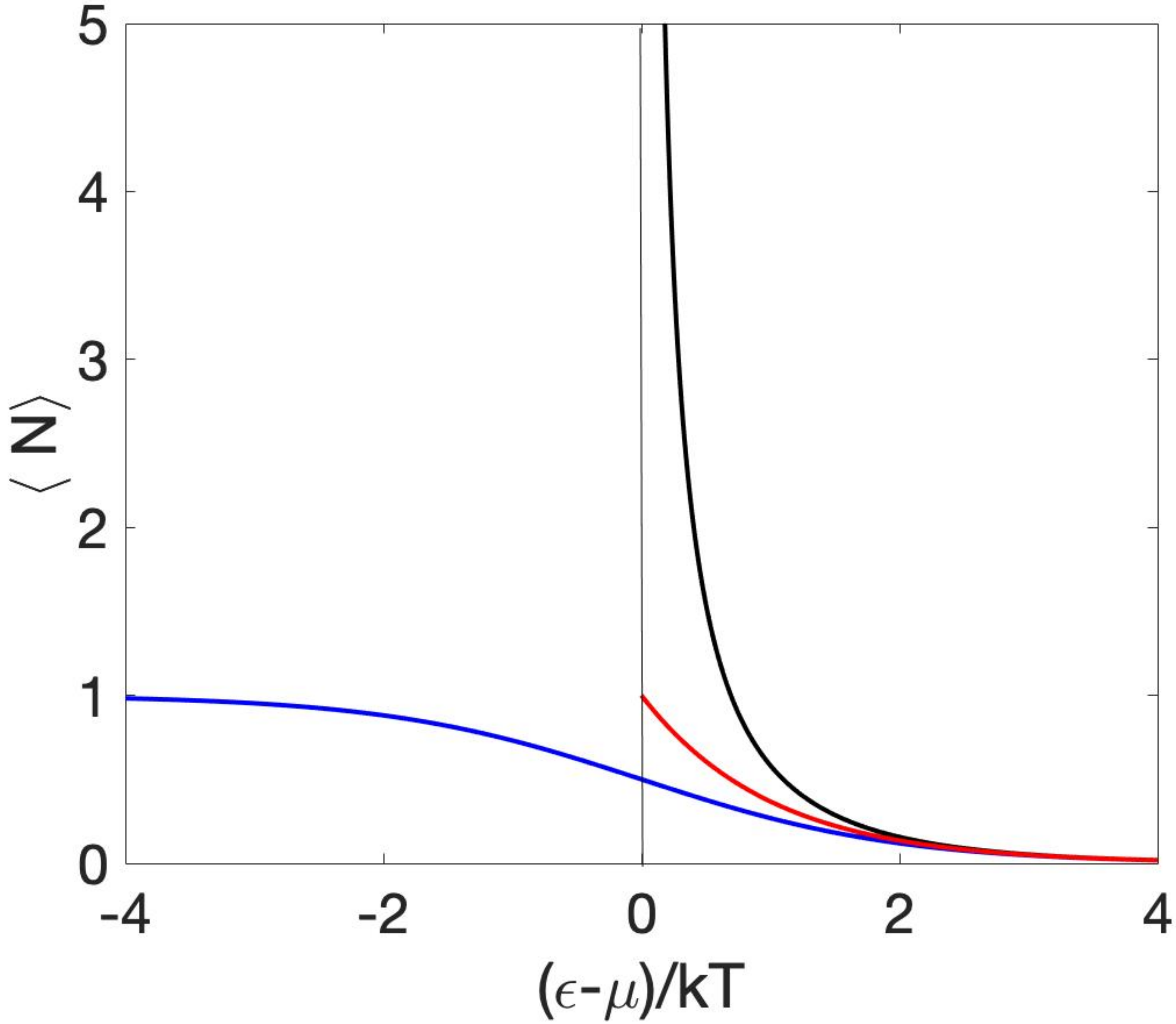
High T limit ($\frac{\mu(T)}{kT} \ll 0$)

