Bose-Einstein distribution and black body radiation

Lecture 19

Quantum gas at constant T, μ

Quantum state: s

$\epsilon(s) = N(s)\varepsilon$ is the energy of and $N(s)$ is the number of particles in the state s.

Gibbs factor of state s: $e^{-\beta(\epsilon(s)-\mu N(s))} = e^{-\beta N(s)(\varepsilon-\mu)}$

Gibbs sum at T, μ, over microstates, s: $Z_G(\varepsilon; T, \mu) = \sum_s e^{-\beta N(s)(\varepsilon-\mu)}$

Probability at T, μ, of N particles in, s: $P(N, \varepsilon; T, \mu) = {e^{-\beta N(s)(\varepsilon-\mu)}}/{Z_G(\varepsilon;T,\mu)}$

Fermions: N=0 or 1 in each quantum state: Fermi-Dirac

Gibbs sum: $Z_G(\varepsilon; T, \mu) = 1 + e^{-\beta(\varepsilon-\mu)}$

Probability: $P(N, \varepsilon; T, \mu) = \frac{e^{-\beta N(\varepsilon-\mu)}}{1+e^{-\beta(\varepsilon-\mu)}}$

Average occupation $\bar{n}_{FD}(\varepsilon; T, \mu) = \sum_N NP(N) = \frac{1}{1+e^{\beta(\varepsilon-\mu)}}$

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Bosons: $N = 1 \ldots \infty$, in each quantum state: Bose-Einstein

Gibbs sum: $Z_G(\varepsilon; T, \mu) = \sum_{N=0}^{\infty} e^{-\beta N(\varepsilon-\mu)}$

(geometric sum for $\varepsilon > \mu$) $= \frac{1}{1-e^{-\beta(\varepsilon-\mu)}}$

Probability: $P(N, \varepsilon; T, \mu) = e^{-\beta N(\varepsilon-\mu)}(1 - e^{-\beta(\varepsilon-\mu)})$

Average occupation $\bar{n}_{BE}(\varepsilon; T, \mu) = \sum_N NP(N)$

$x = \beta(\varepsilon - \mu)$

$= (1 - e^{-x}) \sum_N Ne^{-Nx}$

$= -(1 - e^{-x}) \frac{\partial}{\partial x} \sum_N e^{-Nx}$

$= -(1 - e^{-x}) \frac{\partial}{\partial x} \left(\frac{1}{1-e^{-x}}\right) = \frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^{x}-1}$

$= \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$

Occupation number distributions

Boltzmann distribution

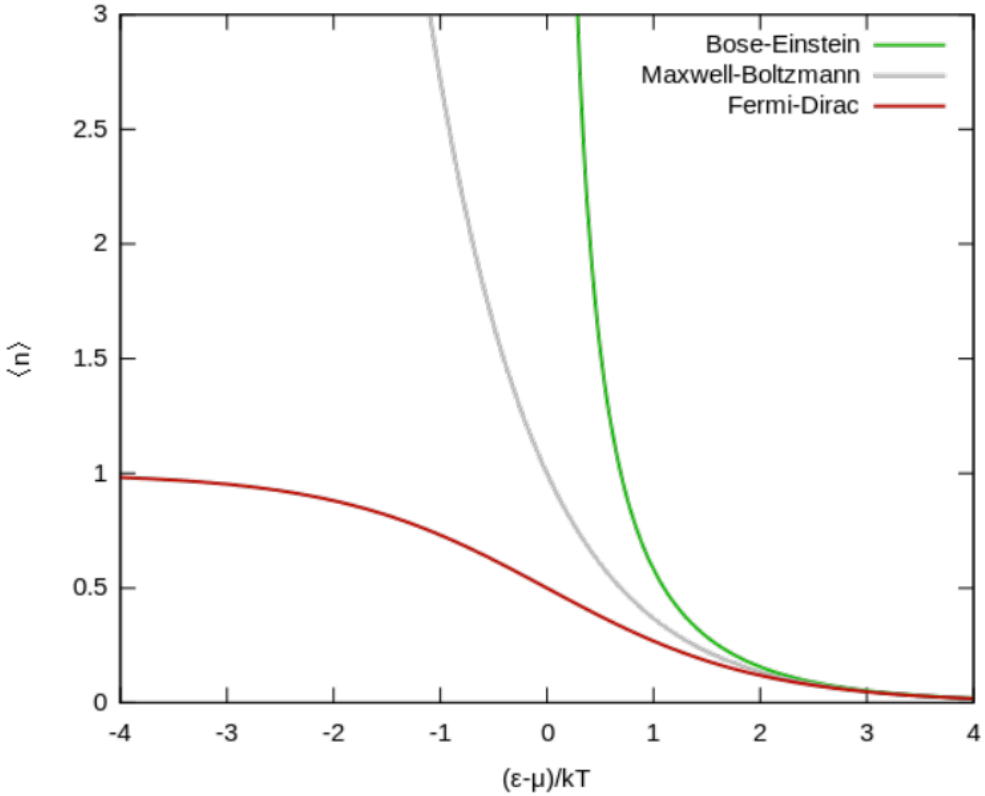
$$\bar{n}_B = e^{-\beta(\varepsilon-\mu)}$$

