

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: FYS3110 – Quantum mechanics

Date and time: Dec. 04 at 09:00 (4 hours)

Permitted aids: Approved calculator, K. Rottmann: "Matematisk formelsamling", One handwritten A4 sheet (2 pages) with your own notes, Textbook: A. Berera and L. Del Debbio, "Quantum Mechanics", and the two printed lecture notes: "Atoms and solids" and "Zeeman effect and hyperfine splitting".

Problem 1

1a) (6 points) Define what is meant by a hermitian operator. What are the properties of hermitian operators which make them suitable for representing physical observables. Explain briefly.

Solution:

A hermitian operator is an operator that is equal to its hermitian conjugate: $\hat{K} = \hat{K}^\dagger$.

- Real eigenvalues. Eigenvalues correspond to measurement values, they should be real.
- Eigenstates with different eigenvalues are orthogonal. Eigenstates are states for which the measurement value is the corresponding eigenvalue with 100% certainty. Physical states that for sure give different measurement values must be orthogonal.
- Eigenstates *span* the Hilbert space¹. It must be possible to assign probabilities for getting a particular measurement value for *any* state. This means that it must be possible to write any state as a superposition of eigenstates. This is the spanning or completeness property.

(from properties of hermitian operators (see B&DD, 3.4)).

Grading:

For full score the answer must include: Definition of hermitian operator (1.5), Real eigenvalues (1.5), Orthogonal eigenkets (1.5), and eigenkets span Hilbert space (1.5).

1b) (3 points) Uncertainty is an inherent part of quantum mechanics. Explain under what circumstances one can still get measurement results with 100% certainty.

Solution:

This occurs whenever the system is prepared in an eigenstate of the physical quantity (operator) being measured. Then the measurement result is guaranteed (to 100% accuracy)

¹The last property is not automatic for hermitian operators in an ∞ -dimensional Hilbert space.

to be the corresponding eigenvalue.

Grading:

To get full score it must be stated that the system is prepared in an *eigenstate* of the physical observable being measured. Heisenberg's uncertainty principle (and commuting operators) is irrelevant in this context as it only *relates* uncertainties in several observables. Partial credit of at most one point is given for (incomplete reasoning) about special examples of measurements that can give 100% accuracy. This includes measuring position when the wave function has collapsed, and measuring a property for which there is only one possibility.

Problem 2

2a) (2 points) Two non-normalized kets: $|A\rangle$ and $|B\rangle$ are specified in terms of an orthonormal basis set $\{|0\rangle, |1\rangle, |2\rangle\}$ as

$$\begin{aligned}|A\rangle &= |0\rangle - i|1\rangle + 2i|2\rangle \\ |B\rangle &= |0\rangle + i|1\rangle + 3i|2\rangle\end{aligned}$$

Calculate the inner product $\langle A|B\rangle$.

Alternatives: -8,-6,-4,-3,-2,-1,0,1,2,3,4,6,8

Solution:

The bra $\langle A|$ is

$$\langle A| = \langle 0| + i\langle 1| - 2i\langle 2|$$

So that

$$\langle A|B\rangle = (\langle 0| + i\langle 1| - 2i\langle 2|)(|0\rangle + i|1\rangle + 3i|2\rangle) = 1\langle 0|0\rangle - 1\langle 1|1\rangle + 6\langle 2|2\rangle = \underline{6}$$

2b) (2 points) A particle in a one dimensional harmonic oscillator potential is in the state $|0\rangle + |1\rangle$ at time $t = 0$. $|n\rangle$ are the standard occupation number states. What is the correct expression for the state at time $t > 0$:

Alternatives: $0, e^{-i\omega t/2}|0\rangle + e^{-i3\omega t/2}|1\rangle, e^{+i\omega t/2}|0\rangle + e^{+i3\omega t/2}|1\rangle, e^{-i\omega t/2}|0\rangle + e^{+i3\omega t/2}|1\rangle, e^{+i\omega t/2}|0\rangle + e^{-i3\omega t/2}|1\rangle, |0\rangle + e^{-i\omega t/2}|1\rangle, |0\rangle + e^{+i\omega t/2}|1\rangle, |0\rangle - e^{-i\omega t/2}|1\rangle, |0\rangle - e^{+i\omega t/2}|1\rangle, |0\rangle + |1\rangle, |0\rangle - |1\rangle$,
None of the alternatives.

