

FYS3120 Classical Mechanics and
Electrodynamics

Discussion 3
Solutions

February 9, 2025

Problem 1

- a) Hamiltonian.

Solution

The Hamiltonian is a (Legendre) transformation of the Lagrangian.

$$H(q, p, t) = \sum_j p_j \dot{q}_j - L, \quad (1)$$

where we have substituted out the velocities of the generalised coordinates $\dot{q}_j$ by using the expression for the generalised momentum $p_j = \frac{\partial L}{\partial \dot{q}_j}$.

- b) Scalar potential, ϕ .

Solution

The scalar potential ϕ is a function whose gradient is a given vector field. From *electrostatics* we have that $\mathbf{E} = -\nabla\phi$.

- c) Vector potential, $\mathbf{A}$.

Solution

A vector potential is a vector field whose curl is a given vector field. For the magnetic field, $\mathbf{B}$ we have that $\mathbf{B} = \nabla \times \mathbf{A}$. In *electrodynamics* we have for the electric field $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$.

- d) Gauge transformations.

Solution

A gauge transformation is a transformation of scalar and vector potentials that does not change the given (physical and observable) vector fields ($\mathbf{E}$ and $\mathbf{B}$ in electrodynamics) and the equations of motion. A gauge transformation also leaves the Lagrangian invariant up to a total time-derivative.

Problem 2 Velocities in the Lagrangian

- a) Explain the contents and the meaning of the equation below:

$$\mathcal{F}_j = \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right) - \frac{\partial U}{\partial q_j}.$$

Solution

q_j is the generalized coordinate, $j = 1, \dots, d$. $\mathcal{F}_j$ is generalized force. U is a potential, that can be velocity dependent.

The equation is the condition a force must satisfy, in order for it to be included in the Lagrangian as a velocity dependent potential.

- b) Assume that $L = K - U$ is the Lagrangian for a particle in a velocity dependent potential. For instance, this could be a charged particle in a magnetic field. Will the total energy be conserved in such a case? Why/why not?

Solution

There is not a straightforward answer to this, as we need to impose some assumptions on U and the constraints to say if the total energy will be conserved.

Since the potential is velocity dependent we will have that $H \neq K + U$. However, if there is no explicit time-independence in L then $\frac{\partial L}{\partial t} = -\frac{dH}{dt} = 0$. We might consider the case where the potential is a static electromagnetic potential ($U(\mathbf{r}, \dot{\mathbf{r}}) = e\phi(\mathbf{r}) - e\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r})$) then,

$$L = K - U = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\phi(\mathbf{r}) + e\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}),$$

where the scalar and vector potentials ($\phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$) are not time-dependent to ensure that $\frac{\partial L}{\partial t} = 0$. In this case the total energy of the particle is the sum of its kinetic and energy and potential energy

$$E = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\phi(\mathbf{r}).$$

This is conserved, as it is exactly the same as the expression for the Hamiltonian, $H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi(\mathbf{r})$, with $\mathbf{p} = m\mathbf{v} + e\mathbf{A}$, and time-independence for L ensures that $\frac{\partial L}{\partial t} = -\frac{dH}{dt} = 0$.

In short, yes, the total energy will certainly be conserved if U is a time-independent electromagnetic potential. If this is not the case, we can not say for sure.

Problem 3 Vector potential.

- a) Show that the vector potential $\mathbf{A} = yB_0\mathbf{e}_z$, gives a constant magnetic field along the x-axis.

Solution

On component form, the vector potential is

$$A_x = A_1 = 0,$$

$$A_y = A_2 = 0,$$

$$A_z = A_3 = yB_0.$$

The magnetic field is given by $\mathbf{B} = \nabla \times \mathbf{A}$. We will calculate this on index form. We use x_1, x_2, x_3 for x, y, z and find that the i -th component of the $\mathbf{B}$ -field is

$$B_i = (\nabla \times \mathbf{A})_i = \sum_{jk} \epsilon_{ijk} \frac{\partial A_k}{\partial x_j}.$$