Fermi-Dirac distribution

$$\bar{n}_{FD} = \frac{1}{1+e^{\beta(\varepsilon-\mu)}}$$

Bose-Einstein distribution $(\varepsilon > \mu)$

$$\bar{n}_{BE} = \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$$



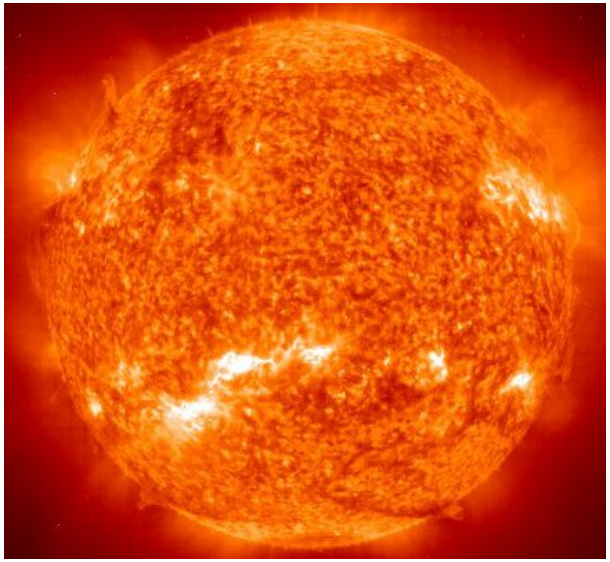
Quantum distributions approach classical distribution at high T

Photons

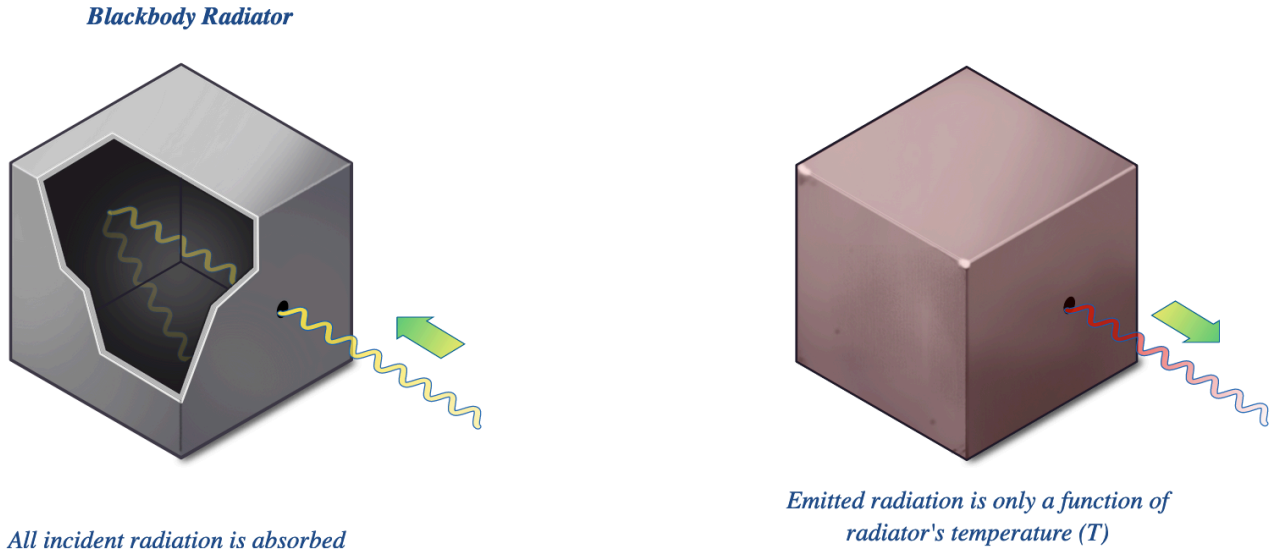
- Photons are bosons
- Photons
  - have no mass at rest
  - can be created/destroyed
  - number, N, not conserved
  - => Helmholtz free energy, F, independent of N
  - => $\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = 0$
- => Chemical potential of photons is zero
- Occupation number distribution: $\bar{n} = \frac{1}{e^{\beta\varepsilon}-1}$