Solution:

The time evolution operator is $e^{-iHt/\hbar}$ where $H = \hbar\omega(a^\dagger a + 1/2)$. The time evolved state is then

$$\begin{aligned}|\psi(t)\rangle &= e^{-iHt/\hbar}|\psi(0)\rangle = e^{-iHt/\hbar}(|0\rangle + |1\rangle) = e^{-i\hbar\omega(0+1/2)t/\hbar}|0\rangle + e^{-i\hbar\omega(1+1/2)t/\hbar}|1\rangle \\ &= \underline{e^{-i\omega t/2}|0\rangle + e^{-i3\omega t/2}|1\rangle}\end{aligned}$$

2c) (2 points) A hermitian operator T is a symmetry of a time-independent Hamiltonian H . Which of the following statements is *not* necessarily true (if you think all are true: select None):

Alternatives: $[H, T] = 0$, T is a unitary operator, $\frac{d}{dt}\langle\psi|T|\psi\rangle = 0$ (for any $|\psi\rangle$), There exist a common complete set of eigenstates for T and H , H has a degenerate spectrum, None.

Solution:

The fact that T is a symmetry implies $[H, T] = 0$. The symmetry conserves also the norm of quantum states, so T must be unitary. The fact that $[H, T] = 0$ and H is time-independent means that the expectation value of T is conserved in time. The fact that there exists a common complete set of eigenstate of T and H follows from $[H, T] = 0$. The fact that H has a degenerate spectrum would follow from $[H, T] = 0$ because then

$$HT|E_i\rangle = TH|E_i\rangle = E_i T|E_i\rangle$$

if also

$$T|E_i\rangle \neq |E_i\rangle$$

but the latter is *not* always true. For instance the 1d harmonic oscillator has a non-degenerate energy spectrum, yet the parity operator which exchanges x to $-x$ is a unitary symmetry-operator that commutes with the Hamiltonian.

2d) (2 points) A hydrogen atom is in the state $|nlm\rangle$ with $n = 2$, $l = 1$ and $m = +1$. What is the probability that its electron has a z coordinate greater than half its distance to the proton ($z > r/2$)?

Select the nearest alternative: 0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0

Solution:

We can express the condition $z > r/2$ in spherical coordinates r, θ, ϕ using $z = r \cos \theta$:

$$\cos \theta > \frac{1}{2}$$

The angular dependence of the $|211\rangle$ state is that of $Y_1^{+1}(\theta, \phi)$. Thus the probability of finding the electron with $z > r/2$ is

$$P_{z>r/2} = \int_0^{2\pi} d\phi \int_{1/2}^1 d\cos \theta Y_1^{+1*} Y_1^{+1}$$

Plugging in $Y_1^{+1} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{+i\phi}$ we get

$$\begin{aligned} P_{z>r/2} &= \frac{3}{8\pi} 2\pi \int_{1/2}^1 d\cos \theta \sin^2 \theta = \frac{3}{4} \int_{1/2}^1 d\cos \theta (1 - \cos^2 \theta) = \frac{3}{4} \left(x - \frac{1}{3} x^3 \right) \Big|_{1/2}^1 \\ &= \frac{3}{4} \left(1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{8} \right) = \frac{3}{4} \left(\frac{12}{24} - \frac{8}{24} + \frac{1}{24} \right) = \frac{3}{4} \cdot \frac{5}{24} = \frac{15}{96} \approx 0.156 \end{aligned}$$

so the closest alternative is 0.2.

2e) (2 points) Two identical non-interacting spin-1/2 fermions, each of mass m , are located in a three dimensional harmonic oscillator potential ($V = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$). What is the degeneracy (including spin degrees of freedom) of the first excited energy level for this system?

