FYS3120 Classical mechanics and electrodynamics
Exam – spring term 2019
Guide for examiners

May 26, 2019

Question 1 Swinging Lagrangian mechanics

A mechanical system consists of a mass m attached, on opposite sides of the mass, to two weightless springs of un-stretched lengths ℓ_1 and ℓ_2, with spring constants k_1 and k_2, fastened between two unmoving walls a distance d apart. Assume there is no gravity affecting the system and that the mass moves in the horizontal direction only. You can also ignore the size of the mass (assume it is a point). However, note that in general $\ell_1 + \ell_2 \neq d$. See illustration in Fig. 1.

We remind you that the potential energy for a spring is given by $V(x) = \frac{1}{2}kx^2$, where x is the displacement of the string length away from the un-stretched length.

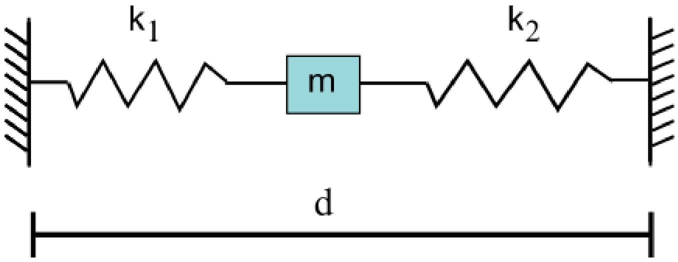

Figure 1: Mass m attached to two springs of length ℓ_1 and ℓ_2, with spring constants k_1 and k_2.

1) Find the number of degrees of freedom and generalised coordinate(s) of the system. [2 points]

Solution: There is a single mass moving in one dimension only, so there is only one degree of freedom. We choose the horizontal coordinate x of the mass, as measured from the left wall, as the generalised coordinate.

Marking: 1 point for realizing that there is only one generalized coordinate, 1 point for giving an appropriate suggestion for coordinate (multiple possible, including the equilibrium position).

2) Find the Lagrangian of the system. [4 points]

Solution: The kinetic energy of the mass is $K = \frac{1}{2}m\dot{x}^2$ and the potential energy of the two springs are $V = \frac{1}{2}k_1(x - \ell_1)^2$ and $V = \frac{1}{2}k_2(d - x - \ell_2)^2$. Thus

$$L = K - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k_1(x - \ell_1)^2 - \frac{1}{2}k_2(d - x - \ell_2)^2. \quad (1)$$

Marking: 1 point for $L = K - V$, 1 point for the right kinetic energy and 1 point for each of the potential energies.

3) Give the equilibrium condition for generalised coordinates and find the equilibrium position of the mass. [4 points]

Solution: The equilibrium condition for generalised coordinates q_i is given by $\frac{\partial V}{\partial q_i} = 0$ for all i, where V is the potential. For our single coordinate we have

$$\frac{\partial V}{\partial x} = k_1(x - \ell_1) - k_2(d - x - \ell_2) = 0, \tag{2}$$

which gives

$$x = \frac{k_1\ell_1 + k_2(d - \ell_2)}{k_1 + k_2}. \tag{3}$$

Marking: 2 points for the general formulation of equilibrium, 1 point for finding $\frac{\partial V}{\partial x}$ and 1 point for solving for the position.

4) Find the equation(s) of motion of the system and give the general solution(s). [5 points]

Solution: Inserting into Lagrange's equation

$$\frac{\partial L}{\partial x} = -k_1(x - \ell_1) + k_2(d - x - \ell_2), \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}, \tag{4}$$

giving

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + k_1(x - \ell_1) - k_2(d - x - \ell_2) = 0, \tag{5}$$

or

$$m\ddot{x} + (k_1 + k_2)x - k_1\ell_1 - k_2(d - \ell_2) = 0. \tag{6}$$

This can be solved either as an inhomogeneous differential equation, or, by realising that the equation is simpler if the coordinate is changed to be relative to the equilibrium position. Using $x' = x - x_0$ where x_0 is the equilibrium position in (3) we have $dx'/dx = 1$ and thus $\ddot{x}' = \ddot{x}$ so that the differential equation can be written

$$m\ddot{x}' + (k_1 + k_2)x - x_0(k_1 + k_2) = m\ddot{x}' + (k_1 + k_2)x' = 0. \tag{7}$$

