

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Solution to exam in: FYS3110 - Quantum mechanics.

Date Nov. 30. 2021 (4 hours)

Max score: 58 points

1a) (6 points)

$$A|1\rangle = |2\rangle$$

$$A|2\rangle = |1\rangle$$

Adding and subtracting the two equations give

$$A(|1\rangle \pm |2\rangle) = \pm(|1\rangle \pm |2\rangle)$$

Thus the normalized eigenvectors and eigenvalues are

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \quad \text{eigenvalue: } +1$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle), \quad \text{eigenvalue: } -1$$

No reasoning is required here. It is enough to just write down the eigenstates and their corresponding eigenvalues. For full score the eigenvectors must be normalized.

1b) (6 points) The measurement at $t = 0$ gives -1. Thus the state immediately after this measurement is $|-\rangle$. $|1\rangle$ and $|2\rangle$ are energy eigenkets with energies E_1 and E_2 respectively. The state at time t is

$$|t\rangle = e^{-iHt/\hbar}|-\rangle = \frac{1}{\sqrt{2}}(e^{-iE_1t/\hbar}|1\rangle - e^{-iE_2t/\hbar}|2\rangle)$$

The probability of measuring A to be -1 at time t is $P(-1) = |\langle -|t\rangle|^2$ where

$$\begin{aligned} \langle -|t\rangle &= \frac{1}{2}(\langle 1| - \langle 2|)(e^{-iE_1t/\hbar}|1\rangle - e^{-iE_2t/\hbar}|2\rangle) = \frac{1}{2}(e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}) \\ &= e^{-i(E_1+E_2)t/2\hbar} \frac{1}{2}(e^{-i(E_1-E_2)t/2\hbar} + e^{i(E_1-E_2)t/2\hbar}) = e^{-i(E_1+E_2)t/2\hbar} \cos \frac{(E_1-E_2)t}{2\hbar} \end{aligned}$$

Thus

$$P(-1) = \cos^2\left(\frac{(E_1-E_2)t}{2\hbar}\right)$$

For full score the final expression must be convincingly explained. Partial credit will be given for the correct time-dependent state. Partial credit will also be given for a correct (intermediate) expression for the probability.

2a) (2 points)

$$\langle \psi | A | \chi \rangle = \langle \chi | A^\dagger | \psi \rangle^* = \lambda^* \langle \chi | \psi \rangle^* = \lambda^* \langle \psi | \chi \rangle$$

2b) (2 points) Using $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ one gets

$$(a^\dagger + a)^3 |0\rangle = (a^\dagger + a)^2 |1\rangle = (a^\dagger + a) \left(\sqrt{2}|2\rangle + |0\rangle \right) = \sqrt{2 \cdot 3}|3\rangle + \sqrt{2 \cdot 2}|1\rangle + |1\rangle = \underline{\sqrt{6}|3\rangle + 3|1\rangle}$$

2c) (2 points) We first combine the spin of two quarks. This gives a total of 0 and 1. Then combine this result with the spin of the third quark. This gives 1/2, and 1/2, 3/2. Then combine this with the spin of the fourth quark which gives 0, 1, 0, 1, and 1, 2. Finally combine all this with the spin of the fifth quark. This yields the following total spin states 1/2, 1/2, 3/2, 1/2, 1/2, 3/2, 1/2, 3/2, 3/2, 5/2. Thus the possible values for the total spin are 1/2, 3/2, 5/2. In fact, counting the number of multiplets, we find that there are 5 different total spin-1/2 multiplets, 4 different total spin-3/2 multiplets and one spin-5/2 multiplet. All together $5 \times 2 + 4 \times 4 + 6 = 32$ states which matches the 2^5 spin states of the individual quarks.