$(\bar{n}_{BE} = \frac{1}{e^{\beta(\varepsilon-\mu)}-1})$

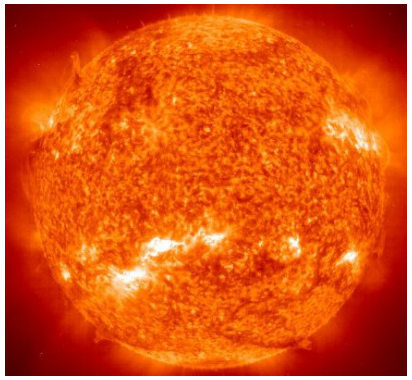
Black body radiation



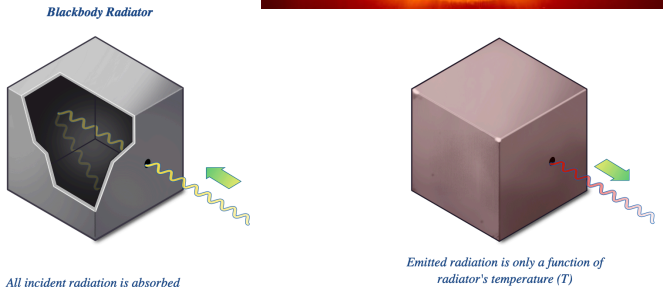
- Black body: body/object that
  - does not reflect photons (i.e. black)
  - emits and absorbs photons
- Thermal (black body) radiation
  - photons emitted by a body at temperature T>0

