UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	**Fys2160,**
Exam date:	**Friday 13 December 2019**
Exam time:	**09.00 - 13.00**
Number of pages:	**2**
Allowed material:	**Electronic calculator** **K. Rottman: Matematisk formelsamling** **Øgrim and Lian or Angell and Lian: Fysiske størrelser og enheter** **One A4 sheet of notes (both sides can be used)**

Make sure that the exam set is complete before you begin to solve the problems. You must justify your answers in the solutions.

Problem 1:

1.1 Suppose you roll a 6-sided die 5 times. What is the probability that you get the same side of the die exactly 3 times?

Solution: The probability of each side is $p = 1/6$. The probability that we have one such occurrence out of 5 times of rolling the die is

$$P(1,1,1,0,0) = p^3(1-p)^2 \tag{1}$$

The total number of such occurrences is $5!/(3!2!) = 10$. Hence the probability is

$$P = \frac{5!}{3!2!}(\frac{1}{6})^3(1-\frac{1}{6})^2 \approx 0.032 \tag{2}$$

1.2 Define temperature and thermodynamic equilibrium between two systems in thermal contact.

Solution: Tendency to exchange energy in order to maximise entropy, $1/T = (\partial S/\partial U)$. Thermodynamic equilibrium is established when temperature is the same.

1.3 Why does a pressure cooker cook food faster than a normal casserole?

Solution: The boiling point of water is higher at higher pressure.

1.4 You have a thermos with a gas tight piston enclosing a monatomic ideal gas inside the thermos. If you use the piston to compress the gas volume by 20% will the temperature of the gas increase, decrease or stay the same?

Solution: This is an adiabatic compression. Hence, the change is the internal energy is due to the work done on the gas $dU = -PdV > 0$ (for compression) and the temperature change is related to the change in the internal energy as $dT = 2dU/3Nk$. The temperature will therefore increase: $dT = -2PdV/3Nk$. Some heat will go to the insulation in the thermos, but the temperature can only rise.

1.5 Inside a similar thermos you have equal masses of ice, liquid water and air at atmospheric pressure. The specific heat capacity of ice is twice that of water vapour and the specific heat capacity of water is twice that of ice. The air is saturated with water vapour and the partial pressure of water is $p_w = 0.61$ kPa. Now you pull the piston slowly to increase the gas volume by 20%. What will the pressure and temperature of the gas be?

Solution: Since the piston is pulled slowly we can assume that the thermos stays in thermal equilibrium at $T = 0°$C. So, this is an isothermal gas expansion. The heat supplied to the gas to keep its temperature constant while expanding comes from the latent heat released from freezing a bit the ice. We assume that the volume change due to this phase transition is very small compared to the volume change in the gas expansion. Since $p_w/p_{atm} \sim 6 \cdot 10^{-3}$ we can neglect the change in particle number due to evaporation. Using the equation of state, $PV = NkT$, with N and T kept constant, it follows that the pressure is therefore decreased by 20% : $\Delta P/P = -\Delta V/V = -0.2$.

Problem 2:

2.1 Consider an ideal gas of non-relativistic particles in two-dimensions (2D). What is the probability of a particle having velocity vector $\vec{v}$? (You do not need to write the normalization constant explicitly.)

Solution: The kinetic energy of the particle is $E = mv^2/2$, with $v = |\vec{v}|^2$. The probability is given by the Boltzmann's factor up to a normalization constant, C.

$$P(\vec{v}) = Ce^{-mv^2/(2kT)} \tag{3}$$

2.2 Derive the Maxwell-Boltzmann expression for the speed distribution and show that it equals

$$D(v) = \left(\frac{m}{2\pi kT}\right) 2\pi v e^{-mv^2/(2kT)}, \tag{4}$$

Solution: Speed density is given by the probability of having a velocity vector times times the number of velocity vectors with a speed $v = |\vec{v}|$. This number is proportional to the circumference of the circle with radius v, which is $2\pi v$. Hence

$$D(v) = C_1 2\pi v e^{-mv^2/(2kT)}, \tag{5}$$

where the normalization constant is determined as

$$C_1^{-1} = 2\pi \int_0^\infty dv v e^{-mv^2/(2kT)} = 2\pi \left(\frac{2kT}{m}\right) \int_0^\infty dx x e^{-x^2} = \left(\frac{2\pi kT}{m}\right) \tag{6}$$

2.3 Determine the most likely speed and show that it corresponds to the equipartition of energy.

Solution: The most likely speed corresponds to the maximum of the speed distribution

$$\frac{dD(v)}{dv} = 0 \rightarrow v_{max} = \sqrt{kT/m}. \tag{7}$$

The kinetic energy is related to the mean-square velocity $E = m\langle v^2\rangle/2$ and, by equipartition of energy theorem, and equals to $E = 2 \cdot kT/2$. Hence $\langle v^2\rangle = 2kT/m = 2v_{max}^2$.

Problem 3:

Consider a collection of N identical quantum harmonic oscillators at a given temperature T. Each quantum harmonic oscillator has an energy spectrum of allowed energies $\epsilon_n = nh\nu$, where $n = 0, 1, 2, \cdots$, h is the Planck's constant and ν is the fixed frequency of the oscillator.

3.1 Show that the partition function of a single quantum harmonic oscillator $Z_1(T, \nu)$ is given by

$$Z_1 = \frac{1}{1 - e^{-\beta h\nu}} \tag{8}$$

(Hint: Make use of the formula $\sum_{n=0}^{\infty} x^n = 1/(1-x), |x| < 1$).

