

# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

**Exam in:** FYS3110 – Quantum mechanics

**Date and time:** Dec. 21 at 15:00 (on Inspira)

*Remember to put your candidate number on your answer sheets (not your name).*

### Problem 1

**1a) (6 points)** Define what is meant by a hermitian operator. What are the properties of hermitian operators which make them suitable for representing physical observables. Explain.

#### Solution:

A hermitian operator is an operator that is equal to its hermitian conjugate:  $\hat{K} = \hat{K}^\dagger$ .  
Properties of hermitian operators (see B&DD, 3.4):

- Real eigenvalues. Eigenvalues correspond to measurement values, they should be real.
- Eigenstates with different eigenvalues are orthogonal. Eigenstates are states for which the measurement value is the corresponding eigenvalue with 100% certainty. Physical states that for sure give different measurement values must be orthogonal.
- Eigenstates *span* the Hilbert space<sup>1</sup>. It must be possible to assign probabilities for getting a particular measurement value for *any* state. This means that it must be possible to write any state as a superposition of eigenstates. This is the spanning or completeness property.

#### Grading:

To get full score the definition of hermitian operator must be stated, and three properties. No statements can be wrong or misleading. Partial score will be given if only some of the properties are present.

**1b) (6 points)** What is the Pauli exclusion principle? Explain the context in which it occurs, and the reason for why it is valid.

#### Solution:

The Pauli exclusion principle states that a quantum many-body state of identical fermions cannot be composed out of more than one copy of each single-particle states. Or briefly: two identical fermions cannot occupy the same single-particle state.

The context is a many-particle state of identical fermions. The reason why it is valid is

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<sup>1</sup>The last property is not automatic for hermitian operators in an  $\infty$ -dimensional Hilbert space.

that a system of identical fermions must have a state that is antisymmetric under the permutation of particle labels. If two fermions are in the same single-particle state, a permutation of their labels leads to the same state. This is only consistent with being antisymmetric if the state itself vanishes ( $0 = -0$ ). Such a state is the zero ket, and is not physical.

**Grading:**

To get full score the answer must state the Pauli principle, mention the context either specifically or generally, and also explain the impossibility of making an antisymmetric combination of identical single-particle states. Partial credit is given if only parts are present. Deductions will be made for statements that are wrong or indicate misunderstandings.

**Problem 2**

**2a) (2 points)** Two kets  $|A\rangle$  and  $|B\rangle$  are specified in terms of an orthonormal basis set  $\{|0\rangle, |1\rangle, |2\rangle\}$  as

$$\begin{aligned}|A\rangle &= i|0\rangle + 2|1\rangle - i|2\rangle \\ |B\rangle &= -i|0\rangle + |1\rangle + 2i|2\rangle\end{aligned}$$

Calculate the inner product  $\langle B|A\rangle$ . Alternatives: -8,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,8,

**Solution:**

The bra  $\langle B|$  is

$$\langle B| = i\langle 0| + \langle 1| - 2i\langle 2|$$

So that

$$\langle B|A\rangle = (i\langle 0| + \langle 1| - 2i\langle 2|)(i|0\rangle + 2|1\rangle - i|2\rangle) = i(i)\langle 0|0\rangle + 2\langle 1|1\rangle - 2i(-i)\langle 2|2\rangle = \underline{-1}$$

**2b) (2 points)** A calculation gives  $\langle\psi|H|\psi\rangle = E_\psi$  where  $H$  is a Hamiltonian and  $|\psi\rangle$  is a normalized ket. Which of the following statements is certainly true: Alternatives:  $E_\psi$  is positive or zero,  $E_\psi$  is bigger than or equal to the ground state energy,  $|\psi\rangle$  is an energy eigenstate,  $|\psi\rangle$  is orthogonal to the ground state,  $|\psi\rangle$  is the ground state,  $E_\psi$  is a good estimate of the ground state energy,  $E_\psi$  is always smaller than the first excited energy level,  $H$  has degenerate energy levels.

**Solution:**

The correct answer is  $E_\psi$  is bigger than or equal to the ground state energy. This is the variational principle.