Alternatives: 1,2,3,4,6,8,9,12,15,16,18,24,27

Solution:

The energy of the system does not depend on spin and is

$$E = \hbar\omega \left(n_{1x} + n_{1y} + n_{1z} + \frac{3}{2} \right) + \hbar\omega \left(n_{2x} + n_{2y} + n_{2z} + \frac{3}{2} \right)$$

where n 's are occupation numbers taking integer (and zero) values. The corresponding eigenstates are products of 1D harmonic oscillator states which we denote $|n_{1x}n_{1y}n_{1z}\rangle \otimes |n_{2x}n_{2y}n_{2z}\rangle$. In addition to the spatial degree of freedom the particles have spin-1/2 and they are fermions. It follows that the states of the system are described by antisymmetric combination of states

$$|n_{1x}n_{1y}n_{1z}\rangle \otimes |n_{2x}n_{2y}n_{2z}\rangle \otimes |m_1\rangle \otimes |m_2\rangle$$

under the permutation of particle labels $1 \leftrightarrow 2$. $|m_i\rangle$ is the spin-1/2 S^z eigenstate of particle i . For the ground state, the spatial part is $|000\rangle \otimes |000\rangle$ which is necessarily symmetric, which means that the spin part must be antisymmetric. This is achieved by forming the singlet combination of the two spins. Thus the ground state is unique; its degeneracy is 1. For the first excited energy level, one of the n 's on one of the particles is 1, the rest are zero. Thus the following symmetric or antisymmetric combinations of the spatial part can be formed

$$|100\rangle \otimes |000\rangle \pm |000\rangle \otimes |100\rangle, \quad |010\rangle \otimes |000\rangle \pm |000\rangle \otimes |010\rangle, \quad |001\rangle \otimes |000\rangle \pm |000\rangle \otimes |001\rangle.$$

Two ensure overall antisymmetry, a symmetric (antisymmetric) spatial part must be combined with a singlet (triplet) spin state, thus each symmetric (antisymmetric) spin state adds one (three) to the degeneracy. With in all three symmetric spatial spin states and three antisymmetric, we thus get in total 12 degenerate states for the first excited energy level.

2f) (2 points) Two spin degrees of freedom; A with spin-1/2 and B with spin-3/2, forms a state where $m_A = +1/2$ and $m_B = +1/2$. What is the expectation value of the total spin squared ($\vec{S}_{tot} = \vec{S}_A + \vec{S}_B$) in this state?

Alternatives: $\frac{1}{2}\hbar^2, \hbar^2, \frac{3}{2}\hbar^2, 2\hbar^2, \frac{5}{2}\hbar^2, 3\hbar^2, \frac{7}{2}\hbar^2, 4\hbar^2, \frac{9}{2}\hbar^2, 5\hbar^2, \frac{11}{2}\hbar^2, 6\hbar^2$

Solution:

Being composed of $S_A = 1/2$ and $S_B = 3/2$, the total spin can be $S_{tot} = \{1, 2\}$. Looking

up the Clebsch-Gordan table $3/2 \times 1/2$, it follows that the state

$$|\frac{3}{2}, +\frac{1}{2}\rangle \otimes |\frac{1}{2}, +\frac{1}{2}\rangle = \sqrt{\frac{3}{4}}|2, +1\rangle - \sqrt{\frac{1}{4}}|1, +1\rangle$$

where we have written the left hand side as $|S_B, m_B\rangle \otimes |S_A, m_A\rangle$. Thus we can evaluate

$$\begin{aligned} \langle S_{tot}^2 \rangle &= \langle \frac{3}{2}, +\frac{1}{2} | \otimes \langle \frac{1}{2}, +\frac{1}{2} | S_{tot}^2 | \frac{3}{2}, +\frac{1}{2} \rangle \otimes | \frac{1}{2}, +\frac{1}{2} \rangle \\ &= \left(\sqrt{\frac{3}{4}} \langle 2, +1 | - \sqrt{\frac{1}{4}} \langle 1, +1 | \right) S_{tot}^2 \left(\sqrt{\frac{3}{4}} | 2, +1 \rangle - \sqrt{\frac{1}{4}} | 1, +1 \rangle \right) \\ &= \left(\sqrt{\frac{3}{4}} \langle 2, +1 | - \sqrt{\frac{1}{4}} \langle 1, +1 | \right) \left(\hbar^2 6 \sqrt{\frac{3}{4}} | 2, +1 \rangle - \hbar^2 2 \sqrt{\frac{1}{4}} | 1, +1 \rangle \right) = \hbar^2 \left(\frac{3}{4} 6 + \frac{1}{4} 2 \right) \\ &= \underline{5\hbar^2} \end{aligned}$$

2g) (2 points) Suppose you have to calculate the hyperfine splitting of the highest $n = 2$ fine structure energy level in hydrogen to first order in degenerate perturbation theory. What is the dimension of the perturbation matrix for this case?