This is a harmonic oscillator with angular frequency $\omega = \sqrt{\frac{k_1+k_2}{m}}$ and solution $x(t) = A\sin(\omega t) + B\cos(\omega t) + x_0$, where A and B are constants to be determined form the initial conditions.

Marking: 1 point for giving Lagrange's equation, 2 points for finding the necessary derivatives, 2 points for identifying the correct solution including the frequency and offset from zero.

Let us now look at the situation with two masses m and three springs with the same spring constant k, shown in Fig. 2. Assume that the distance between the walls is such that the strings are un-stretched in the equilibrium position.

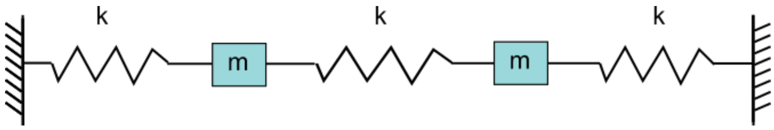

Figure 2: Two masses attached to springs with identical spring constants.

5) Show that the Lagrangian for this system can be written

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}kx_2^2, \tag{8}$$

where x_1 and x_2 are the displacements of the two masses from their equilibrium position, and find the equations of motion. [4 points]

Solution: Since the springs are un-stretched we can easily use the distance from the equilibrium position as the coordinates of the masses. In these coordinates the kinetic energy is the sum of the kinetic energies for each mass, $K = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$, and the potential energy for the three springs is $V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}kx_2^2$, where the perturbation of the middle string from equilibrium depends on the movement of both masses from their equilibrium position. Thus the Lagrangian for the problem is

$$L = K - V = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}kx_2^2. \tag{9}$$

Finding the Lagrange equations

$$\frac{\partial L}{\partial x_1} = -kx_1 + k(x_2 - x_1), \qquad \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m\ddot{x}_1, \tag{10}$$

and

$$\frac{\partial L}{\partial x_2} = -kx_2 - k(x_2 - x_1), \qquad \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = m\ddot{x}_2, \tag{11}$$

gives

$$\begin{aligned} m\ddot{x}_1 + kx_1 - k(x_2 - x_1) &= 0, \\ m\ddot{x}_2 + kx_2 + k(x_2 - x_1) &= 0, \end{aligned} \tag{12}$$

or

$$\begin{aligned}\ddot{x}_1 + \frac{2k}{m}x_1 - \frac{k}{m}x_2 &= 0, \\ \ddot{x}_2 + \frac{2k}{m}x_2 - \frac{k}{m}x_1 &= 0.\end{aligned}$$

Marking: 2 points for demonstrating the correct Lagrangian (0.5 point for $L = K - V$, 0.5 point for the kinetic energies, 1 point for the potential energies), 2 points for finding the equations of motion (1 point for doing the derivatives, 1 point for correct equations).

6) Show that

$$x_i(t) = A_i e^{i\omega t}, \tag{13}$$

are mathematical solutions of the equation of motion and find what the allowed values of A_i and ω are. Briefly discuss the physical interpretation of these solutions. [6 points]

Solution: We can insert the solutions into the equations of motion and get the following set of equations for A_i and ω

$$-\omega^2 A_1 + \frac{2k}{m}A_1 - \frac{k}{m}A_2 = 0, \tag{14}$$

$$-\omega^2 A_2 + \frac{2k}{m}A_2 - \frac{k}{m}A_1 = 0. \tag{15}$$

This set of equations has a non-trivial solution if and only if the corresponding determinant is zero

$$\begin{vmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \omega^2 \end{vmatrix} = 0, \tag{16}$$

giving

$$\left(\omega^2 - \frac{2k}{m}\right)^2 - \frac{k^2}{m^2} = 0, \tag{17}$$

which has solutions $\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{3k}{m}}$. For ω_1 inserted into (14) and (15) we must have $A_1 = A_2$, while ω_2 requires $A_1 = -A_2$. The absolute value of the A_i is not fixed since it depends on the initial conditions.

