Exam Fys2160 2023

December 7

	water	ice
Density [kg/m^3]	1000	916.2
Speed of sound [m/s]	1500	3800
Latent heat of melting [kJ/kg]		334
Latent heat of vaporization [kJ/kg]	2260	
Heat capacity [kJ/kg K]	4.186	2.108
Thermal conductivity [W / m K]	0.54	2.22
	eV/K	J/K
Boltzmann constant	$8.62 \cdot 10^{-5}$	$1.38 \cdot 10^{-23}$

Table 1: Numerical values you may need.

Problem 1: Conceptual Questions

(a) Explain in words and mathematically how air pressure changes with altitude.

(b) Explain in words how the boiling temperature of water changes with altitude

(c) Two identical cups are each filled with water at different temperatures and brought in perfect thermal contact. The water in one cup is at 20^oC and the other one is at 80^oC. What is the total entropy change when the water reaches the equilibrium temperature between the two cups? Express your answer in units of the heat capacity of the water, C, assumed constant.

(d) State the equipartition of energy theorem. Using the equipartition theorem, predict the energy of an ideal gas of N quantum harmonic oscillators.

Problem 2: Ice on a lake

You want to go skating on a beautiful lake. It has been cold for a long time, the water temperature in the lake is close to 0°C and a thin layer of about a millimeter of ice has formed. The weather forecast says it is going to be around -8°C and no wind the next days.

You want to estimate when you can expect the ice to be safe (5 cm thickness). You know that in steady state the magnitude of the heat flux J (with units [J]=J m^{-2} s^{-1} =W m^{-2}) through the ice is

$$J = \lambda \frac{\Delta T}{\Delta z}, \tag{1}$$

where λ is the heat conductivity, ΔT is the temperature difference across the ice layer and Δz is the thickness of the ice.

(a) In order to make the calculation as simple as possible, what will you assume the temperature difference ΔT across the ice layer is? What assumptions are you then implicitly making?

(b) What will you assume is the main source of heat that is transported through the ice? Which heat sources will you neglect?

(c) Find the constant of proportionality between the heat production rate and the speed of freezing v

$$\frac{1}{A}\frac{dQ}{dt} \propto v \tag{2}$$

from freezing of the ice over an area A of 1 m^2. The freezing speed is how fast the ice thickness Δz is growing.

The heat production can not be larger than the heat conduction. The fastest is when they are equal:

$$\frac{1}{A}\frac{dQ}{dt} = \lambda \frac{\Delta T}{\Delta z}. \tag{3}$$

(d) Use the relations above to estimate how long time it takes for the ice to reach a thickness of $\Delta z = 5$ cm.

(e) Discuss if you find the answer reasonable. (Don't include in exam, but save for later)

Problem 3: Quantum ideal gas in 2D

Consider an ideal gas of quantum particles in a two-dimensional domain of linear size L. Each particle has the energy levels

$$\epsilon_n = \frac{h^2}{8mL^2} n^2, \text{ where } n^2 = n_x^2 + n_y^2.$$

(a) Compute the one particle partition function at a given temperature, T and area L^2. Show that it reduced to

$$Z_1 = \frac{A}{\Lambda^2}$$

where $\Lambda(T)$ is the thermal wavelength.

Hint: Use the Gaussian integral $\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\pi/a}$ where $a > 0$.

(b) Using a), compute the total Helmholtz free energy $F_N(T)$ of the ideal gas. Discuss how this is different from the expression in three dimensions.

(c) Compute the chemical potential $\mu(T, N)$ of this gas. Find the condition for the temperature T such that the chemical potential is negatively defined (classical limit).

Problem 4: Photon Gas Engine

Let us consider an ideal photon gas at equilibrium described as the black-body radiation in a cubic cavity of size $V = L^3$. The average energy of the radiation per unit volume is given by

$$\frac{U}{V} = aT^4 \ , \ a = \frac{\pi^2 k^4}{15\hbar^3 c^3} \ . \tag{4}$$

We construct a cyclic, reversible machine using this photon gas as its medium. The cyclic process consists of the following three parts: an isothermal process from 1 to 2, an isochoric process ($V = const.$) from 2 to 3, and an adiabatic expansion from 3 to 1. Assume the sign convention used in the course that the work done by the machine is negative valued.

(a) Demonstrate that the entropy S of the photon gas as function of a, volume V, and temperature T is

$$S = \frac{4}{3} aVT^3.$$

(b) Using the result in a), derive the Helmholtz free energy F and show that it results in the equation of state

$$P = \frac{1}{3} aT^4.$$

(c) Draw the cyclic process schematically in a P-V diagram and in a S-T diagram. Use the expressions from a) and b) to draw the curves more realistically.

(d) The efficiency of the machines is generally given by

$$e = \frac{-W_{\text{net}}}{Q_{\text{in}}} \ , \tag{5}$$

where $-W_{\text{net}}$ is the net work done by the machine and Q_{in} is the total *absorbed* during one cycle. Determine the efficiency of the photon gas machine. Express your answer in terms of $x = V_3/V_1$, where V_1 and V_3 are the the two volume states explored by the cycle.