2d) (2 points) K is unitary: $K^{-1} = K^\dagger$. The relation $KK = I$ implies that $K = K^{-1}$ and together with the unitariness: $K = K^{-1} = K^\dagger$. I.e. K is hermitian and thus has real eigenvalues. Eigenvalue equation

$$K|\lambda\rangle = \lambda|\lambda\rangle, \quad \implies \underbrace{KK}_I|\lambda\rangle = \lambda^2|\lambda\rangle \implies \lambda^2 = 1 \implies \lambda = \pm 1$$

where we have used the fact that λ is real as follows from the hermiticity of K . Thus true statements are: Eigenvalues are real and ± 1 .

2e) (2 points)

$$\begin{aligned}\langle\psi|a|\psi\rangle &= \sum_{m=0}^{\infty} c_m^* \langle m|a \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n,m=0}^{\infty} c_m^* c_n \langle m|a|n\rangle = \sum_{n,m=0}^{\infty} c_m^* c_n \sqrt{n} \langle m|n-1\rangle \\ &= \sum_{n,m=0}^{\infty} c_m^* c_n \sqrt{n} \delta_{m,n-1} = \sum_{n=0}^{\infty} c_{n-1}^* c_n \sqrt{n}\end{aligned}$$

2f) (2 points)

$$E_{n,j} = \frac{E_1}{n} \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) \right]$$

This fine-structure formula is derived from relativistic corrections to the kinetic energy, the spin-orbit coupling and the Darwin term, none of which has anything to do with the proton spin. Thus the fine-structure is not related to the proton spin.

For $n = 5$ the possible values for l are $l \in \{0, 1, 2, 3, 4\}$. Combining this with the spin-1/2 of the electron the possible values for j are $j \in \{1/2, 3/2, 5/2, 7/2, 9/2\}$. Since the formula above gives distinct values for different values of j when n is fixed we get five different values for the five possible values of j . Thus the unperturbed $n = 5$ level is split into five different fine-structure levels.

The fine-structure constant $\alpha \approx 1/137$. From the formula above one sees that the fine structure correction to the energy levels is proportional to $\alpha^2 \approx \frac{1}{137^2} \approx 5 \cdot 10^{-5}$, which is much smaller than 1%.

The square parenthesis is always positive. This follows from the fact that n is the principal quantum number and is 1 larger than the maximum value of l , l_{max} . The maximum value of j is $l_{max} + 1/2$. Therefore $\frac{n}{j + \frac{1}{2}} = \frac{l_{max} + 1}{j + \frac{1}{2}} = \frac{j_{max} + \frac{1}{2}}{j + \frac{1}{2}} > 1$. Thus the square parenthesis in the formula for $E_{n,j}$ is always positive. This implies that $|E_{n,j}| > |\frac{E_1}{n^2}|$.

2g) (2 points) The possible ground states for this system is achieved by making a slater determinant out of the two harmonic oscillator ground states $|000\rangle \otimes |\uparrow\rangle$ and $|000\rangle \otimes |\downarrow\rangle$ and a third state which can be either of the six states $|100\rangle \otimes |\uparrow\rangle$, $|100\rangle \otimes |\downarrow\rangle$, $|010\rangle \otimes |\uparrow\rangle$, $|010\rangle \otimes |\downarrow\rangle$, $|001\rangle \otimes |\uparrow\rangle$, or $|001\rangle \otimes |\downarrow\rangle$. These slater determinants are all fully antisymmetric under the interchange of particle labels and satisfy therefore the requirements on a system of indistinguishable fermions. These six slater determinant states are all orthogonal to each other and have the same energy. The degeneracy of the ground state is six.

2h) (2 points) The equation can be solved graphically by plotting the right hand side vs. z . This gives an oscillating function that starts on the value $1 + \beta$ at $z = 0$. Thus for larger values of β the oscillating function has larger amplitudes. The energy bands are the regions of z which gives a function value in the interval $[-1, +1]$. For large values of N the discretization of the levels within this interval is of less importance. For large values of β the right hand side is generally steeper when the function value is between -1 and +1. Therefore the width of the energy bands decreases as β increases. Since the amplitude gets larger with β there is also a larger spacing between the energy bands. Therefore the gaps between the energy bands increases when β increase. One can also arrive at these conclusions by realizing that larger values of β means stronger delta-barriers which will make the spectrum look more like that of isolated atoms, which has discrete energy levels, rather than energy bands.