Solution:

$$Z_1 = \sum_{n=0}^{\infty} e^{-n\beta h\nu} = \frac{1}{1 - e^{-\beta h\nu}} \tag{9}$$

3.2 Show that the average energy of a single quantum harmonic oscillator at a given temperature is

$$\langle\epsilon\rangle(T, \nu) = \frac{h\nu}{e^{\beta h\nu} - 1}. \tag{10}$$

Express this in terms of the Bose-Einstein distribution. Provide an example of a Bose system with this average energy and discuss it.

Solution:

$$\langle\epsilon\rangle(T, \nu) = -\frac{1}{Z_1}\frac{\partial Z_1}{\partial \beta} = \frac{h\nu}{e^{\beta h\nu} - 1} = h\nu n_\nu(T), \tag{11}$$

where $n_\nu(T) = 1/(e^{\beta h\nu} - 1)$ is the Bose-Einstein distribution for the average occupation number of a frequency mode. For example, we can apply this formula to calculate the average energy of given mode at frequency ν for black body radiation or thermal vibrations in crystals. Then we are dealing with a system of photons or phonons which are Bose particles at zero chemical potential. The average energy of a specific mode is the photon energy $h\nu$ times the average number of photons occupying that frequency mode.

3.3 Calculate the total average energy and heat capacity of N independent harmonic oscillators as functions of frequency and temperature. Show that, in the high temperature limit, the energy and heat capacity follow the equipartition of energy theorem.

Solution: Total energy is an extensive function hence,

$$U_N = N\langle\epsilon\rangle(T, \nu) = N\frac{h\nu}{e^{\beta h\nu} - 1}. \tag{12}$$

For $\beta \to 0$ (high T limit), we can Taylor's expand it $U_N = NkT$ which follows also from the equipartition of energy of N indendent classical oscillators. The heat capacity is

$$C_V = \frac{\partial U_N}{\partial T} = \frac{N}{kT^2}\frac{(h\nu)^2 e^{\beta h\nu}}{(e^{\beta h\nu} - 1)^2}. \tag{13}$$

For high T, $C_V = Nk$.

3.4 Calculate Helmholtz free energy of a single harmonic oscillator and then of N independent harmonic oscillators.

Solution:

$$F_1 = -kTlnZ_1 = kTln\left(1 - e^{-\beta h\nu}\right). \tag{14}$$

Helmholtz free energy is an extensive potential hence, $F_N = NF_1$.

3.5 Calculate the entropy of N independent harmonic oscillators.

Solution:

$$S = \frac{U_N - F_N}{T} = \frac{N}{T}\frac{h\nu}{e^{\beta h\nu} - 1} - Nkln\left(1 - e^{-\beta h\nu}\right). \tag{15}$$

Entropy is an extensive potential hence, $S_N = NS_1$.

Problem 4:

4.1 The heat capacity is defined as the absorbed heat per unit increase in temperature $C \equiv \frac{Q}{\Delta T}$. Use the first law of thermodynamics to derive the expressions for the heat capacity at constant volume

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \tag{16}$$

and the heat capacity at constant pressure

$$C_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P \tag{17}$$

in terms of changes in the internal energy, U.

Solution: The first law: $\Delta U = Q + W = TdS - PdV$, thus $Q = \Delta U + PdV$, $C_V = (\Delta U/\Delta T)_V$ and $C_P = (\Delta U/\Delta T + PdV/dT)_P$

4.2 Use the thermodynamic identities of the internal energy $U(S, V)$ and the enthalpy $H(S, P)$ to show that

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V \tag{18}$$

and

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_P \tag{19}$$

Solution: $dU = TdS - PdV$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{T\partial S - P\partial V}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$

and $dH = dU + PdV + VdP$, thus

$$\left(\frac{\partial H}{\partial T}\right)_P = \left(\frac{\partial U + P\partial V}{\partial T}\right)_P$$

and $dH = TdS + VdP$:

$$C_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P = \left(\frac{\partial H}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P$$

4.3 From the thermodynamic identity of the Helmholtz energy $F(T, V)$ derive the Maxwell relation

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T \tag{20}$$

Solution:

$$dF = -PdV - SdT = \left(\frac{\partial F}{\partial V}\right)_T dV + \left(\frac{\partial F}{\partial T}\right)_V dT$$

This yields

$$-P = \left(\frac{\partial F}{\partial V}\right)_T$$

and

$$-S = \left(\frac{\partial F}{\partial T}\right)_V$$

The double derivatives

$$\frac{\partial^2 F}{\partial T \partial V} = \frac{\partial^2 F}{\partial V \partial T}$$

which yields the desired result.

4.4 Using the equipartition of energy, determine the internal energy of a monoatomic ideal gas in 3D, i.e. $U(T, N)$. Determine the heat capacities C_V and C_P for the ideal gas using 4.1, and show that $C_P - C_V = Nk$. Explain why the heat capacity is larger at constant pressure than at constant volume.

Solution: $U = 3NkT/2$. Hence, $C_V = 3Nk/2$ and $C_P = 3Nk/2 + Nk$, using the ideal gas equation of state $PV = NkT$. Thus, $C_P - C_V = Nk$. The heat capacity at constant pressure is larger because more heat needs to be absorbed and used for volume expansion and temperature increase temperature.