**2c) (2 points)** A one-dimensional system is periodic with period  $l$ . Which of the following alternatives describes best  $\langle\psi|\hat{T}_l|\psi\rangle$ , where  $\hat{T}_l$  is the translation operator that translates the

system a distance  $l$ , and  $|\psi\rangle$  is an energy eigenstate of the system. Alternatives: 0, 1, a real number, a phase factor, a positive real number, a negative real number, an imaginary number.

**Solution:**

We have

$$\langle\psi|\hat{T}_l|\psi\rangle = \int_{-\infty}^{\infty} dx \psi^*(x) \psi(x+l) = \int_{-\infty}^{\infty} dx \psi^*(x) e^{iKl} \psi(x) = e^{iKl}$$

where we used Bloch's theorem  $\psi(x+l) = e^{iKl}\psi(x)$  which holds in a periodic system.  $K$  is a real number. Thus the correct answer is: a phase factor.

**2d) (2 points)** What is the best estimate of the ionization energy (energy to remove the electron) of  $Li^{2+}$  in its ground state (Li has atomic number 3) of the following alternatives? Alternatives: 0 eV, 10eV, 15eV, 25eV, 40eV, 60eV, 90eV, 120eV, 180eV, 240eV

**Solution:**

$Li$  normally has three electrons, but in  $Li^{2+}$  two electrons are already removed.  $Li^{2+}$  resembles therefore hydrogen, but with a nuclear charge  $Z = 3$ . The ground state energy is therefore  $-13.6\text{eV} \times 3^2 = -122.4\text{eV}$ , and the energy needed to remove the electron is 122.4eV, meaning that the closest alternative is 120eV.

**2e) (2 points)** The energy correction due to fine-structure is

$$E_{nj} = -E_1 \alpha^2 \left( \frac{n}{j + 1/2} - \frac{3}{4} \right).$$

To how many different energy levels will this correction split the  $n = 3$  Coulomb potential energy level?

Alternatives: 1,2,3,4,6,8,9,10,12,18.

**Solution:**

For  $n = 3$  the orbital angular momentum takes the values  $l = \{0, 1, 2\}$ . For each of these the angular momentum takes the values  $j = \{|l - \frac{1}{2}|, l + \frac{1}{2}\}$ . Therefore the possible values of  $j$  are  $j \in \{1/2, 3/2, 5/2\}$ , and therefore the  $n = 3$  level will be split into 3 distinct levels.

**2f) (2 points)** Two spin-1/2 d-electrons ( $l = 2$  each) are measured to be in a spin triplet state. The same state is also known to be an eigenstate of the total orbital angular momentum operator. What are the possible values of the total orbital angular momentum in this state. Alternatives:  $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{0,1\}, \{0,2\}, \{0,4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{0,2,4\}, \{0,1,2,3,4\}$ , None of the above.

**Solution:**

Electrons are identical fermions so their many-body state must be antisymmetric under

the interchange of particle labels. The spin triplet state is symmetric under particle interchange, and therefore the orbital state must be antisymmetric. Inspecting the Clebsch-Gordan table for combining orbital angular momentum 2 with 2, we find that the states with total orbital angular momentum 0, 2 and 4 are all symmetric, while the states with total orbital angular momentum 1 and 3 are antisymmetric. Therefore, the allowed values for the total orbital angular momentum of this state are 1, 3.

**2g) (2 points)** What is the result of

$$(a, b) e^{ik\sigma^x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where  $a, b$  and  $k$  are complex numbers, and  $\sigma^x$  is the Pauli matrix

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Alternatives:  $ae^{ik}$ ,  $be^{ik}$ ,  $ae^{-ik}$ ,  $be^{-ik}$ ,  $ae^{ik}+be^{-ik}$ ,  $ae^{ik}-be^{-ik}$ ,  $a \cos k - ib \sin k$ ,  $a \cos k + ib \sin k$ ,  $a \cos k + b \cos k$ ,  $a \cos k - b \cos k$ , None of the above.