2i) (2 points) The ground state of the unperturbed problem is a singlet. Thus one can use non-degenerate perturbation theory here. Let us first calculate the action of the perturbation on the singlet state

$$\left(\frac{2\alpha}{\hbar} S_1^z + \frac{2\beta}{\hbar} S_2^z \right) \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = (\alpha - \beta) \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

Thus we see that the perturbation on the singlet state gives a number times the triplet state with $m_s = 0$. It follows that the first order energy correction is 0 because

$$\begin{aligned} E^1 &= \frac{1}{\sqrt{2}} (\langle\uparrow\downarrow| - \langle\downarrow\uparrow|) \left(\frac{2\alpha}{\hbar} S_1^z + \frac{2\beta}{\hbar} S_2^z \right) \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ &= (\alpha - \beta) \frac{1}{2} (\langle\uparrow\downarrow| - \langle\downarrow\uparrow|) (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = 0. \end{aligned}$$

It also follows that the second order correction gets its only non-zero contribution from the excited state $\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$. Thus

$$E^2 = \frac{(\alpha - \beta)^2}{0 - E} = -\frac{(\alpha - \beta)^2}{E}$$

2j) (2 points) Looking up the Clebsch-Gordan table $1 \times 1/2$ for combining $l = 1$ with $s = 1/2$, we see that the state

$$|j = \frac{1}{2}, m_j = +\frac{1}{2}\rangle = \sqrt{\frac{2}{3}} |l = 1, m = +1\rangle \otimes |s = \frac{1}{2}, m_s = -\frac{1}{2}\rangle - \sqrt{\frac{1}{3}} |l = 1, m = 0\rangle \otimes |s = \frac{1}{2}, m_s = \frac{1}{2}\rangle$$

Thus the probability of measuring the spin to be $+\hbar/2$ in this state is $\left(-\sqrt{\frac{1}{3}}\right)^2 = \underline{\underline{\frac{1}{3}}}$.

2k) (2 points) According to first-order time-dependent perturbation theory, the probability amplitude for finding the atom in the excited state at time t is

$$\begin{aligned} a_{210}(t) &= \frac{1}{i\hbar} \int_0^t dt \langle 210 | (-eA\hat{n} \cdot \vec{r}) | 100 \rangle e^{-t/\tau} e^{i(E_{210}-E_{100})t/\hbar} \\ &= -\frac{1}{i\hbar} eA \langle 210 | \hat{n} \cdot \vec{r} | 100 \rangle \frac{e^{[i(E_{210}-E_{100})/\hbar - 1/\tau]t} - 1}{i(E_{210} - E_{100})/\hbar - 1/\tau} \end{aligned}$$

The energy of $|210\rangle$ is $E_{210} = E_1/4$, therefore $E_{210} - E_{100} = E_1/4 - E_1 = -3E_1/4$. The matrix element is between states with the same m quantum number, thus $\langle 210|x|100\rangle = \langle 210|y|100\rangle = 0$, and only the $\langle 210|z|100\rangle$ is non-zero. Taking $t \gg \tau$, we get

$$a_{210}(t) = \frac{1}{i\hbar} eA \langle 210|z|100\rangle \hat{n}_z \frac{1}{-i3E_1/4\hbar - 1/\tau}$$

Then taking the modulus, we find the probability to be in the state $|210\rangle$ at time t :

$$P_{210}(t) = e^2 A^2 |\langle 210|z|100\rangle|^2 \frac{\hat{n}_z^2}{9E_1^2/16 + \frac{\hbar^2}{\tau^2}}$$