Alternatives: 2×2 , 4×4 , 6×6 , 8×8 , 10×10 , 12×12 , 14×14 , 16×16 , 18×18 , 20×20 .

Solution:

The fine structure energies are

$$E_{n,j} = \frac{E_1}{n^2} \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j+1/2} - \frac{3}{4} \right) \right]$$

where $E_1 = -13.6\text{eV}$. For $n = 2$ the possible quantum numbers for $l \in \{0, 1\}$. Since the spin of the electron is $s_e = 1/2$, $j \in \{1/2, 3/2\}$. Thus for $n = 2$ there are two distinct fine structure energy levels:

$$\begin{aligned} E_{2,1/2} &= \frac{E_1}{4} \left[1 + \frac{\alpha^2}{4} \frac{5}{4} \right] \\ E_{2,3/2} &= \frac{E_1}{4} \left[1 + \frac{\alpha^2}{4} \frac{1}{4} \right] \end{aligned}$$

The highest of these is $E_{2,3/2}$ (as $E_1 < 0$) which has $j = 3/2$. This level is $2j + 1 = 4$ fold degenerate because of the different m_j levels and only one value of $l = 1$. The four fine structure states $|nljm_j\rangle = |21\frac{3}{2}m_j\rangle$ are thus fully describing the electron including its spin. In addition the proton spin $s_p = 1/2$ must be taken into account as the interaction involves the proton spin. It takes two values. Therefore the perturbation matrix is an 8×8 -matrix in this case.

Problem 3

A particle with mass m is located in a one dimensional harmonic oscillator potential with Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, where m is the mass of the particle and ω is a fixed frequency. In this problem use a trial wave function of the form

$$\psi(x) = \begin{cases} A(a^2 - x^2), & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

where A is a normalization constant and a is a positive parameter. It can be shown that the integral

$$\int_{-\infty}^{\infty} dx \psi^*(x) H \psi(x) = A^2 \left(\frac{4}{3} \frac{\hbar^2}{m} a^3 + \frac{8}{105} m \omega^2 a^7 \right).$$

3a) (6 points) Use this trial wave function to find the best upper bound on the ground state energy. Compare it with the exact ground state energy.

Solution:

To employ the variational method, we need to also find the normalization constant A

$$\begin{aligned} 1 &= \int_{-a}^a dx A^2 (a^2 - x^2)^2 = 2A^2 \int_0^a dx (a^2 - x^2)^2 = 2A^2 \int_0^a dx (a^4 - 2a^2 x^2 + x^4) \\ &= 2A^2 a^5 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{16}{15} a^5 A^2 \implies A^2 = \frac{15}{16a^5} \end{aligned}$$

The trial energy is therefore

$$E_{\text{trial}} = \frac{4}{3} \frac{\hbar^2}{ma^2} \frac{15}{16} + m\omega^2 a^2 \frac{15}{16} \cdot \frac{8}{105} = \frac{5}{4} \frac{\hbar^2}{ma^2} + \frac{1}{14} m\omega^2 a^2$$

To find the best upper bound on the ground state energy, we minimize the trial energy with respect to a . Differentiating and setting equal to zero gives

$$-2 \frac{5}{4} \frac{\hbar^2}{ma^3} + \frac{2}{14} m\omega^2 a = 0 \implies a^4 = \frac{35}{2} \frac{\hbar^2}{m^2 \omega^2} \implies a^2 = \sqrt{\frac{35}{2}} \frac{\hbar}{m\omega}$$

which gives the best trial energy

$$\begin{aligned} E_{\text{upperbound}} &= \frac{5}{4} \frac{\hbar^2}{ma^2} + \frac{1}{14} m\omega^2 a^2 = \frac{5}{4} \sqrt{\frac{2}{35}} \hbar\omega + \frac{1}{14} \sqrt{\frac{35}{2}} \hbar\omega = \hbar\omega \frac{1}{2} \left(\frac{5}{2} \sqrt{\frac{2}{35}} + \frac{1}{7} \sqrt{\frac{35}{2}} \right) \\ &= \hbar\omega \sqrt{\frac{5}{14}} \approx \hbar\omega \cdot 0.59761 \end{aligned}$$

