

FYS3120 Classical Mechanics and Electrodynamics

Bonus discussion before midterm

March 12, 2026

Remember the formula sheet on the exam!

Problem 1 A very, very fast train.

A train moves with speed $v = 0.8c$. The train has length $L_0 = 100$ m at rest. The train approaches a straight tunnel with length $d_0 = 60$ m. Let us call the rest frame of the tunnel S , and the frame comoving with the train (i.e. the rest frame of the train) is called S' .

- a) Calculate the length of the train in the rest frame of the tunnel, S .

Solution

In S the train is length contracted, $L_S = \frac{1}{\gamma} L_0$. For $v = 0.8c$ we get

$$L_S = \frac{100 \text{ m}}{1/\sqrt{1-0.8^2}} = 0.6 \cdot 100 \text{ m} = 60 \text{ m}. \quad (1)$$

- b) Draw the Minkowski diagram for reference frame S including the tunnel and the train.
- c) Calculate the length of the tunnel in the rest frame of the train, S' .

Solution

In S' the tunnel is length contracted, $d_{S'} = \frac{1}{\gamma} d_0$. In S' the tunnel moves with velocity $-v = -0.8c$, so we get:

$$d_{S'} = \frac{60 \text{ m}}{1/\sqrt{1-0.8^2}} = 0.6 \cdot 60 \text{ m} = 36 \text{ m}. \quad (2)$$

- d) Draw the Minkowski diagram for the situation in S' .
- e) Is the train ever completely (with both ends) inside the tunnel? Explain your answer.

Solution

Yes and no. The train is completely inside the tunnel in S , but not in S' . In S the train length is 60 m – the same as the length of the tunnel. This means at a time t_S both ends of the train will be inside the tunnel simultaneously. Note that t_S then serves as a measurement of the (contracted) train length, L_S , in S . In S' it is the tunnel that gets length contracted, $d_{S'} = 36 \text{ m} < L_0$. Hence there can not be a time $t_{S'}$ when both ends of the train is inside the tunnel simultaneously. This is a perfect example of the relativity of simultaneity – two simultaneous spacetime-events (both ends of the train inside the tunnel is one spacetime-event each) in one frame are not necessarily simultaneous in all other frames. *Bonus challenge:* draw a Minkowski diagram of the setup and prove to yourself that the two events at t_S in S occurs at two different points in time in S' .

Problem 2

- a) What are the conserved quantities in a relativistic collision?

Solution

All components of the total four-momentum of the systems in the collisions are conserved separately. This is equivalent to energy conservation and momentum conservation. This also implies that the invariant mass is conserved for a relativistic collision,

$$(P_{tot,i})^\mu = (P_{tot,f})^\mu \Rightarrow (P_{tot,i})^\mu (P_{tot,i})_\mu = (P_{tot,f})^\mu (P_{tot,f})_\mu.$$

- b) Show that $E^2 = p^2 c^2 + m^2 c^4$ using $E = \gamma m c^2$ and $p = \gamma m v$.

Solution

Solve $p = \gamma mv$ for v

$$p = \frac{mv}{\sqrt{1-v^2/c^2}} \implies v = \frac{c}{\sqrt{1+(mc/p)^2}}.$$

Inserting this into $\gamma = (1 - v^2/c^2)^{-1/2}$ gives

$$\gamma = \sqrt{1 + \left(\frac{p}{mc}\right)^2},$$

which we can then insert into $E = \gamma mc^2$

$$E = \sqrt{p^2 c^2 + m^2 c^4}.$$

Problem 3 Relativistic collision between two particles.

Consider a relativistic elastic collision between two particles, forming two output particles. Particle i has four-momentum P_i^μ , where $i \in \{1, 2\}$ label the input particles and $i \in \{3, 4\}$ the output particles. Each particle has (rest) mass m_i , momentum $\mathbf{p}_i$, and energy E_i . The momentum of particle 1 and 2 are considered to be along the same direction, whereas the momentum of particle 3 is at an angle θ compared to this direction, and particle 4 is at an angle ϕ .

- Draw the setup for this problem.
- Using the conservation laws from the previous exercise, what is the difference in four-momentum between particle 3 and 1 equal to?

Solution

Using conservation of four-momentum ($P_1^\mu + P_2^\mu = P_3^\mu + P_4^\mu$) gives

$$P_3^\mu - P_1^\mu = P_2^\mu - P_4^\mu.$$

- Calculate the invariant scalar product on both sides of the equation obtained in b) in terms of the energy and the momentum of the particles.

Solution

It follows that

$$(P_i^\mu - P_j^\mu)((P_i)_\mu - (P_j)_\mu) = (m_i c)^2 + (m_j c)^2 - 2(E_i E_j / c^2 - \mathbf{p}_i \cdot \mathbf{p}_j).$$

Using the angles θ and ϕ , we therefore have that

$$\begin{aligned} & (m_3 c)^2 + (m_1 c)^2 - 2(E_3 E_1 / c^2 - p_3 p_1 \cos \theta) \\ &= (m_2 c)^2 + (m_4 c)^2 - 2(E_2 E_4 / c^2 - p_2 p_4 \cos \phi) \end{aligned}$$

- d) Now we consider ourselves to be in the rest frame of particle 2. What can we now immediately eliminate from the equation we had in b)?

Solution

We now have that $E_2 = m_2 c^2$ and that $\mathbf{p}_2 = \mathbf{0}$, so we can eliminate one term on the right-hand side of the equation

$$(m_3 c)^2 + (m_1 c)^2 - 2(E_3 E_1 / c^2 - p_3 p_1 \cos \theta) = (m_2 c)^2 + (m_4 c)^2 - 2(m_2 E_4).$$

- e) Use energy conservation to eliminate E_4 from the equation.

Solution

Energy conservation tells us that $E_1 + E_2 = E_3 + E_4$, so we have that $E_4 = E_1 + m_2 c^2 - E_3$. Inserting this into our expression gives

$$(m_3 c)^2 + (m_1 c)^2 - 2(E_3 E_1 / c^2 - p_3 p_1 \cos \theta) = (m_4 c)^2 - (m_2 c)^2 - 2(m_2 (E_1 - E_3)).$$

- f) We now seek to apply this result to a particular system. Consider the input particles to be a photon and an electron at rest, and the output particles to be a photon and an electron with some unknown momentum $\mathbf{p}$. What is the rest energy and the momentum of the photon in terms of its energy?

Solution

The rest energy of a photon is zero, and the momentum of the photon is $p = E/c$.

- g) Label the particles and insert into the equation from e). What is the angle between the directions of the incoming and outgoing photons in terms of their energies?

Solution

We label the incoming photon as particle 1 and electron as particle 2, and the outgoing photon as particle 3 and electron as particle 4. The angles θ and ϕ now relate the momentum of the outgoing particles to the momentum of the photon. Then our equation reads

$$-\frac{2}{c^2}(E_3E_1 - E_3E_1 \cos \theta) = (m_4c)^2 - (m_2c)^2 - 2(m_2(E_1 - E_3)).$$

Since particles two and four are both electrons, their rest energy is equal, so we can simplify further

$$E_3E_1(1 - \cos \theta) = m_2c^2(E_1 - E_3).$$

Moving things around lets us find the angle

$$\theta = \arccos\left(1 + \frac{E_3 - E_1}{E_3E_1}m_2c^2\right).$$