From dimensional considerations and the fact that the length scale of the hydrogen atom is set by the Bohr radius a_0 : $|\langle 210|z|100\rangle|^2 \propto a_0^2$. Thus according to first order time-dependent perturbation theory the probability to find the atom in the $|210\rangle$ state at time $t \gg \tau$ is

$$P_{210}(t \gg \tau) \propto e^2 A^2 a_0^2 \frac{\hat{n}_z^2}{9E_1^2/16 + \hbar^2/\tau^2}$$

For the subproblems 2a),2b),2c),2e),2g),2i),2j) with "Select one alternative" among four possibilities: The correct answer gives 2 points. A wrong answer gives -2/3=-0.67 points, so that random guessing gives on average 0 points. Similarly 2k) with nine alternatives gives 2 points for the correct answer and -2/8=-0.25 point for a wrong answer. For 2d),2f),2h) with "Select one or more alternatives" among four possibilities, each correct alternative gives +1 point (there are two correct ones), and each wrong gives -1 point, so that random guessing or checking all alternatives gives zero points.

3a)(6 points) The commutation relation $[L^2, L_x] = 0$ implies that L^2 and L_x share a common complete set of eigenstates. Similarly $[L^2, L_y]$ implies that L^2 and L_y share a common complete set of eigenstates. However these common sets of eigenstates cannot both be the same set because $[L_x, L_y] \neq 0$ which implies that L_x and L_y do not share a common

complete set of eigenstates. Together this means that there are at least one eigenvalue of L^2 which is shared by different states, one in the L^2, L_x set and another in the L^2, L_y set. Thus at least one eigenvalue of L^2 must be degenerate.

To obtain full score, the formulation of the answer must be complete and clear, and not indicate any misunderstandings and/or mixup of concepts, such as for instance mixing eigenstates and eigenvalues. Partial credit will be given for mentioning that two commuting operators share a common (complete) set of eigenstates. No credit will be given if other criteria than the commutation relations are used to prove the non-degeneracy.

3b) (6 points) Choose the exact ground state $\psi_2(\vec{r})$ of $\frac{\vec{p}^2}{2m} + V_2(\vec{r})$ as trial wavefunction for system 1. Then the trial energy for system 1 is

$$E_{tr1} = \langle \psi_2 | H_1 | \psi_2 \rangle = \int d^3r \psi_2^*(\vec{r}) \left(\frac{\vec{p}^2}{2m} + V_1(\vec{r}) \right) \psi_2(\vec{r}) \leq \int d^3r \psi_2^*(\vec{r}) \left(\frac{\vec{p}^2}{2m} + V_2(\vec{r}) \right) \psi_2(\vec{r}) = E_2$$

where the middle inequality follows from

$$\int d^3r |\psi_2(\vec{r})|^2 V_1(\vec{r}) \leq \int d^3r |\psi_2(\vec{r})|^2 V_2(\vec{r})$$

which holds because $V_1(\vec{r}) \leq V_2(\vec{r})$ for all $\vec{r}$. Then using the variational principle: $E_1 \leq E_{tr1}$, we get

$$\underline{E_1 \leq E_{tr1} \leq E_2 \quad q.e.d.}$$

To get full score on this problem it must be clearly stated where the variational principle is being used. 1 point is deducted if the kinetic term in the Hamiltonian has not been taken into account. Partial credit is given for correctly stating the variational principle in the context of this problem. A maximum of 2 points is given for attempts that make correct conclusions based on specific choices for the potentials and wavefunctions.