BOLZMANN distribution

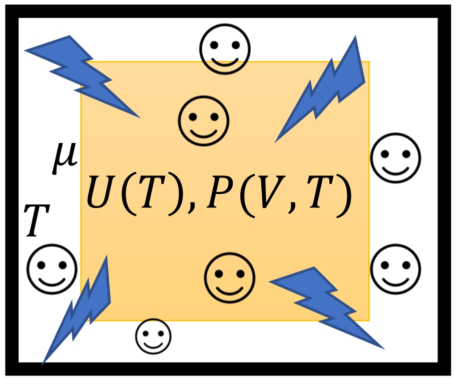
$$\langle \boldsymbol{N} \rangle(\boldsymbol{\epsilon}) = \frac{e^{\beta\mu}}{e^{\beta\epsilon} \pm e^{\beta\mu}} \rightarrow_{e^{\beta\mu} \to 0} \langle \boldsymbol{N} \rangle(\boldsymbol{\epsilon}) = e^{-\beta(\epsilon-\mu)}$$

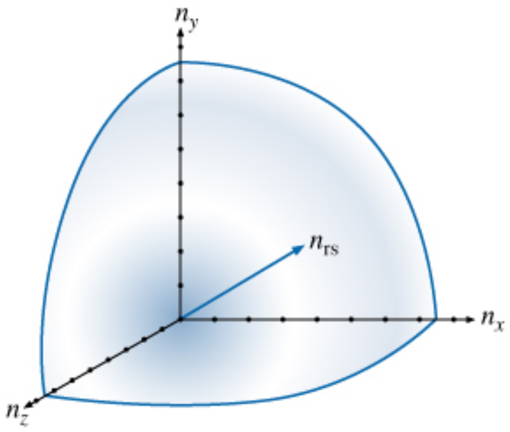


μ
T
$U(T), P(V,T)$



5
4
3
2
1
0
-4
-2
0
2
4
$\langle N \rangle$
$(\epsilon-\mu)/kT$





THERMODYNAMIC PROPERTIES AND DENSITY OF STATES

Average energy

$$U = \sum_{n_x}\sum_{n_y}\sum_{n_z}\langle N\rangle(\epsilon)\cdot\epsilon(n_x,n_y,n_x) = \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z\ \epsilon\cdot\langle N\rangle = \int_0^\infty d\epsilon\ g(\epsilon)\epsilon\cdot\langle N\rangle$$

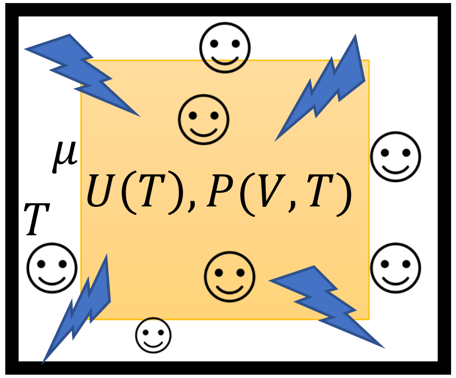
- Density of states $g(\epsilon)$ comes become we need to count all the quantum states at a given energy ϵ. Remember that the quantum state is given by the state of the wavefunction

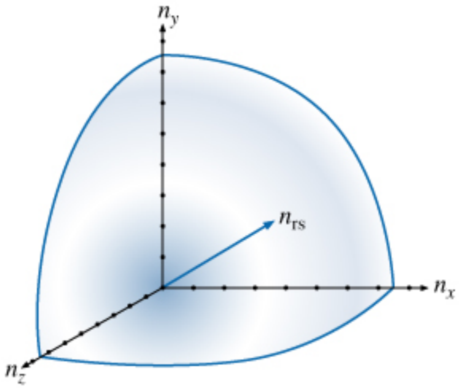
- Number of states with energy between ϵ and $\epsilon + d\epsilon \equiv$ Number of states with state number between n and $n + dn$ (*positive quadrant*)

$$(3D)\ g(\epsilon)\mathrm{d}\epsilon = \frac{1}{8}4\pi n^2 dn, \qquad (2D) g(\epsilon)\mathrm{d}\epsilon = \frac{1}{4}2\pi n dn, (2D), \qquad (1D)\ g(\epsilon)\mathrm{d}\epsilon = dn$$