The exact ground state energy is $\hbar\omega/2$. Our result is about 20% higher. This is to be expected as the true ground state wave function is a Gaussian wave function that is not part of this class of trial wave functions.

Although not required in order to solve this problem, we include here also the steps behind the given formula in the problem: The second derivative of the trial wave function in the interval $-a < x < a$ is

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -2A$$

The expectation value of the kinetic energy in the trial wave function is therefore

$$\begin{aligned} -\frac{\hbar^2}{2m} \int_{-a}^a dx \psi(x) \frac{\partial^2}{\partial x^2} \psi(x) &= -\frac{\hbar^2}{2m} A^2 \int_{-a}^a dx (a^2 - x^2) (-2) = +4 \frac{\hbar^2}{2m} A^2 \int_0^a dx (a^2 - x^2) \\ &= 4 \frac{\hbar^2}{2m} A^2 \left(1 - \frac{1}{3}\right) a^3 = \frac{4}{3} \frac{\hbar^2}{m} A^2 a^3. \end{aligned}$$

The potential energy is

$$\begin{aligned} \int_{-a}^a dx \psi(x) V(x) \psi(x) &= \frac{1}{2} m \omega^2 A^2 \int_{-a}^a dx (a^2 - x^2)^2 x^2 = m \omega^2 A^2 \int_0^a dx (a^4 x^2 - 2a^2 x^4 + x^6) \\ &= m \omega^2 A^2 \int_0^a dx a^7 \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = m \omega^2 A^2 a^7 \frac{8}{105}. \end{aligned}$$

Adding these gives the stated formula.

Grading:

To get full score on this subproblem, the best upper bound must be correct, and compared to the exact ground state energy. Up to 2 points will be subtracted for simple calculational mistakes. It is essential in this problem to understand that one must calculate how A depends on a in order to obtain the trial energy. Attempts at setting $A = 1$ directly will not give a minimum energy (for positive a) and is flawed. Such an approach will be awarded max 1 points for performing a minimization procedure, and possibly another max 1 point for comparing to the exact answer and recognize that the result is unphysical or fundamentally flawed..

Problem 4

A hydrogen atom is in its lowest hyperfine state ($n = 1$, the singlet state). A weak magnetic field pulse $\vec{B} = B \hat{e}_z$ along the z-axis is turned on at $t = 0$ and stays constant in magnitude and direction until it is turned off at a time $t = T$. The magnetic field interacts with the magnetic moments of the electron and proton:

$$H_{\text{int}} = -\vec{B} \cdot (\vec{\mu}_e + \vec{\mu}_p)$$

where $\vec{\mu}_e = -g_e \frac{e}{2m_e} \vec{S}_e$ and $\vec{\mu}_p = g_p \frac{e}{2m_p} \vec{S}_p$. $g_e = 2.00$ and $g_p = 5.59$. $m_p/m_e = 1836$.

4a) (3 points) Show (up to a phase factor) that

$$H_{\text{int}}|0, 0\rangle = \mu_B B \frac{1}{2} \left(g_e + \frac{g_p}{m_p/m_e}\right) \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle).$$

where $|0,0\rangle$ is the singlet state of the electron and proton spins and μ_B is a constant. In your derivation give the expression for the constant μ_B .