The physical interpretation of the solutions are sine and cosine oscillations with two different frequencies ω_1 and ω_2, where the first solution has the amplitudes of the two masses in phase, which means that the middle spring does not compress, while in the second higher frequency, the amplitudes are opposite.

Marking: 1 point for finding the equations that must be satisfied for this to be solutions, 1 point for finding the equation for ω, 0.5 point for each of the two allowed values of ω, 0.5 point for each of the corresponding allowed values of A_i. 2 points for giving the correct physical interpretation.

$\boldsymbol{\sum_{n=1}^{\infty} n+6)}$ Find at least one solution for a sequence of $2n+1$ masses and $2n+2$ springs. [$3\left(\frac{1}{2}\right)^n$ points]

Solution: The Lagrangian for a sequence of m masses is

$$L = \sum_{i=1}^{m} \frac{1}{2} m \dot{x}_i^2 - \sum_{i=1}^{m-1} \frac{1}{2} k (x_{i+1} - x_i)^2 - \frac{1}{2} k x_1^2 - \frac{1}{2} k x_m^2. \tag{18}$$

Finding the Lagrange equation

$$\frac{\partial L}{\partial x_i} = k(x_{i+1} - 2x_i + x_{i-1}), \quad \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = m\ddot{x}_i, \tag{19}$$

gives

$$m\ddot{x}_i - k(x_{i+1} - 2x_i + x_{i-1}) = 0, \tag{20}$$

which has the solutions $x_i(t) = A_i e^{i\omega t}$ if the corresponding determinant is zero:

$$D_m = \begin{vmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} & 0 & \cdots & \cdots & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \omega^2 & -\frac{k}{m} & 0 & \cdots & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} - \omega^2 & -\frac{k}{m} & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & 0 & -\frac{k}{m} & \frac{2k}{m} - \omega^2 \end{vmatrix} = 0. \tag{21}$$

This determines the allowed frequencies and the relationship between the amplitudes. Expanding the determinant around the first row gives

$$D_m = \left(\frac{2k}{m} - \omega^2\right) D_{m-1} - \frac{k^2}{m^2} D_{m-2}. \tag{22}$$

If m is odd, as it will be with $2n+1$ masses, an iterative expansion of D_m ends with the term $D_1 = \frac{2k}{m} - \omega^2$, which is then a common factor in all terms. Thus one solution must be $\omega_1 = \sqrt{\frac{2k}{m}}$. Inserted into the equation of motion (20) this gives $A_2 = 0$ and $A_{i+1} = -A_{i-1}$, so that every other mass is at rest and the two surrounding masses oscillate with equal amplitudes and opposite phase.

This question was inspired by this year's (2019) Abel prize. Interestingly, the solution found here can also be found to exist for an infinitly long sequence of springs. Karen Uhlenbeck's work on integrable systems, for example in the form of soliton wave solutions that can occur on such a set-up, has had a great impact on physics since the 1970ies.

Marking: 1 point for finding the general equation of motion. 1 point for finding the general requirement for a solution, 0.5 point for finding ω_1 and 0.5 point for finding the correspoding A_i. We will give up to 1.5 points for providing a correct solution for $n = 1$ (three masses) only.