4a) (6 points) The eigenfunctions of the unperturbed problem are the eigenstates of L^2 : $|l, m\rangle$ with energy eigenvalues $E_l^0 = J l(l+1)$. Their position representation are the spherical harmonics

$$\langle \theta, \phi | l, m \rangle \equiv Y_l^m(\theta, \phi)$$

The unperturbed ground state is $|0, 0\rangle$ and is not degenerate. Therefore the first correction to the ground state energy is

$$E_0^1 = \langle 0, 0 | H^1 | 0, 0 \rangle = \kappa \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi Y_0^{0*}(\theta, \phi) \cos^2\phi Y_0^0(\theta, \phi) = \kappa \frac{1}{4\pi} 2 \underbrace{\int_0^{2\pi} d\phi \cos^2\phi}_\pi = \underline{\underline{\frac{\kappa}{2}}}$$

where we used $Y_0^0 = \frac{1}{\sqrt{4\pi}}$. The second order correction to the ground state energy is found by computing the following

$$E_0^2 = \sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{|\langle l, m | H^1 | 0, 0 \rangle|^2}{E_0^0 - E_l^0}$$

As stated in the problem text, the sum is approximated by only keeping states belonging to the two lowest lying excited energy *levels*. This means that only $l = 1$ and $l = 2$ levels are being kept in the sum. The matrix elements are

$$\langle l, m | H^1 | 0, 0 \rangle = \kappa \langle l, m | \cos^2 \phi | 0, 0 \rangle = \kappa \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi Y_l^{*m}(\theta, \phi) \cos^2 \phi Y_0^0(\theta, \phi)$$

where $l = 1$ and $l = 2$. Now we write the $\cos(\phi)$ on exponential form

$$\cos^2 \phi = \left[\frac{1}{2} (e^{i\phi} + e^{-i\phi}) \right]^2 = \frac{1}{4} (e^{i2\phi} + e^{-i2\phi}) + \frac{1}{2}$$

and insert. We get for $l = 1$ and $l = 2$

$$\langle l, m | H^1 | 0, 0 \rangle = \frac{\kappa}{4} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi Y_l^{*m}(\theta, \phi) (e^{i2\phi} + e^{-i2\phi}) Y_0^0(\theta, \phi)$$

where the constant term $1/2$ of the perturbation vanished because of the orthogonality of Y_l^m and Y_0^0 for $l \neq 0$. Since the perturbation is independent of θ it is convenient to use $Y_l^m(\theta, \phi) = \Theta_l^m(\theta) e^{im\phi}$, which gives

$$\langle l, m | H^1 | 0, 0 \rangle = \frac{\kappa}{4} \int_0^\pi d\theta \sin \theta \Theta_l^{*m}(\theta) \Theta_0^0(\theta) \int_0^{2\pi} d\phi e^{-im\phi} (e^{i2\phi} + e^{-i2\phi})$$

The integral over ϕ only gives a non-zero contribution when $m = -2$ or $m = 2$.

$$\int_0^{2\pi} d\phi e^{-im\phi} (e^{i2\phi} + e^{-i2\phi}) = 2\pi (\delta_{m,2} + \delta_{m,-2})$$

This means that for the $l = 1$ and $l = 2$ levels, we only get non-zero contributions from the states $|2, +2\rangle$ and $|2, -2\rangle$. Using $Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm i2\phi}$, we find

$$\langle 2, \pm 2 | H^1 | 0, 0 \rangle = \frac{\kappa}{4} \sqrt{\frac{15}{32\pi}} \frac{1}{\sqrt{4\pi}} 2\pi \int_0^\pi d\theta \sin \theta \sin^2 \theta = \frac{\kappa}{4} \sqrt{\frac{15}{32}} \underbrace{\int_{-1}^1 d(\cos \theta) (1 - \cos^2 \theta)}_{2 - \frac{2}{3} = \frac{4}{3}} = \kappa \sqrt{\frac{5}{96}}$$

The $l = 2$ level has unperturbed energy $E_2^0 = 6J$. Thus the second order perturbative correction is

$$E_0^2 = 2 \frac{\left| \kappa \sqrt{\frac{5}{96}} \right|^2}{-6J} = -\kappa \frac{\kappa}{J} \frac{5}{288}$$

We have left out the terms with $l > 2$:

$$- \sum_{l>2} \left(\frac{|\langle l, -2 | H^1 | 0, 0 \rangle|^2}{E_l^0} + \frac{|\langle l, 2 | H^1 | 0, 0 \rangle|^2}{E_l^0} \right)$$

and we see that they always give a further negative contribution to E_0^2 . Thus our result keeping only $l = 1$ and $l = 2$ levels underestimates the magnitude of the second order energy correction.