Energy $\epsilon(n)$ is determined by the *quantum mechanics:*

- *Particle in a box* $\epsilon(n) = \frac{h^2}{8mL^2}n^2$
- *Quantum harmonic oscillator* $\epsilon(n) = n\hbar\omega$
- *Relativistic particles* $\epsilon(n) = hf = \frac{hc}{2L}n$





Density of states

Number of states with energy between ϵ and $\epsilon + d\epsilon \equiv$ Number of states with state number between n and $n + dn$

$$(3D)\ g(\epsilon)\mathrm{d}\epsilon = \frac{\pi}{2} n^2 dn, \qquad (2D) g(\epsilon)\mathrm{d}\epsilon = \frac{\pi}{2} n dn, (2D), \qquad (1D)\ g(\epsilon)\mathrm{d}\epsilon = dn$$

FERMIONS: remember to multiply by factor 2 because there are two electrons per energy level (spin up and spin down)

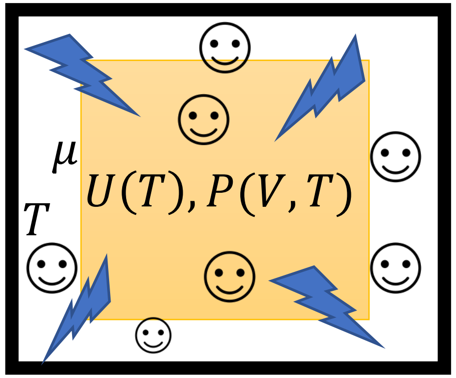
$$(3D)\ g(\epsilon)\mathrm{d}\epsilon = 2\times\frac{\pi}{2} n^2 dn, \qquad (2D) g(\epsilon)\mathrm{d}\epsilon = 2\times\frac{\pi}{2} n dn, (2D), \qquad (1D) g(\epsilon)\mathrm{d}\epsilon = 2\times dn$$

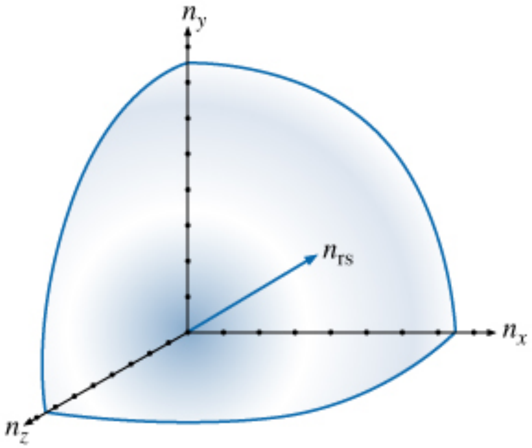
PHOTONS : remember to multiply by factor 2 for the two transverse polarizations of the EM waves

$$(3D)\ g(\epsilon)\mathrm{d}\epsilon = 2\times\frac{\pi}{2} n^2 dn,$$

PHONONS: remember to multiply by factor 3 for the three polarizations of the sound waves

$$(3D)\ g(\epsilon)\mathrm{d}\epsilon = 3\times\frac{\pi}{2} n^2 dn$$





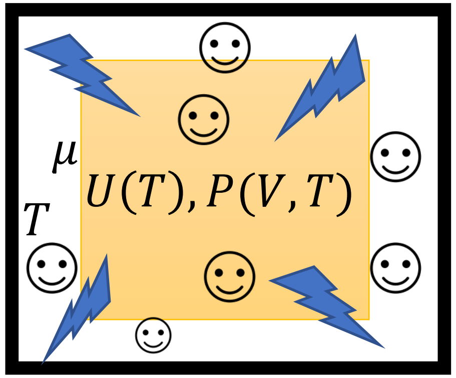
Thermodynamic properties and density of states