Solution:

$\vec{B} = B\hat{e}_z$: We use the convention where we write the electron spin state to the left and proton spin state to the right. Then we can write the singlet state as

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

We could also choose to multiply this with an arbitrary phase factor, therefore the phase factor ambiguity referred to in the problem text. Note that when the z-component of the electron (proton) spin operator acts on this state one gets

$$\begin{aligned} S_e^z|0,0\rangle &= \frac{\hbar}{2} \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \\ S_p^z|0,0\rangle &= -\frac{\hbar}{2} \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle). \end{aligned}$$

Using this, we get

$$\begin{aligned} H_{int}|0,0\rangle &= \frac{eB}{2} \left(\frac{g_e}{m_e} S_e^z - \frac{g_p}{m_p} S_p^z \right) \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ &= \frac{e\hbar}{2} B \frac{1}{2} \left(\frac{g_e}{m_e} + \frac{g_p}{m_p} \right) \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ &= \frac{e\hbar}{2m_e} B \frac{1}{2} \left(g_e + \frac{g_p}{m_p/m_e} \right) \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ &= \mu_B B \frac{1}{2} \left(g_e + \frac{g_p}{m_p/m_e} \right) \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle). \end{aligned}$$

with $\mu_B = \frac{e\hbar}{2m_e}$.

Grading:

In this "show that" subproblem it is important to show essential steps in order to get full credit. The only really essential step in this problem is how the singlet state turns into the $m = 0$ triplet state when acted upon by spin operators. If this is not convincingly shown, up to 2 points will be subtracted. Also the expression for the Bohr magneton must be correct. Just writing down the final result without proofs gives no credits. Some start with writing the singlet state as the $m = 0$ triplet state. This is erroneous.

4b) (6 points) Use first order time-dependent perturbation theory to calculate the probability that the hydrogen atom has transitioned to one of the hyperfine triplet states after the magnetic field has been turned off at time T . Use this expression to determine the smallest T that ensures maximum transition probability. For this value of T , find the transition probability as a function of B .

Solution:

In first order time-dependent perturbation theory the probability amplitude for finding the system in the excited state f at a time later than T is

$$a_f = \frac{1}{\hbar} \int_0^T dt \langle f | H_{\text{int}} | i \rangle e^{i\omega_{fi}t} = \frac{1}{\hbar} \langle f | H_{\text{int}} | i \rangle \frac{1}{i\omega_{fi}} (e^{i\omega_{fi}T} - 1)$$

where $\hbar\omega_{fi} = \Delta E = 5.88 \cdot 10^{-6} \text{eV}$ is the hyperfine singlet-triplet gap which is given in the note "Zeeman effect and hyperfine splitting"

The probability is obtained by taking the modulus squared of this

$$|a_f|^2 = \frac{1}{\hbar^2} |\langle f | H_{\text{int}} | i \rangle|^2 \frac{1}{\omega_{fi}^2} (2 - 2 \cos \omega_{fi}T) = 4 \frac{|\langle f | H_{\text{int}} | i \rangle|^2}{(\hbar\omega_{fi})^2} \sin^2 \left(\frac{\omega_{fi}T}{2} \right)$$

The optimal duration is to choose T such that the sine term is 1. This happens first time when $\omega_{fi}T/2 = \pi/2$, i.e.

$$T = \pi/\omega_{fi}$$

for which the probability is

$$P_{i \rightarrow f} = 4 \left(\frac{|\langle f | H_{\text{int}} | i \rangle|}{\hbar\omega_{fi}} \right)^2.$$

When the field is oriented along the z-axis, it follows from the previous subproblem that the only candidate for the final state is the triplet state with $m = 0$, $|f\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ and that the matrix element is

$$\langle f | H_{\text{int}} | i \rangle = \mu_B B \frac{1}{2} \left(g_e + \frac{g_p}{m_p/m_e} \right)$$

Inserting this, we get

$$P_{i \rightarrow f} = \left(\frac{\mu_B B}{\hbar\omega_{fi}} \right)^2 \left(g_e + \frac{g_p}{m_p/m_e} \right)^2.$$

The constant $\mu_B = 5.8 \cdot 10^{-5} \text{eV/T}$, $\hbar\omega_{fi} = 5.88 \cdot 10^{-6} \text{eV}$ and the expression

$$\left(g_e + \frac{g_p}{m_p/m_e} \right)^2 = \left(2.00 + \frac{5.59}{1836} \right)^2 \approx 4.012$$

So with the magnetic field is given in units of Teslas, then

$$P_{i \rightarrow f} = 390.38 B^2$$

For B big enough, $P_{i \rightarrow f}$ becomes larger than unity. This happens when $B \approx 0.0506$, thus B should be significantly less than 0.05T for this result to be valid.