**Solution:**

The unnormalized eigenvectors of the  $\sigma^x$  matrix are

$$\sigma^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} = +1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \sigma^x \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then writing the right hand column vector as a sum of eigenvectors, we can write

$$\begin{aligned} (a, b) e^{ik\sigma^x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= (a, b) e^{ik\sigma^x} \left( \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{1}{2} (a, b) \left( e^{ik\sigma^x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{ik\sigma^x} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \\ &= \frac{1}{2} (a, b) \left( e^{ik} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-ik} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{1}{2} (a, b) \begin{pmatrix} e^{ik} + e^{-ik} \\ e^{ik} - e^{-ik} \end{pmatrix} \\ &= (a, b) \begin{pmatrix} \cos k \\ i \sin k \end{pmatrix} = \underline{a \cos k + ib \sin k} \end{aligned}$$

### Problem 3

Consider the following spin-1/2 Hamiltonian with 2 spins

$$H = \frac{J}{\hbar^2} (S_1^x S_2^x + S_1^y S_2^y)$$

where  $J$  is a positive constant with dimensions energy.

**3a) (... points)** Find the eigenstates and associated eigenvalues of  $H$ .

**Solution:**

To find the eigenstates and eigenvalues it is helpful to write the Hamiltonian in terms of ladder operators:

$$H = \frac{J}{2\hbar^2} (S_1^+ S_2^- + S_1^- S_2^+)$$

and then apply it to the four basis states. One then gets

$$H|\uparrow\uparrow\rangle = 0, \quad H|\uparrow\downarrow\rangle = \frac{J}{2}|\downarrow\uparrow\rangle, \quad H|\downarrow\uparrow\rangle = \frac{J}{2}|\uparrow\downarrow\rangle, \quad H|\downarrow\downarrow\rangle = 0.$$

Then making the symmetric and antisymmetric combinations of the up-down states, one sees that the eigenstates and associated eigenvalues are

| # | state                                                                         | energy |
|---|-------------------------------------------------------------------------------|--------|
| 0 | $\frac{1}{\sqrt{2}}( \uparrow\downarrow\rangle -  \downarrow\uparrow\rangle)$ | $-J/2$ |
| 1 | $ \uparrow\uparrow\rangle$                                                    | 0      |
| 2 | $ \downarrow\downarrow\rangle$                                                | 0      |
| 3 | $\frac{1}{\sqrt{2}}( \uparrow\downarrow\rangle +  \downarrow\uparrow\rangle)$ | $J/2$  |

An alternative solution is to form the total spin operator:  $\vec{S} = \vec{S}_1 + \vec{S}_2$ . So that  $\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2(S_1^x S_2^x + S_1^y S_2^y) + 2S_1^z S_2^z$ . Then

$$H = \frac{J}{2\hbar^2} (\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2 - 2S_1^z S_2^z)$$

Thus the eigenstates of  $H$  are therefore eigenstates of the total spin, namely the triplet and the singlet state:

$$\begin{aligned} H|0,0\rangle &= \frac{J}{2} \left( -\frac{3}{2} + \frac{1}{2} \right) |0,0\rangle = -\frac{J}{2} |0,0\rangle \\ H|1,+1\rangle &= \frac{J}{2} \left( 2 - \frac{3}{2} - \frac{1}{2} \right) |1,+1\rangle = 0 \\ H|1,-1\rangle &= \frac{J}{2} \left( 2 - \frac{3}{2} - \frac{1}{2} \right) |1,-1\rangle = 0 \\ H|1,0\rangle &= \frac{J}{2} \left( 2 - \frac{3}{2} + \frac{1}{2} \right) |1,0\rangle = \frac{J}{2} |1,0\rangle \end{aligned}$$

same result as above.

A small perturbation is induced such that the Hamiltonian becomes

$$H = \frac{J}{\hbar^2} ((1 + \lambda) S_1^x S_2^x + S_1^y S_2^y)$$

where  $\lambda \ll 1$  is a small dimensionless positive number.

**3b) (... points)** The unperturbed ground state of  $H$  for  $J > 0$  is the state  $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ . Use time-independent perturbation theory to calculate the first and second order correction (in  $\lambda$ ) to the ground state energy when  $J > 0$ .

