

FYS3120 Classical Mechanics and
Electrodynamics

Discussion problems 12

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Problem 1 Klein-Gordon equation.

- a) How can we write down the action and the Euler-Lagrange equations knowing a Lagrangian density of a real-valued scalar field?

Solution

The definition of the action in terms of the Lagrangian density is always

$$S = \int d^3r dt \mathcal{L} = \int d^4x \mathcal{L}, \quad (1)$$

where the latter version is the more commonly used one. The Euler-Lagrange equations for field theories are

$$\frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0. \quad (2)$$

- b) If we want to consider a complex-valued scalar field χ , would we need to modify these equations? If so, how?

Solution

The equations remain the same, but we need to take into account that there are two real field degrees of freedom for each complex-valued scalar field. This is essentially saying that you need two real values to describe a complex one. Two common approaches to this is to take χ and χ^* to be the two degrees of freedom, resulting in two Euler-Lagrange equations. Another one is to define real scalar fields ϕ_1 and ϕ_2 so that $\chi = \phi_1 + i\phi_2$. Then we can use the two Euler-Lagrange equations from these two real scalar fields instead.

- c) Find the Euler-Lagrange equation for the Lagrangian density for a massive real scalar field

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2. \quad (3)$$

The resulting equation here is called the Klein-Gordon equation.

Solution

We have that $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi = \square \phi$ and that $\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$, so

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = (\square + m^2)\phi = 0, \quad (4)$$

which is the Klein Gordon equation.

- d) Find the Hamiltonian density corresponding to (3).

Solution

The Hamiltonian density is related to the Lagrangian density by $\mathcal{H} = \pi(x) \partial_0 \phi(x) - \mathcal{L}(\phi, \partial_\mu \phi, x)$, where $\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$. In this case $\pi(x) = \partial^0 \phi(x)$, and

$$\mathcal{L} = \frac{1}{2}(\partial_0 \phi)(\partial^0 \phi) - \frac{1}{2}(\partial_i \phi)(\partial^i \phi) - \frac{1}{2}m^2 \phi^2, \quad (5)$$

so then

$$\mathcal{H} = (\partial_0 \phi)(\partial^0 \phi) - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2 \phi^2. \quad (6)$$

- e) What is the requirement for conservation of energy density for field theories? What is the energy density and is it conserved here? Is the Hamiltonian density equal to the energy density?

Solution

The energy density is related to the time-time component of the energy-momentum tensor $u = T^{00}/c^2$.

A Lagrangian density that is not explicitly dependent on the coordinates, it is invariant under translations. By Noether's theorem, we know that the energy-momentum tensor is conserved in this case, i.e. $\partial_\mu T^{\mu\nu} = 0$. Therefore, energy density is conserved.

Lastly, we can check that

$$T^{00} = \pi^0 \partial^0 \phi - \mathcal{L} g^{00} = \mathcal{H}, \quad (7)$$

so the Hamiltonian density is equal to the energy density.

- f) Find the Euler-Lagrange equations also for this similar Lagrangian density in terms of a complex-valued scalar field

$$\mathcal{L}(\phi) = (\partial_\mu \chi)^* (\partial^\mu \chi) - m^2 \chi^* \chi.$$

Solution

We take the path where we choose to treat χ and χ^* as our degrees of freedom. Then we get two Euler-Lagrange equations, the first one being

$$(\square + m^2)\chi = 0 \quad (8)$$

and secondly

$$(\square + m^2)\chi^* = 0, \quad (9)$$

which are both the Klein-Gordon equation.

Let us now modify the Klein-Gordon equation by adding a source term $J(x)$. This becomes the inhomogeneous Klein-Gordon equation.

- g) What is a Green's function, and how can we use it to find a solution to the inhomogeneous Klein-Gordon equation? Can we simplify this by doing a Fourier transform?

Solution

The Green's function for a linear differential operator L is the solution to the equation $LG = \delta$ where δ is the Dirac delta function. This can be used to generate a solution to the equation $L\phi = J$ is the convolution $G*J$.

For the Klein-Gordon equation we have that $L = (\square + m^2)$, so if we know the Green's equation can generate a solution for ϕ on the form $\phi = G * J = \int d^4y G(x - y)J(y)$.

The Klein-Gordon equation for the fourier transformed field $\tilde{\phi}(p^\mu)$ simplifies into $(m^2 - p_\mu p^\mu)\tilde{\phi}(p^\mu)$.