Question 2 Acceleration by photons

We will look at the kinematics of a process where a photon hits and is absorbed by an object with mass m.

a) Assume that we start in the rest frame of the massive object. Draw a sketch of the process, give explicit expressions for the involved four-momenta, and give the equations for the conservation of relativistic energy and momentum in the collision in terms of the four-momenta. [4 points]

Answer: The four-momenta are

$$p_\gamma^\mu = (|\vec{p}_\gamma|, \vec{p}_\gamma), \quad (23)$$
$$p_m^\mu = (mc, 0), \quad (24)$$
$$p_m'^\mu = (E_m'/c, \vec{p}_m'), \quad (25)$$

where the primes denote the four-momenta and mass after collision and where $E_m' = \sqrt{|\vec{p}_m'|^2c^2 + m'^2c^4}$. The conservation of four-momentum is then given by

$$p_\gamma^\mu + p_m^\mu = p_m'^\mu. \quad (26)$$

Marking: 1.5 points for a correct sketch, 1.5 points for a correct list of four-momenta, 1 point for the conservation of four-momenta equation.

b) Find the change in relativistic energy, relativistic momentum and mass of the object due to the absorbed photon, given in terms of the initial momentum of the photon. [4 points]

Answer: From the conservation of relativistic energy and momentum the change in the energy is $\Delta E = E_\gamma = |\vec{p}_\gamma|c$ and the change in momentum is $\Delta\vec{p} = \vec{p}_\gamma$. The change in mass can be found from the invariant

$$\begin{aligned} p_m'^2 = m'^2c^2 &= (p_\gamma + p_m)^2 \\ &= p_\gamma^2 + 2p_\gamma p_m + p_m^2 \\ &= 2|\vec{p}_\gamma|mc + m^2c^2, \end{aligned} \tag{27}$$

where we have used that the square of the four-momentum of a particle of mass m is $p^2 = m^2c^2$. This gives

$$\Delta m = m' - m = m\left(\sqrt{1 + \frac{2|\vec{p}_\gamma|}{mc}} - 1\right). \tag{28}$$

Marking: 0.5 points each for the change in relativistic energy and momentum, 3 points for the calculation of the change in mass (of which 1 point for using the invariant mass relationship, and 1 point for calculating it correctly if that solution is followed).

c) The relativistic force is given as $\vec{F} = \frac{d\vec{p}}{dt}$, where $\vec{p}$ is the relativistic momentum. Use this to show that the relativistic force is related to the velocity and acceleration in a given reference frame as

$$\vec{F} = \frac{\gamma^3 m}{c^2}(\vec{v}\cdot\vec{a})\vec{v} + \gamma m\vec{a}. \tag{29}$$

[4 points]

Answer: We use that the time-derivative of the gamma factor is

$$\frac{d\gamma}{dt} = \frac{\gamma^3}{2c^2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{\gamma^3}{c^2}(\vec{v}\cdot\vec{a}). \tag{30}$$

We then carry out the differentiation:

$$\begin{aligned} \vec{F} &= m\frac{d}{dt}(\gamma\vec{v}) = m\left(\frac{d\gamma}{dt}\vec{v} + \gamma\vec{a}\right) \\ &= m\left(\frac{\gamma^3}{c^2}(\vec{v}\cdot\vec{a})\vec{v} + \gamma\vec{a}\right). \end{aligned} \tag{31}$$

Marking: 1 point for using the right expression for relativistic momentum, 2 points for the correct derivative of γ (1 point for keeping the vector property), 1 point for correct product differentiation and insertion.

d) The four-force K^μ is similarly given as the proper time derivative of the four-momentum, $K^\mu = \frac{dp^\mu}{d\tau}$. Show that the four-force can be written in terms of the relativistic force as

$$K^\mu = \left(\gamma \frac{\vec{F} \cdot \vec{v}}{c}, \gamma \vec{F}\right). \tag{32}$$

[5 points]