Grading:

Full score is given if reasoning is sound and the (underlined) answers are correct. It is not necessary to give numerical answers for full score. Partial scores will be given. How many

depends on how far one has gotten towards the correct answers. Since this is an open book exam, no credits are given for starting formulae alone.

Problem 5

The Hamiltonian for a particle of mass m moving in two dimensional harmonic oscillator potential characterized by a frequency ω is

$$\hat{H}^0 = \hbar\omega \left(a_x^\dagger a_x + a_y^\dagger a_y + 1 \right)$$

where $a_x^\dagger, a_x$ and $a_y^\dagger, a_y$ are the standard harmonic oscillator ladder operators in directions x and y respectively.

5a) (3 points) Use the definition $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ to show that

$$\hat{L}_z = i\hbar \left(a_y^\dagger a_x - a_x^\dagger a_y \right)$$

Solution:

We use

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_x^\dagger + a_x), \quad \hat{p}_x = i\sqrt{\frac{m\omega\hbar}{2}}(a_x^\dagger - a_x).$$

Then

$$\begin{aligned} \hat{L}_z &= i\frac{\hbar}{2} \left[(a_x^\dagger + a_x) (a_y^\dagger - a_y) - (a_y^\dagger + a_y) (a_x^\dagger - a_x) \right] \\ &= i\frac{\hbar}{2} \left[a_x^\dagger a_y^\dagger - a_x^\dagger a_y + a_x a_y^\dagger - a_x a_y - (a_y^\dagger a_x^\dagger - a_y^\dagger a_x + a_y a_x^\dagger - a_y a_x) \right] \\ &= i\frac{\hbar}{2} \left[a_x^\dagger a_y^\dagger - a_x^\dagger a_y + a_x a_y^\dagger - a_x a_y - (a_x^\dagger a_y^\dagger - a_x a_y^\dagger + a_x^\dagger a_y - a_x a_y) \right] \\ &= i\frac{\hbar}{2} \left[-2a_x^\dagger a_y + 2a_x a_y^\dagger \right] = \underline{i\hbar (a_y^\dagger a_x - a_x^\dagger a_y)} \end{aligned}$$

where we used that $a_x^\dagger, a_x$ commutes with $a_y^\dagger, a_y$ in the third and fifth equation signs.

Grading:

To get full score in this "show that" problem, the proof must be convincing.

A term $H^1 = \lambda \hat{L}_z$ is added to the hamiltonian H^0 . (λ is a constant with appropriate dimension.)

5b) (6 points) Use perturbation theory with to determine how the first excited energy level of H^0 changes to first order in λ . Give also the dimensionless parameter that governs the perturbation expansion.

Solution:

The degenerate unperturbed states belonging to the first excited level are $|1, 0\rangle$ and $|0, 1\rangle$,

both with energies $2\hbar\omega$. Thus one needs to apply degenerate perturbation theory. The matrix elements of the perturbation matrix are:

$$\langle 1, 0 | H^1 | 1, 0 \rangle = 0, \quad \langle 0, 1 | H^1 | 0, 1 \rangle = 0, \quad \langle 0, 1 | H^1 | 1, 0 \rangle = i\lambda\hbar, \quad \langle 1, 0 | H^1 | 0, 1 \rangle = -i\lambda\hbar.$$

and the energy corrections are given as solutions to the matrix equation

$$\begin{pmatrix} E^1 & i\lambda\hbar \\ -i\lambda\hbar & E^1 \end{pmatrix} = 0 \implies E^1 = \pm|\lambda\hbar|.$$

This means that to first order in perturbation theory, the first excited energy level splits into two distinct energy levels with energies

$$E_{\pm} = \underline{2\hbar\omega \pm |\lambda\hbar|} = 2\hbar\omega \left(1 \pm \frac{|\lambda|}{2\omega} \right)$$

The dimensionless parameter characterizing the perturbation expansion is $\underline{\lambda/2\omega}$ (or $\underline{\lambda/\omega}$).