**Solution:**

The perturbation Hamiltonian is

$$H^1 = \frac{J\lambda}{\hbar^2} S_1^x S_2^x = \frac{J\lambda}{4\hbar^2} (S_1^+ S_2^- + S_1^- S_2^+ + S_1^+ S_2^+ + S_1^- S_2^-)$$

where the spin-operators has been rewritten using spin ladder operators. The ground state of the unperturbed problem is unique and so we may use non-degenerate perturbation theory. The first order energy correction to the ground state is therefore

$$\begin{aligned} E_0^1 &= \langle 0 | H^1 | 0 \rangle \\ &= \frac{J\lambda}{8\hbar^2} (\langle \uparrow\downarrow | - \langle \downarrow\uparrow |) (S_1^+ S_2^- + S_1^- S_2^+ + S_1^+ S_2^+ + S_1^- S_2^-) (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ &= \frac{J\lambda}{8} (\langle \uparrow\downarrow | - \langle \downarrow\uparrow |) (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) = -\frac{J\lambda}{4}. \end{aligned}$$

The second order correction involves matrix elements

$$E_0^2 = \sum_{m \neq 0} \frac{|\langle m | H^1 | 0 \rangle|^2}{E_0 - E_m}$$

where the sum over  $m$  goes over all the excited states, not the ground state. We just showed that  $H^1|0\rangle = -|0\rangle$ , and therefore the 2. order correction will be 0 as a consequence of orthogonality of the excited states and the ground state.

**3c) (... points)** Use time-independent perturbation theory to calculate the first order energy correction in  $\lambda$  to the rest of the unperturbed energy levels.

**Solution:**

The unperturbed energy level with energy 0 is two-fold degenerate. We must therefore use *degenerate* perturbation theory. Since there are just two states,  $|1\rangle = |\uparrow\uparrow\rangle$  and  $|2\rangle = |\downarrow\downarrow\rangle$ , we can just calculate the perturbation matrix by brute force. We have

$$\begin{aligned} H^1 |\uparrow\uparrow\rangle &= \frac{J\lambda}{4} |\downarrow\downarrow\rangle \\ H^1 |\downarrow\downarrow\rangle &= \frac{J\lambda}{4} |\uparrow\uparrow\rangle \end{aligned}$$

The energy corrections are therefore found as the solution to the following determinant equation

$$\begin{vmatrix} -E_{1,2}^1 & J\lambda/4 \\ J\lambda/4 & -E_{1,2}^1 \end{vmatrix} = 0 \implies (E_{1,2}^1)^2 = \left(\frac{J\lambda}{4}\right)^2 \implies \underline{E_{1,2}^1 = \pm \frac{J\lambda}{4}}.$$

and so this energy level is being split as a result of the perturbation.

The unperturbed highest energy level with  $E_3 = J/2$  is not degenerate, and the unperturbed state is  $|3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ . The energy correction is therefore

$$\begin{aligned} E_3^1 &= \frac{1}{2} (\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) H^1 (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ &= \frac{J\lambda}{8\hbar^2} (\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) (S_1^+ S_2^- + S_1^- S_2^+ + S_1^+ S_2^+ + S_1^- S_2^-) (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ &= \frac{J\lambda}{8} (\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) = \underline{\underline{\frac{J\lambda}{4}}} \end{aligned}$$

### Grading:

Subproblems 3a), b) and c) are treated as one problem with a maximum total of **18 points**. To get full score all answers must be correct. Partial credit will be given if some of the energy eigenstates/eigenvalues and perturbation methods are done correctly. A maximum of two points is awarded for just writing down general formulae from the book without actually employing them.

### Problem 4

A one-dimensional harmonic oscillator with Hamiltonian  $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$  is in its ground state at  $t = 0$ . For times  $0 < t \leq T$  it is perturbed by a potential

$$V = \lambda\hat{x}(1 - t/T), \quad 0 < t \leq T$$

where  $\lambda$  has dimensions energy divided by length.

**4a) (... points)** Use first order time-dependent perturbation theory in  $\lambda$  to show that the probability *amplitude* to find the harmonic oscillator in the first excited state at time  $t = T$  is

$$c_1(T) = \alpha \int_0^T dt (1 - t/T) e^{i\omega t}$$

where  $\alpha$  is a constant. Find also the value of  $\alpha$ .