Answer: The four-momentum is $p^\mu = (\gamma mc, \gamma m\vec{v})$. Taking the derivative to find the four-force we have

$$\begin{aligned} K^\mu &= \frac{dp^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dp^\mu}{dt} = \gamma \frac{dp^\mu}{dt} \\ &= \gamma \left(\frac{d\gamma}{dt} mc, \frac{d\gamma}{dt} m\vec{v} + \gamma m\vec{a}\right) \\ &= \gamma \left(\frac{\gamma^3}{c^2}(\vec{v}\cdot\vec{a})mc, \frac{\gamma^3}{c^2}(\vec{v}\cdot\vec{a})m\vec{v} + \gamma m\vec{a}\right) \\ &= \gamma \left(\frac{\vec{F}\cdot\vec{v}}{c}, \vec{F}\right), \end{aligned} \tag{33}$$

where we have used that

$$\begin{aligned} \vec{F}\cdot\vec{v} &= m\left(\frac{\gamma^3}{c^2}(\vec{v}\cdot\vec{a})\vec{v} + \gamma\vec{a}\right)\cdot\vec{v} \\ &= m\gamma\left(\frac{\gamma^2}{c^2}v^2 + 1\right)(\vec{a}\cdot\vec{v}) \\ &= m\gamma\left(\frac{\beta^2}{1-\beta^2} + \frac{1-\beta^2}{1-\beta^2}\right)(\vec{a}\cdot\vec{v}) \\ &= m\gamma\left(\frac{1}{1-\beta^2}\right)(\vec{a}\cdot\vec{v}) \\ &= m\gamma^3(\vec{a}\cdot\vec{v}). \end{aligned} \tag{34}$$

Marking: 1 point for starting from the correct four-momentum form, 1 point for changing the differentiation to proper time, 1 point for carrying out the differentiation, 2 points for finding $\vec{F}\cdot\vec{v}$.

e) Given that a mass m is hit by a flux of n photons per second with momentum $\vec{p}_\gamma$ in a particular direction in a particular reference frame. Find the velocity dependent acceleration in that frame. You can ignore any change in the mass of the object found in **b)**. [4 points]

Answer: With a change in relativistic momentum $|\vec{p}_\gamma|$ from n photons the force is $\vec{F} = \frac{d\vec{p}}{dt} = n|\vec{p}_\gamma|$. There are multiple ways to go from the relativistic force to the acceleration, the easiest is probably to use the expression for the power in (34),

$$\vec{F} \cdot \vec{v} = m\gamma^3 \vec{v} \cdot \vec{a}, \tag{35}$$

using that the force (and acceleration) is linear (parallel to the velocity) so that $\vec{F} \cdot \vec{v} = Fv$ and $\vec{v} \cdot \vec{a} = va$, so that

$$a = \frac{F}{m\gamma^3} = \frac{n|\vec{p}_\gamma|}{m\gamma^3}. \tag{36}$$

Marking: 1 point for finding the realtivistic force on the mass, 1 point for realizing that the acceleration is linear, 1 point for using a known relationship between relativistic force and acceleration, 1 point for solving for a.

Question 3 Classical atom

In the classical atom model an electron moves in a scalar potential

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{Ze}{r}, \tag{37}$$

where Z is the (effective) number of positive charges in the nucleus seen by the electron and e is the unit elementary positive charge. Assume for now that the vector potential is $\vec{A} = 0$.

a) Find the electric field affecting the electron. [3 points]

Answer: With $\vec{A} = 0$ the electric field is given as

$$\vec{E}(r) = -\vec{\nabla}\phi(r) = -\frac{Ze}{4\pi\epsilon_0}\vec{\nabla}\frac{1}{r} = -\frac{Ze}{4\pi\epsilon_0}\frac{\partial}{\partial r}\frac{1}{r}\hat{r} = \frac{Ze}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}. \tag{38}$$

Marking: 1 point for the correct expression for $\vec{E}$, 2 points for carrying out the differentiation.

b) Find the magnitude of the acceleration for an electron in a circular orbit around the nucleus. Give your answer in terms of the **classical electron radius**

$$r_0 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mc^2} = 2.872 \cdot 10^{-15}\,\mathrm{m}, \tag{39}$$

where m is the mass of the electron. [3 points]