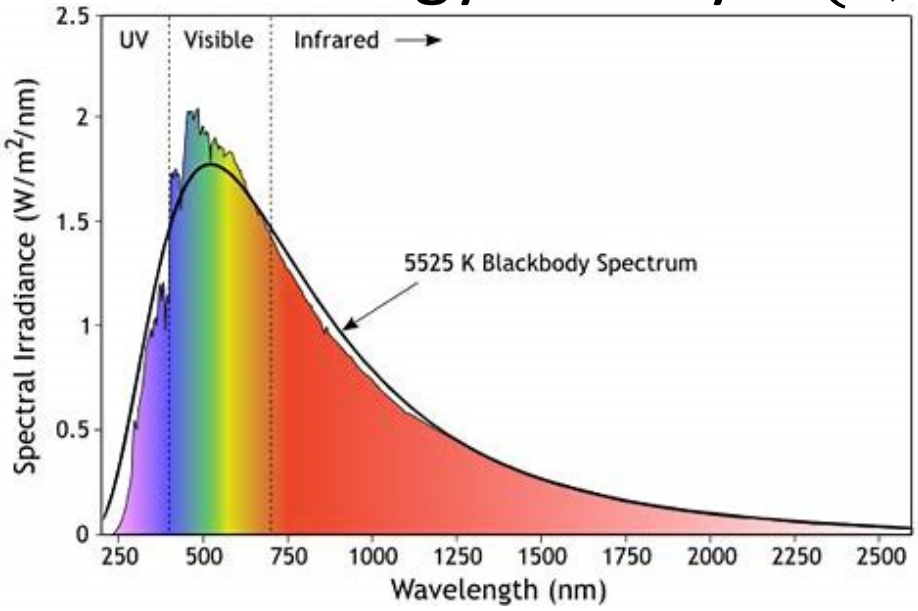


Black body radiation

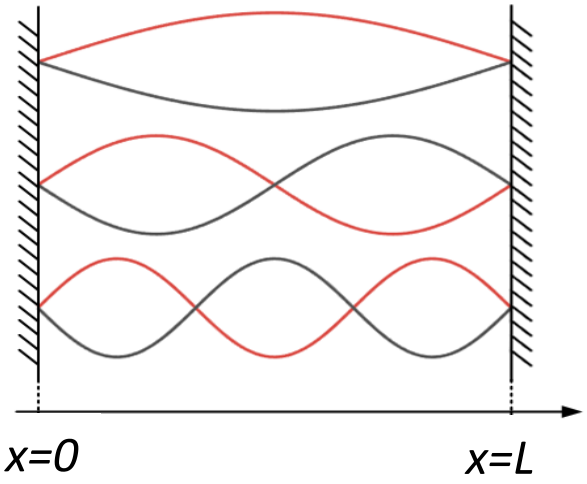


- Black body: body/object that
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- Thermal (black body) radiation
  - photons emitted by a body at temperature T>0
- Planck distribution: Energy density $\varepsilon(\nu, T)$, ν- frequency





Black body at temperature T



Standing electromagnetic waves

- wavevector $\vec{k} = \frac{\pi}{L}\vec{n},\ n_{x,y,z} = 0,1,..$
- momentum $\vec{p} = \hbar\vec{k}$
- frequency $\nu = \frac{c}{2\pi}\left|\vec{k}\right| = \frac{c}{2L}\left|\vec{n}\right| = \frac{cn}{2L}$
- wavelength $\lambda = \frac{2L}{n}, \quad n = |\vec{n}| = \sqrt{n_x^2 + n_y^2 + n_z^2}$
- energy $\varepsilon = pc = h\nu = \frac{hcn}{2L}$
- 2 modes (electric, $\vec{k}\cdot\vec{E} = 0$, magnetic, $\vec{k}\cdot\vec{B} = 0$), $\vec{B}\cdot\vec{E} = 0$
- => 2 polarizations (degrees of freedom)

Total energy $U = 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\varepsilon\bar{n}$

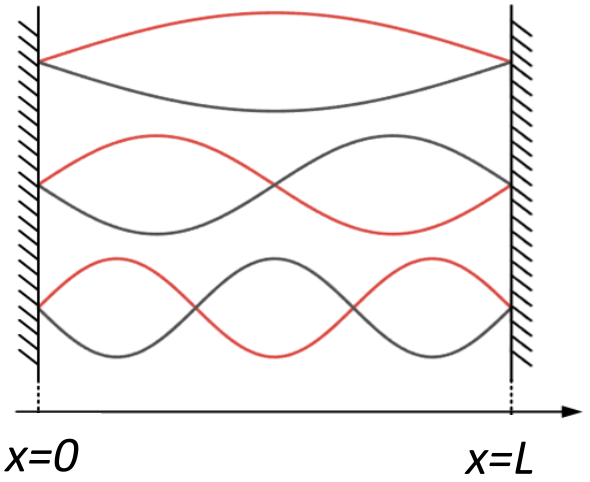
$$= \sum_{n_x,n_y,n_z}\frac{hcn}{L}\frac{1}{e^{\beta\frac{hcn}{2L}}-1}$$

$$\bar{n} = \frac{1}{e^{\beta\varepsilon}-1}$$

Black body at temperature T

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Total energy $U = 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\varepsilon\bar{n}$

$$\mathrm{U} = \sum_{n_x,n_y,n_z} \frac{hcn}{L}\ \frac{1}{e^{\beta\frac{hcn}{2L}} - 1}$$

$$= \int_0^{\infty} dn \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi\, n^2 \sin\theta \frac{hcn}{L}\ \frac{1}{e^{\beta\frac{hcn}{2L}} - 1}$$

$$\varepsilon = \frac{hcn}{2L} \qquad = L^3 \int_0^{\infty} d\varepsilon \frac{8\pi\varepsilon^3}{h^3c^3(e^{\beta\varepsilon} - 1)} = L^3 \int_0^{\infty} d\varepsilon\, u(\varepsilon)$$

Energy spectrum $u(\varepsilon) = \frac{8\pi\varepsilon^3}{h^3c^3(e^{\beta\varepsilon} - 1)}$ = Planck distribution

Cosmic background radiation

- Universe was filled with ionized gas & EM radiation at 3000K
- Expansion of universe Doppler shifted wavelength
- Leftover background radiation can be measured now
- Spectral intensity can be fitted with Planck distribution $\frac{x^3}{e^x-1}$