### Solution:

The general formula from first order time-dependent perturbation theory for the probability amplitude to find the perturbed system in an energy eigen state  $f$  at time  $t$  given an initial energy eigenstate  $i$  at  $t = 0$  is

$$c_1(t) = \frac{1}{i\hbar} \int_0^t dt' \langle f | V(t') | i \rangle e^{i(E_f - E_i)t'/\hbar}$$

Inserting the ground state energy  $E_i = \hbar\omega/2$ , the excited energy  $E_f = 3\hbar\omega/2$ , the form of

the perturbing potential and  $t = T$ , we get

$$c_1(T) = \frac{\lambda}{i\hbar} \int_0^T dt \langle 1|\hat{x}|0\rangle (1 - t/T) e^{i\omega t}$$

where we have also changed the name of the integration variable from  $t' \rightarrow t$ . Using the fact that the position operator  $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a)$ , the matrix element becomes  $\langle 1|\hat{x}|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}$ , and we get

$$c_1(T) = \frac{\lambda}{i\hbar} \sqrt{\frac{\hbar}{2m\omega}} \int_0^T dt (1 - t/T) e^{i\omega t}$$

which is of the desired form with

$$\alpha = \frac{\lambda}{i\hbar} \sqrt{\frac{\hbar}{2m\omega}} = -i\lambda \sqrt{\frac{1}{2m\hbar\omega}}.$$

**4b) (... points)** Find the leading order  $T$  dependence of  $c_1(T)$  when  $\omega T \ll 1$ . Discuss the validity of this perturbation theory result for long times  $T$  (but not so long that you cannot assume  $\omega T \ll 1$ ).

**Solution:**

To see the behavior as a function of  $T$  we change variable in the integral to  $\tau = t/T$ . Then

$$c_1(T) = \alpha T \int_0^1 d\tau (1 - \tau) e^{i(\omega T)\tau} = \alpha T f(\omega T)$$

where

$$f(\omega T) \equiv \int_0^1 d\tau (1 - \tau) e^{i(\omega T)\tau}$$

is a function of  $\omega T$ .

For  $\omega T \ll 1$  the perturbation is very sudden and short compared to the wavelength of the oscillator. The exponential factor in the integral can be approximated to leading order in  $\omega T$  as 1, and the integral becomes  $1/2$ , and so  $c_1(T) = \alpha T/2$ , i.e. a linearly increasing function of  $T$ .

Therefore the probability of finding the system in the excited state increases as the square of  $T$ , and can surpass 1 if  $T$  becomes big enough. This is an artefact of using first order perturbation theory, and so the result will only hold for times which are significantly shorter than the time at which the amplitude becomes 1:

$$T \ll \frac{2}{|\alpha|} = \frac{\sqrt{8m\hbar\omega}}{\lambda}.$$

**4c) (... points)** Use first order time-dependent perturbation theory to calculate the *probability* (not the probability amplitude) of finding the harmonic oscillator in the first excited state at time  $t = T$ . (You may state your answer in terms of  $\alpha$ ).

**Solution:**

We must then do the integral

$$c_1 = \alpha T \int_0^1 d\tau (1 - \tau) e^{i(\omega T)\tau} = \alpha T \int_0^1 d\tau (1 - \tau) e^{i\Omega\tau}$$

and take the modulus square in order to find the probability. To make the notation simpler we introduced  $\Omega \equiv \omega T$ . The integral can for instance be carried out using partial integration with  $v = 1 - \tau$  and  $u' = e^{i\Omega\tau}$ . This gives

$$\int_0^1 d\tau (1 - \tau) e^{i\Omega\tau} = (1 - \tau) \frac{e^{i\Omega\tau}}{i\Omega} \Big|_0^1 - \int_0^1 d\tau (-1) \frac{e^{i\Omega\tau}}{i\Omega} = -\frac{1}{i\Omega} + \frac{e^{i\Omega} - 1}{(i\Omega)^2}$$

and the probability amplitude becomes

$$c_1(T) = \frac{\alpha}{i\omega} \left( -1 + \frac{e^{i\omega T} - 1}{i\omega T} \right) = -\frac{\alpha}{i\omega} \left( 1 + \frac{1 - e^{i\omega T}}{i\omega T} \right)$$

(Although not necessary for credit, we can check our result in the previous problem: Taking the  $\omega T \rightarrow 0$  limit of this by expanding the exponential function to second order in  $\omega T$ , we get  $c_1(T) = -\frac{\alpha}{i\omega} \left( 1 + \frac{1 - 1 - i\omega T - (i\omega T)^2/2}{i\omega T} \right) = \alpha T/2$  which corresponds to the result we got.)