Answer: With no magnetic field the Lorentz force causing the acceleration reduces to $\vec{F} = -e\vec{E}$, with the sign due to the sign of the electron's charge. From $m\vec{a} = -e\vec{E}$, using (38), we have

$$a = \frac{Ze^2}{4\pi\epsilon_0 m}\frac{1}{r^2} = \frac{Ze^2}{4\pi\epsilon_0 mc^2}\frac{c^2}{r^2} = \frac{Zr_0c^2}{r^2}. \tag{40}$$

Marking: 1 point for the acceleration and field relationship, 1 point for solving for $\vec{a}$, and 1 point for substituting in r_0.

c) Find the electric dipole moment $\vec{p}_e$ of the electron's orbit. [3 points]

Answer: The charge density of a single charged particle is given by $\rho(\vec{r}, t) = q\delta(\vec{r} - \vec{r}(t))$, where $\vec{r}(t)$ is the path of the particle. The electric dipole moment of the electron is then

$$\vec{p}_e = \int \vec{r}\rho(\vec{r}, t)dV = -\int e\vec{r}\delta(\vec{r} - \vec{r}(t))dV = -e\vec{r}(t). \tag{41}$$

Marking: 1 point for the correct definition, 1 point for the charge density, 1 point for the integration.

d) Find the magnetic dipole moment $\vec{m}_e$ of the electron's orbit. [3 points]

Answer: The current density of a single charged particle is given by $\vec{j}(\vec{r}, t) = q\vec{v}(t)\delta(\vec{r} - \vec{r}(t))$, where $\vec{v}(t)$ is the velocity of the particle. The magnetic dipole moment of the electron is then

$$\vec{m}_e = \frac{1}{2}\int \vec{r}\times\vec{j}(\vec{r}, t)dV = -\frac{1}{2}\int \vec{r}\times e\vec{v}(t)\delta(\vec{r} - \vec{r}(t))dV = -\frac{e}{2}\vec{r}(t)\times\vec{v}(t). \tag{42}$$

Marking: 1 point for the correct definition, 1 point for the current density, 1 point for the integration.

e) Find (numerically) the energy radiated per second from this electron assuming an arbitrary Z and an orbit with radius given by the Bohr radius $a_0 = 5.292 \cdot 10^{-11}$ m. *Hint:* Some constants that may be needed here are the elementary charge $e = 1.60 \cdot 10^{-19}$ C, the permittivity $\epsilon_0 = 8.85 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$, and the speed of light $c = 3.00 \cdot 10^8$ m/s. [3 points]

Answer: Due to the acceleration in the circular orbit the charged electron radiates. Larmor's formula gives the energy radiated per second for a point particle with acceleration a as:

$$\begin{aligned} P &= \frac{\mu_0 e^2}{6\pi c} a^2 = \frac{\mu_0 e^2}{6\pi c}\left(\frac{Z r_0 c^2}{a_0^2}\right)^2 = \frac{\mu_0 e^2}{6\pi c}\frac{r_0^2 c^4}{a_0^4} Z^2 = \frac{e^2}{6\pi\epsilon_0}\frac{r_0^2 c}{a_0^4} Z^2 \\ &= \frac{(1.60\cdot 10^{-19}\,\mathrm{C})^2\cdot(2.872\cdot 10^{-15}\,\mathrm{m})^2\cdot(3.00\cdot 10^8\,\mathrm{m/s})}{6\cdot 3.14\cdot(8.85\cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}})\cdot(5.292\cdot 10^{-11}\,\mathrm{m})^4} Z^2 \\ &= Z^2\cdot 4.8\cdot 10^{-8}\,\mathrm{W}. \end{aligned} \tag{43}$$

Marking: 1 point for Larmor's formula, 1 point for insertion of the acceleration and rewrite into known constants, 1 point for evaluation.

f) Estimate the time it takes for the electron orbit to decay completely due to this radiation. *Hint:* It may be useful to find an expression for how r changes with time using the conservation of energy. [4 points]