Grading:

For full score on this subproblem the answer must be correct and be obtained using degenerate perturbation theory. Also the dimensionless perturbation parameter must be stated (up to numerical factors). If the dimensionless perturbation parameter is missing 1 point will be subtracted. Answers with imaginary energy corrections will be awarded at most 3 points, and possibly 1 extra for the perturbation parameter.

5c) (3 points) Find all of the energy eigenvalues for the Hamiltonian $H = H^0 + H^1$ *exactly*. For which values of λ will the system be unstable (not having a ground state with finite energy)? Hint: Form suitable linear combinations of a_x and a_y .

Solution:

To simplify the notation, let us introduce the dimensionless variable $\tilde{\lambda} = \lambda/\omega$. The Hamiltonian becomes then

$$\hat{H} = \hbar\omega \left(a_x^\dagger a_x + a_y^\dagger a_y + 1 + i\tilde{\lambda} \left(a_y^\dagger a_x - a_x^\dagger a_y \right) \right)$$

The perturbation couples the oscillators. We can then try the linear combinations $a_+ \equiv (a_x + ia_y)/\sqrt{2}$ and $a_- \equiv (a_x - ia_y)/\sqrt{2}$. Inspecting the commutation relations of the new operators, we find

$$\begin{aligned} [a_+, a_+]^\dagger &= \frac{1}{2} [a_x + ia_y, a_x^\dagger - ia_y^\dagger] = 1 \\ [a_-, a_-]^\dagger &= \frac{1}{2} [a_x - ia_y, a_x^\dagger + ia_y^\dagger] = 1 \\ [a_+, a_-]^\dagger &= \frac{1}{2} [a_x + ia_y, a_x^\dagger + ia_y^\dagger] = 0 \\ [a_-, a_+]^\dagger &= \frac{1}{2} [a_x - ia_y, a_x^\dagger - ia_y^\dagger] = 0 \end{aligned}$$

and a_+ and a_- are therefore also proper independent ladder operators. With these new ladder operators we get

$$\begin{aligned} a_+^\dagger a_+ &= \frac{1}{2} (a_x^\dagger - i a_y^\dagger) (a_x + i a_y) = \frac{1}{2} [a_x^\dagger a_x + a_y^\dagger a_y - i (a_y^\dagger a_x - a_x^\dagger a_y)] \\ a_-^\dagger a_- &= \frac{1}{2} (a_x^\dagger + i a_y^\dagger) (a_x - i a_y) = \frac{1}{2} [a_x^\dagger a_x + a_y^\dagger a_y + i (a_y^\dagger a_x - a_x^\dagger a_y)] \end{aligned}$$

and therefore

$$\begin{aligned} a_x^\dagger a_x + a_y^\dagger a_y &= a_+^\dagger a_+ + a_-^\dagger a_- \\ i (a_y^\dagger a_x - a_x^\dagger a_y) &= a_-^\dagger a_- - a_+^\dagger a_+ \end{aligned}$$

and therefore the Hamiltonian becomes

$$\hat{H} = \hbar\omega [a_+^\dagger a_+ + a_-^\dagger a_- + 1 + \tilde{\lambda} (a_-^\dagger a_- - a_+^\dagger a_+)] = \hbar\omega [(1 - \tilde{\lambda}) a_+^\dagger a_+ + (1 + \tilde{\lambda}) a_-^\dagger a_- + 1]$$

The energy eigenstates are therefore $|n_+, n_- \rangle$ where n_+ and n_- are occupation numbers $\in [0, 1, 2, \dots]$ for the $\pm$ oscillators respectively. The corresponding energy eigenvalues are

$$E_{n_+, n_-} = \hbar\omega [(n_+ + n_-) + 1 + \tilde{\lambda} (n_- - n_+)] = \underline{\hbar\omega (n_+ + n_- + 1) + \hbar\lambda (n_- - n_+)}$$

For $|\tilde{\lambda}| > 1$, or equivalently $|\lambda| > \omega$, one of the oscillators will have a negative coefficient (negative effective frequency), and the energy can go to negative infinity by increasing the occupation number of it. Therefore the Hamiltonian is unstable for $|\lambda| > \omega$.

Grading:

For full score on this subproblem, the eigenvalues must be correct and the criterion for when the system becomes unstable must be correct.

THE END