Average energy

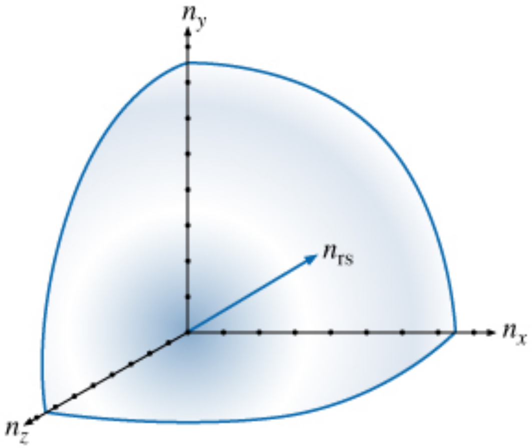
$$U(T,V,\mu) = \int_0^\infty d\epsilon \; g(\epsilon)\, \langle N \rangle\, \epsilon = \int_0^\infty d\epsilon \; g(\epsilon) \frac{\epsilon}{e^{\beta(\epsilon-\mu)} \pm 1}$$

Average number of particles

$$N(T,V,\mu) = \int_0^\infty d\epsilon \; g(\epsilon)\, \langle N \rangle = \int_0^\infty d\epsilon \; g(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1}$$



DEGENERATE FERMIONS



$$\epsilon(n) = \frac{h^2}{8mL^2}n^2 \rightarrow g^{(3D)}(\epsilon)d\epsilon = \pi n^2 dn \rightarrow g^{(3D)}(\epsilon) = \frac{\pi}{2}\left(\frac{8m}{h^2}\right)^3 \sqrt{\epsilon}$$

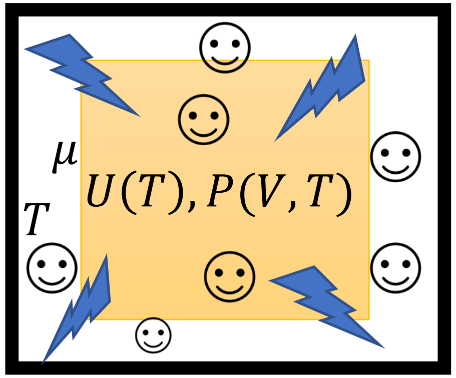
$$\epsilon_F(N) = \frac{h^2}{8mL^2}n_{max}^2 = \frac{h^2}{8mL^2}\left(\frac{N}{2}\right)^2$$

Average energy

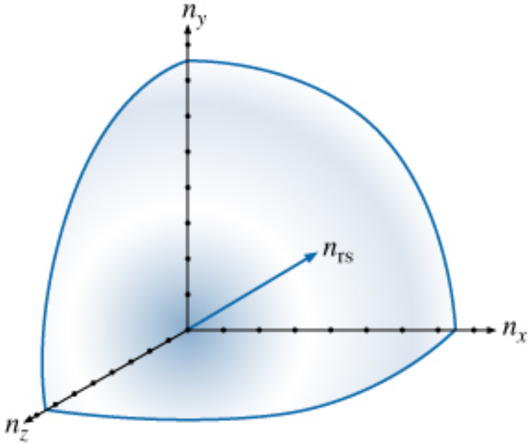
$$U(T,V,\epsilon_F) = \int_0^{\epsilon_F} d\epsilon \; g(\epsilon)\; \epsilon$$

Average number of particles

$$N(T,V,\epsilon_F) = \int_0^{\epsilon_F} d\epsilon \; g(\epsilon)$$



Photons



$$\boldsymbol{\epsilon_n = \frac{hc}{2L} n \rightarrow g(\epsilon)d\epsilon = \pi n^2 dn \rightarrow g(\epsilon) = \frac{8\pi\, V}{(hc)^3}\epsilon^2}$$

Average energy

$$U(T,V) = \int_0^\infty d\epsilon \;\; g(\epsilon) \frac{\epsilon}{e^{\beta\epsilon} - 1} = \frac{8\pi\, V}{(hc)^3} \int_0^\infty d\epsilon \frac{\epsilon^3}{e^{\beta\epsilon} - 1} = \frac{8\pi^5 (kT)^4}{15\, (hc)^3}$$

Average number of particles

$$N(T,V) = \int_0^\infty d\epsilon \;\; g(\epsilon) \frac{1}{e^{\beta\epsilon} - 1} = \frac{8\pi\, V}{(hc)^3} \int_0^\infty d\epsilon \frac{\epsilon^2}{e^{\beta\epsilon} - 1}$$