To get full score both results for the first order and the second order correction must be correct. Partial credit of 2 points is given for realizing that the spherical harmonics are the unperturbed eigenstates. Up to 2 points is given for getting the first order correction correct. Up to 2 points is given for getting the second order correction correct. There is a lot to do on this subproblem, so no credit is given for general perturbation formulae without attempting to connect them to this subproblem.

4b) (6 points) The unperturbed first excited level has $l = 1$ and is threefold degenerate. Therefore we need to use degenerate perturbation theory to calculate the first order energy correction. Either we can find the good states, or we can compute all the matrix elements in the degenerate subspace. Let us do the latter. We must then compute the matrix elements $\langle 1, m' | H^1 | 1, m \rangle$ with $m', m \in \{-1, 0, 1\}$. Writing $Y_1^m(\theta, \phi) = \Theta_1^m(\theta) e^{im\phi}$, we get

$$\langle 1, m' | H^1 | 1, m \rangle = \frac{\kappa}{4} \int_0^\pi d\theta \sin \theta \Theta_1^{*m'}(\theta) \Theta_1^m(\theta) \int_0^{2\pi} d\phi e^{-im'\phi} (e^{i2\phi} + e^{-i2\phi} + 2) e^{im\phi}$$

The integral over ϕ is

$$\int_0^{2\pi} d\phi e^{-im'\phi} (e^{i2\phi} + e^{-i2\phi} + 2) e^{im\phi} = 2\pi (\delta_{m-m', -2} + \delta_{m-m', 2} + 2\delta_{m, m'})$$

Therefore the only non-zero matrix elements are the diagonal ones $m' = m$, and the two offdiagonal $m' = +1, m = -1$ and $m' = -1, m = +1$. Using

$$Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_1^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

the non-zero diagonal matrix elements are

$$\langle 1, 0 | H^1 | 1, 0 \rangle = \frac{\kappa}{4} \int_0^\pi d\theta \sin \theta \frac{3}{4\pi} 4\pi \cos^2 \theta = \kappa \frac{3}{4} \underbrace{\int_{-1}^1 d(\cos \theta) \cos^2 \theta}_{\frac{2}{3}} = \frac{\kappa}{2}$$

$$\langle 1, \pm 1 | H^1 | 1, \pm 1 \rangle = \frac{\kappa}{4} \int_0^\pi d\theta \sin \theta \frac{3}{8\pi} 4\pi \sin^2 \theta = \kappa \frac{3}{8} \underbrace{\int_{-1}^1 d(\cos \theta) \sin^2 \theta}_{\frac{4}{3}} = \frac{\kappa}{2}$$

The offdiagonal matrix elements are

$$\langle 1, \pm 1 | H^1 | 1, \mp 1 \rangle = -\frac{\kappa}{4} \int_0^\pi d\theta \sin \theta \frac{3}{8\pi} 2\pi \sin^2 \theta = -\kappa \frac{3}{16} \underbrace{\int_{-1}^1 d(\cos \theta) \sin^2 \theta}_{\frac{4}{3}} = -\frac{\kappa}{4}$$

The first order energy corrections E_1^1 are gotten by solving the determinant relation

$$\begin{vmatrix} \frac{\kappa}{2} - E_1^1 & 0 & -\frac{\kappa}{4} \\ 0 & \frac{\kappa}{2} - E_1^1 & 0 \\ -\frac{\kappa}{4} & 0 & \frac{\kappa}{2} - E_1^1 \end{vmatrix} = 0$$

Computing the determinant, we find

$$\left(\frac{\kappa}{2} - E_1^1\right) \left(\left(\frac{\kappa}{2} - E_1^1\right)^2 - \left(\frac{\kappa}{4}\right)^2\right) = 0$$

which has solutions

$$\underline{E_1^1 = \frac{\kappa}{2} \text{ and } E_1^1 = \frac{\kappa}{2} \pm \frac{\kappa}{4}}$$

Thus to first order in the perturbation the first order excited energy level is split into three levels.

