UNIVERSITY OF OSLO

Obligatory assignment 3: FYS2160, Thermodynamics and statistical physics, Fall 2019
Due: October 29, 2019

1 Equipartition of energy

Consider an ultra-relativistic particle that can move in one dimension and is coupled to a thermal bath at temperature T. The kinetic energy is the ultra-relativistic particle is $E = c|p|$ where c is the speed of light and p is the momentum of the particle. This energy is linear rather than quadratic in momentum as for a classical non-relativistic particle. The aim of this exercise is to derive the equipartition theorem corresponding to a *linear* degree of freedom and also to compute the standard deviation of fluctuations in energy.

1. Compute the partition function $Z(\beta)$ as function of $\beta = 1/(kT)$ assuming that there are very many microstates available for the ultra-relativistic particle at a given temperature (continuous energy spectrum).
2. Derive the particle mean energy $\overline{E}$ from the partition function computed in 1.1 and show that $\overline{E} = kT$. Discuss this expression in relation to the equipartition theorem for a quadratic degree of freedom.
3. Show that the mean square energy $\overline{E^2}$ can be in general derived from the partition function
$$\overline{E^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} \tag{1}$$
4. Using the expression of the partition function computed in 1.1, determine the dependence of mean square energy $\overline{E^2}(\beta)$ on temperature or β.
5. The standard deviation defined as $\sigma_E = \sqrt{\overline{E^2} - \overline{E}^2}$ measures the size of the energy fluctuations around the average value. Show that the size of these fluctuations relative to the average is $\sigma_E/\overline{E} = 1$.

2 Partition function

Let us have a particle which can only have five states, with energies: $\epsilon_1 = -0.1, \epsilon_2 = -0.05, \epsilon_3 = 0, \epsilon_4 = 0.05$ and $\epsilon_5 = 0.1$ in units of eV. The particle is in thermal equilibrium with a heat bath at room temperature, $T = 300$ K.

1. Find the partition function for this particle Z_1.
2. Find the probability $P(\epsilon_i)$ for this particle to be in each of the energy states ϵ_i, $i = 1 \cdots 5$.
3. The reference point for measuring the energy is arbitrary, so we could shift the energies so that the ground state starts at 0: $\epsilon_1 = 0, \epsilon_2 = 0.05, \epsilon_3 = 0.1, \epsilon_4 = 0.15, \epsilon_5 = 0.2$. Repeat steps (1) and (2) for these energies. What will change and what will remain unchanged?

3 Maxwell-Boltzmann distribution

Let us take the Maxwell-Boltzmann distribution of the speed $D(v)$ for a nitrogen gas N_2 (three dimensional case).

1. Calculate the average speed $\langle v \rangle$, the root-mean-squared speed v_{rms} and the most probable speed v_p for a N_2 molecule at temperatures of $T = 300$K and $T = 600$K. (The mass of a mole of N_2 is $M = 0.028$kg/mol)

2. Plot the Maxwell-Boltzmann distribution $D(v)$ for these two temperatures, on the same graph.

3. At room temperature $T = 300$K, what fraction of the N_2 molecules are moving at speeds less than 300m/s?

4. The temperature of earth's upper atmosphere is around $T = 1000$ K. Calculate the probability of a nitrogen molecule at this temperature moving faster than the earth's escape speed 11km/s. Comment on this result. (Hint: evaluate the integral numerically)

5. Repeat the previous calculation at $T = 1000$K for a hydrogen molecule (H_2) and for helium atom (He) and discuss the results. (use that $M_{H_2} = 0.002$ kg/mol and $M_{He} = 0.004$ kg/mol)

6. Consider on a nitrogen molecule on the moon and assume that the temperature is about 1000K the same as on the earth's upper atmosphere. The escape speed from the moon's surface is about 2.4 km/s, hence smaller than that on earth. Calculate the probability of a nitrogen molecule moving faster than the moon's escape speed and, consequently explain why the moon has no atmosphere.