Answer: The energy of an electron orbiting with a classical radius r is given by

$$E = K + V = \frac{1}{2}mv^2 - \frac{Ze^2}{4\pi\epsilon_0 r} = \frac{1}{2}mv^2 - \frac{Z r_0 m c^2}{r}, \tag{44}$$

where the potential energy is given from the scalar potential as $V = e\phi$. Since $v^2 = ar$ for a circular orbit, using (40) we have

$$v^2 = \frac{Z r_0 c^2}{r^2} r = \frac{Z r_0 c^2}{r}, \tag{45}$$

so that

$$E = \frac{1}{2}m\frac{Z r_0 c^2}{r} - \frac{Z r_0 m c^2}{r} = -\frac{Z r_0 m c^2}{2r}. \tag{46}$$

The radiated power P we found in (43) controls how this energy changes, so $P = -\frac{dE}{dt}$. Differentiating the energy with respect to the time and equating it with the radiated power gives an expression for $\dot{r}$

$$-\frac{dE}{dt} = -\frac{Z r_0 m c^2}{2r^2}\dot{r} = P = \frac{e^2}{6\pi\epsilon_0}\frac{r_0^2 c}{r^4} Z^2 = \frac{2 r_0 m c^2}{3}\frac{r_0^2 c}{r^4} Z^2. \tag{47}$$

Solving for $\dot{r}$,

$$\dot{r} = -\frac{4}{3}\frac{r_0^2 c}{r^2} Z. \tag{48}$$

If we start from an orbit with the Bohr radius, $r = a_0$, and a nucleus with $Z = 1$, this takes the value $\dot{r} \simeq 1.18$ m/s. For a rough estimate of the decay time, assuming $\dot{r}$ is constant,[1] we get $t \approx a_0/\dot{r} \simeq 4.5\cdot 10^{-11}$ s. This precision was sufficient to get a full score.

For a more precise estimate of the time to decay T we can use that $\dot{r}r^2$ is a time-independent constant. The integral of $\dot{r}r^2$ gives

$$\int_0^T r^2\dot{r}\,dt = \int_{r(0)}^{r(T)} r^2\,dr = \left[\frac{1}{3}r^3\right]_{r(0)}^{r(T)} = \frac{1}{3}(r(T)^3 - r(0)^3) = -\frac{4}{3}r_0^2 cZT. \tag{49}$$

Inserting the boundary conditions $r(T) = 0$ and $r(0) = a_0$ and solving for T gives

$$T = \frac{a_0^3}{4r_0^2 cZ} = \frac{4\pi^2\epsilon_0 c m_e^2 a_0^3}{\mu_0 q^4} \approx 1.50\cdot 10^{-11}\text{ s.} \tag{50}$$

Marking: 1 point for making the connection between the radiated power and the derivative of the total energy E, 1 point for finding E, 1 point for finding $\dot{r}$, 1 point for estimating the time to decay.

Atomic nuclei have a magnetic dipole moment due to the angular momentum (and spin) of their nucleons. This is given as $\vec{m}_n = g\frac{e}{2m_p}\vec{\ell}$, where $\vec{\ell}$ is the (constant) total angular momentum of the nucleus, g is a factor depending on the structure of the nucleus, and m_p is the proton mass.

g) Show that the resulting magnetic field seen by the electron is given by

$$\vec{B} = \frac{\mu_0}{2\pi r^3}\left(3(\vec{m}_n\cdot\hat{r})\hat{r} - \vec{m}_n\right). \tag{51}$$

[5 points]