Taking the modulus square of  $c_1(T)$  we find

$$\begin{aligned} P_1(T) &= \frac{|\alpha|^2}{\omega^2} \left( 1 + \frac{1 - e^{i\omega T}}{i\omega T} \right) \left( 1 - \frac{1 - e^{-i\omega T}}{i\omega T} \right) \\ &= \frac{|\alpha|^2}{\omega^2} \left( 1 + \frac{1}{i\omega T} (1 - e^{i\omega T} - 1 + e^{-i\omega T}) + \frac{1}{(\omega T)^2} (1 - e^{i\omega T}) (1 - e^{-i\omega T}) \right) \\ &= \frac{|\alpha|^2}{\omega^2} \left( 1 - \frac{1}{i\omega T} (e^{i\omega T} - e^{-i\omega T}) + \frac{1}{(\omega T)^2} (2 - e^{i\omega T} - e^{-i\omega T}) \right) \\ &= \frac{|\alpha|^2}{\omega^2} \left( 1 - 2 \frac{\sin \omega T}{\omega T} + 2 \frac{1 - \cos \omega T}{(\omega T)^2} \right) \end{aligned}$$

**Grading:**

4a), 4b) and 4c) are treated as one problem with a max. total of **12 points**. For full score, the proof in a) must be convincing and contain the correct expression for  $\alpha$ , and the behaviour in b) must be found to be linear in  $T$  with a brief statement about the validity of first order perturbation theory, and in c) the correct answer must be gotten and written in an explicit way using ordinary trigonometric functions. Partial credit is given based on how well these goals have been achieved.

**Problem 5**

An operator  $\hat{b}$  is expressed in terms of one dimensional harmonic oscillator ladder operators  $\hat{a}$  and  $\hat{a}^\dagger$  as

$$\hat{b} = \hat{a} \cosh \theta + \hat{a}^\dagger \sinh \theta$$

where  $[\hat{a}, \hat{a}^\dagger] = 1$  and  $\theta$  is a real number such that the hyperbolic trigonometric functions obey  $\cosh^2 \theta - \sinh^2 \theta = 1$ . Define the normalized state  $|0\rangle$  as the state which is annihilated by  $\hat{b}$ . i.e.  $\hat{b}|0\rangle = 0$ .

**5a) (4 points)** Calculate  $\langle 0|\hat{a}\hat{a}|0\rangle$

**Solution:**

Taking the hermitian conjugate, we find that

$$\hat{b}^\dagger = \hat{a}^\dagger \cosh \theta + \hat{a} \sinh \theta$$

Then multiplying  $\hat{b}$  with  $\cosh \theta$  and  $\hat{b}^\dagger$  with  $\sinh \theta$  and subtracting, we can express  $\hat{a}$  in terms of  $\hat{b}$  and  $\hat{b}^\dagger$ :

$$\hat{b} \cosh \theta - \hat{b}^\dagger \sinh \theta = \underbrace{(\cosh^2 \theta - \sinh^2 \theta)}_1 \hat{a} \implies \hat{a} = \hat{b} \cosh \theta - \hat{b}^\dagger \sinh \theta$$

The matrix element is therefore

$$\begin{aligned} \langle 0|\hat{a}\hat{a}|0\rangle &= \langle 0| \left( \hat{b} \cosh \theta - \hat{b}^\dagger \sinh \theta \right) \left( \hat{b} \cosh \theta - \hat{b}^\dagger \sinh \theta \right) |0\rangle \\ &= \cosh^2 \theta \langle 0|\hat{b}\hat{b}|0\rangle + \sinh^2 \theta \langle 0|\hat{b}^\dagger \hat{b}^\dagger|0\rangle - \sinh \theta \cosh \theta \left( \langle 0|\hat{b}^\dagger \hat{b}|0\rangle + \langle 0|\hat{b}\hat{b}^\dagger|0\rangle \right) \\ &= \sinh^2 \theta \langle 0|\hat{b}^\dagger \hat{b}^\dagger|0\rangle - \sinh \theta \cosh \theta \langle 0|\hat{b}\hat{b}^\dagger|0\rangle \end{aligned}$$

where we used that  $\hat{b}|0\rangle = 0$ . The first term will also vanish for the same reason because  $\hat{b}^\dagger \hat{b}^\dagger = (\hat{b}\hat{b})^\dagger$  which leads to

$$\langle 0|\hat{b}^\dagger \hat{b}^\dagger|0\rangle = \langle 0| \left( \hat{b}\hat{b} \right)^\dagger |0\rangle = \langle 0|\hat{b}\hat{b}|0\rangle^* = 0.$$