Alternatively one can find the good states that diagonalize the perturbation. A road to finding the good states is to look for eigenstates of an operator $\hat{A}$ that commutes with both L^2 and the perturbation, i.e. $\hat{A}$ describes a symmetry of the full Hamiltonian. Since the perturbation is $\cos^2\phi$ one can try the reflection operators about the xy- and the xz-planes. The action of these operators on a state $f(\theta, \phi)$ are

$$R_{xy}f(\theta, \phi) = f(\pi - \theta, \phi)$$

$$R_{xz}f(\theta, \phi) = f(\theta, -\phi)$$

Note that $\cos^2\phi$ is unchanged by $\phi \rightarrow -\phi$ and so is L^2 . Now putting in the $l = 1$ spherical harmonics we see that

$$R_{xy}Y_1^0(\theta, \phi) = -Y_1^0(\theta, \phi), \quad R_{xy}Y_1^{\pm 1}(\theta, \phi) = Y_1^{\pm 1}(\theta, \phi)$$

$$R_{xz}Y_1^0(\theta, \phi) = Y_1^0(\theta, \phi), \quad R_{xz}Y_1^{\pm 1}(\theta, \phi) = Y_1^{\mp 1}(\theta, \phi)$$

By inspection we see that the eigenstates of these two operators are Y_1^0 and $\frac{1}{\sqrt{2}} (Y_1^{-1} \pm Y_1^{+1})$. Then if we choose $\hat{A} = aR_{xy} + bR_{xz}$ we see that the eigenstates and eigenvalues of $\hat{A}$ are

$$Y_1^0, \quad \text{eigenvalue: } -a + b$$

$$\frac{1}{\sqrt{2}} (Y_1^{-1} \pm Y_1^{+1}), \quad \text{eigenvalue: } a \pm b.$$

Thus we can get distinct eigenvalues with for instance $a = 2$ and $b = 1$. Therefore the good states are $\frac{1}{\sqrt{2}} (Y_1^{-1} \pm Y_1^{+1})$ and Y_1^0 . The energy corrections can therefore be obtained as in non-degenerate perturbation theory

$$E_1^1 = \langle 1, 0 | H^1 | 1, 0 \rangle = \underline{\underline{\frac{\kappa}{2}}},$$

and the other two corrections are

$$\begin{aligned} E_1^1 &= \frac{1}{2} (\langle 1, -1 | \pm \langle 1, +1 |) H^1 (|1, -1\rangle \pm |1, +1\rangle) \\ &= \frac{1}{2} (\langle 1, -1 | H^1 | 1, -1 \rangle + \langle 1, +1 | H^1 | 1, +1 \rangle \pm \langle 1, -1 | H^1 | 1, +1 \rangle \pm \langle 1, +1 | H^1 | 1, -1 \rangle) \\ &= \frac{1}{2} \left(\frac{\kappa}{2} + \frac{\kappa}{2} \pm \left(-\frac{\kappa}{4} - \frac{\kappa}{4} \right) \right) = \underline{\underline{\frac{\kappa}{2} \mp \frac{\kappa}{4}}} \end{aligned}$$

where all the integrals are carried out as above.

Partial credit of 1 point is given for realizing that one needs degenerate perturbation theory. In addition to this partial credit of up to 3 points is given for getting the W matrix elements, or for identifying the good states in a convincing way. Full score is given for the correct answers. Minor algebraic mistakes are penalized by up to 2 points.