Answer: In the multipole expansion the vector potential far away from a magnetic dipole (and the electron is indeed far from the nucleus) is

$$\vec{A} = \frac{\mu_0\vec{m}_n\times\hat{r}}{4\pi r^2}. \tag{52}$$

We use the relation between a $\vec{B}$-field and the vector potential, and the double cross product relationship for $\vec{\nabla}$, to write,

$$\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\left(\frac{\mu_0\vec{m}_n\times\hat{r}}{4\pi r^2}\right) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\vec{m}_n\times\frac{\vec{r}}{r^3}\right)\\ &= \frac{\mu_0}{4\pi}\left(\vec{m}_n(\vec{\nabla}\cdot\frac{\vec{r}}{r^3}) - \frac{\vec{r}}{r^3}(\vec{\nabla}\cdot\vec{m}_n) + (\frac{\vec{r}}{r^3}\cdot\vec{\nabla})\vec{m}_n - (\vec{m}_n\cdot\vec{\nabla})\frac{\vec{r}}{r^3}\right)\\ &= \frac{\mu_0}{4\pi}\left(\vec{m}_n(\vec{\nabla}\cdot\frac{\vec{r}}{r^3}) - (\vec{m}_n\cdot\vec{\nabla})\frac{\vec{r}}{r^3}\right)\end{aligned} \tag{53}$$

[1] It is not, as we see the speed increases as the electron comes closer to the nucleus.

where we have used that $\vec{m}$ is a constant vector. We then use

$$\vec{\nabla}\cdot\frac{\vec{r}}{r^3}=\frac{\partial}{\partial x_i}\frac{x_i}{r^3}=\frac{3}{r^3}-\frac{3x_i}{r^4}\frac{x_i}{r}=\frac{3}{r^3}-\frac{3r^2}{r^5}=0, \tag{54}$$

and

$$(\vec{m}\cdot\vec{\nabla})\frac{\vec{r}}{r^3}=m_i\frac{\partial}{\partial x_i}\frac{\vec{r}}{r^3}=m_i\frac{\hat{e}_i}{r^3}-m_i\vec{r}\frac{3}{r^4}\frac{x_i}{r}=\frac{\vec{m}}{r^3}-\frac{3(\vec{m}\cdot\vec{r})}{r^5}\vec{r}. \tag{55}$$

Thus

$$\vec{B}=\frac{\mu_0}{2\pi r^3}\left(3(\vec{m}_n\cdot\hat{r})\hat{r}-\vec{m}_n\right). \tag{56}$$

Marking: 1 point for the vector potential, 1 point for the relationship between the B-field and the vector potential, 1 point for an explicit expression for the double cross product in terms of scalar products, 2 points for the evaluation of the remaining scalar products .

h) Explain why the force resulting from the magnetic dipole moment of the nucleus can be ignored for the electron. [2 points]

Answer: First, notice that no electric field will be generated because $\frac{\partial\vec{A}}{\partial t}=0$, since the magnetic dipole moment is constant. If we compare the scaling of the forces the magnetic field contributes

$$F_m=evB=ev\frac{\mu_0}{r^3}m_n=ev\frac{\mu_0}{r^3}\frac{e}{m_p}\ell_n=v\frac{e^2\mu_0}{r^3}\frac{\ell_n}{m_p}=\frac{e^2}{\epsilon_0}\frac{v}{c}\frac{1}{r^2}\frac{\ell_n}{rm_pc}, \tag{57}$$

while the electric field gives

$$F_e=eE=\frac{e^2}{\epsilon_0}\frac{1}{r^2}. \tag{58}$$

This show that the contribution from the magnetic dipole moment of the nucleus is both suppressed by a factor v/c, and by the nucleus' total angular momentum compared to rm_pc, where r are distances of the scale of an atom compared to the nucleus' distance scales.

In fact, a much more important (quantum mechanical) contribution is the magnetic field in the electron's rest frame (generated by a reference frame shift of the electric field from the nucleus), which interacts with the electrons's dipole moment from its spin (spin-orbit coupling).

Marking: 1 point for valid arguments that the force from the magnetic dipole moment is suppressed by β, 1 point for arguments that the scale of the motion of the nucleus (angular momentum) is small compared to the atomic scale, leading to an additional suppression.