Therefore we are left with

$$\langle 0|\hat{a}\hat{a}|0\rangle = -\sinh \theta \cosh \theta \langle 0|\hat{b}\hat{b}^\dagger|0\rangle.$$

To find how to evaluate this, we calculate the commutator

$$\begin{aligned} [\hat{b}, \hat{b}^\dagger] &= [\hat{a} \cosh \theta + \hat{a}^\dagger \sinh \theta, \hat{a}^\dagger \cosh \theta + \hat{a} \sinh \theta] = \cosh^2 \theta [\hat{a}, \hat{a}^\dagger] + \sinh^2 \theta [\hat{a}^\dagger, \hat{a}] \\ &= \underbrace{(\cosh^2 \theta - \sinh^2 \theta)}_1 \underbrace{[\hat{a}, \hat{a}^\dagger]}_1 = 1. \end{aligned}$$

It follows that  $\hat{b}$  and  $\hat{b}^\dagger$  are themselves 1D harmonic oscillator ladder operators, with  $\hat{b}\hat{b}^\dagger = \hat{b}^\dagger\hat{b} + 1$ , implying that

$$\langle 0|\hat{a}\hat{a}|0\rangle = -\sinh\theta \cosh\theta \langle 0|\hat{b}\hat{b}^\dagger|0\rangle = -\sinh\theta \cosh\theta \langle 0|(\hat{b}^\dagger\hat{b} + 1)|0\rangle = \underline{-\sinh\theta \cosh\theta}.$$

Alternatively, and a simpler solution, is to form the combinations  $\hat{a}\hat{b}$  and  $\hat{b}\hat{b}$  which both have  $\hat{b}$  to the right ensuring that they are zero when acting on  $|0\rangle$ .

$$\begin{aligned}\hat{a}\hat{b} &= \hat{a}^2 \cosh\theta + \hat{a}\hat{a}^\dagger \sinh\theta \\ \hat{b}\hat{b} &= \hat{a}^2 \cosh^2\theta + (\hat{a}^\dagger)^2 \sinh^2\theta + (\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \cosh\theta \sinh\theta \\ &= \hat{a}^2 \cosh^2\theta + (\hat{a}^\dagger)^2 \sinh^2\theta + (2\hat{a}\hat{a}^\dagger - 1) \cosh\theta \sinh\theta\end{aligned}$$

where we used the commutation relation for the  $\hat{a}$ s. Then multiplying  $\hat{a}\hat{b}$  by  $2\cosh\theta$  and subtracting, we get

$$2\cosh\theta\hat{a}\hat{b} - \hat{b}\hat{b} = \hat{a}^2 \cosh^2\theta - (\hat{a}^\dagger)^2 \sinh^2\theta + \cosh\theta \sinh\theta$$

Then sandwiching both sides between  $\langle 0|$  and  $|0\rangle$ , and using the fact that  $\langle 0|\hat{a}^\dagger\hat{a}^\dagger|0\rangle = \langle 0|\hat{a}\hat{a}|0\rangle^*$ , gives

$$\langle 0|\hat{a}^2|0\rangle \cosh^2\theta - \langle 0|\hat{a}^2|0\rangle^* \sinh^2\theta = -\cosh\theta \sinh\theta$$

This means that

$$\begin{aligned}\Re(\langle 0|\hat{a}^2|0\rangle) \overbrace{(\cosh^2\theta - \sinh^2\theta)}^1 &= -\cosh\theta \sinh\theta \\ \Im(\langle 0|\hat{a}^2|0\rangle) \underbrace{(\cosh^2\theta + \sinh^2\theta)}_{\neq 0} &= 0\end{aligned}$$

where  $\Re$  and  $\Im$  stands for the real and imaginary parts respectively. The last equation implies that the imaginary part is 0, thus the answer is the same as gotten above:

$$\langle 0|\hat{a}^2|0\rangle = \underline{-\cosh\theta \sinh\theta}.$$

### Grading:

For full score on this subproblem the answer must be correct and reached with sound logic. Approaches that assume that the  $|0\rangle$  is annihilated by  $\hat{a}$  are wrong and will not be rewarded.

THE END