Where ϵ_{ijk} is the Levi-Civita symbol: $\epsilon_{ijk} = +1$ if (i, j, k) is an even permutation of $(1, 2, 3)$ (either $(1, 2, 3)$, $(2, 3, 1)$ or $(3, 1, 2)$). And $\epsilon_{ijk} = -1$ if (i, j, k) is an odd permutation of $(1, 2, 3)$ (either $(1, 3, 2)$, $(3, 2, 1)$ or $(2, 1, 3)$). $\epsilon_{ijk} = 0$ for all other values of (i, j, k) . We then find

$$B_1 = \epsilon_{123} \frac{\partial A_3}{\partial x_2} + \epsilon_{132} \frac{\partial A_2}{\partial x_3} = \frac{\partial}{\partial y} yB_0 - 0 = B_0.$$

$$B_2 = \epsilon_{231} \frac{\partial A_1}{\partial x_3} + \epsilon_{213} \frac{\partial A_3}{\partial x_1} = 0 - 0 = 0.$$

$$B_3 = \epsilon_{312} \frac{\partial A_2}{\partial x_1} + \epsilon_{321} \frac{\partial A_1}{\partial x_2} = 0 - 0 = 0.$$

So we see that, indeed, $\mathbf{B} = B_0\mathbf{e}_x$ is constant and directed along the x -axis (provided that B_0 is constant).

- b) Show that $\mathbf{A}' = -zB_0\mathbf{e}_y$ gives the same magnetic field as in a).

Solution

On component form the vector potential is

$$A'_1 = 0,$$

$$A'_2 = -zB_0,$$

$$A'_3 = 0.$$

We then find the components of $\mathbf{B}$

$$B_1 = \epsilon_{123} \frac{\partial A_3}{\partial x_2} + \epsilon_{132} \frac{\partial A_2}{\partial x_3} = 0 - \frac{\partial}{\partial z}(-zB_0) = B_0.$$

$$B_2 = \epsilon_{231} \frac{\partial A_1}{\partial x_3} + \epsilon_{213} \frac{\partial A_3}{\partial x_1} = 0 - 0 = 0.$$

$$B_3 = \epsilon_{312} \frac{\partial A_2}{\partial x_1} + \epsilon_{213} \frac{\partial A_1}{\partial x_2} = 0 - 0 = 0.$$

So we see that, indeed, $\mathbf{B} = B_0 \mathbf{e}_x$ is constant and directed along the x -axis (provided that B_0 is constant).

- c) Find a function $\chi(\mathbf{r}, t)$ such that $\mathbf{A} = \mathbf{A}' + \nabla\chi$.

Solution

Inserting for the expressions for $\mathbf{A}$ and $\mathbf{A}'$ we see that

$$\underbrace{yB_0 \mathbf{e}_z}_{\mathbf{A}} = \underbrace{-zB_0 \mathbf{e}_y}_{\mathbf{A}'} + (zB_0 \mathbf{e}_y + yB_0 \mathbf{e}_z).$$

So $\chi(\mathbf{r}, t)$ can be solution to the equations

$$\frac{\partial \chi}{\partial y} = zB_0,$$

$$\frac{\partial \chi}{\partial z} = yB_0.$$

We then find that $\chi(\mathbf{r}, t) = yzB_0 + f(t)$, where f is some function of time that can not be decided from the differential equations above.

Problem 4 Poisson brackets Let $f(q, p, t)$ and $g(q, p, t)$ be two functions

of phase space variables $q = \{q_1, \dots, q_d\}$ and $p = \{p_1, \dots, p_d\}$. The Poisson bracket, $\{\cdot, \cdot\}$ is defined as

$$\{f, g\} = \sum_{i=1}^d \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

Use Hamilton's equations and find $\{q_i, H\}$, $\{p_i, H\}$ and $\{q_i, p_j\}$. Can you make an educated guess of what the "quantum version" of the Poisson bracket is?

Solution

We find that

$$\{q_k, H\} = \sum_{i=1}^d \left(\underbrace{\frac{\partial q_k}{\partial q_i}}_{\delta_{ki}} \underbrace{\frac{\partial H}{\partial p_i}}_{\dot{q}_i} - \underbrace{\frac{\partial q_k}{\partial p_i}}_0 \frac{\partial H}{\partial q_i} \right) = \frac{d}{dt} q_k$$

and in the same fashion

$$\{p_k, H\} = \frac{d}{dt} p_k, \quad \{q_k, p_l\} = \delta_{kl}.$$

In quantum mechanics, we replace the Poisson bracket with the commutator $[\cdot, \cdot]$. For the quantum operators $\hat{q}_i$ and $\hat{p}_j$ we then have

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}.$$

Furthermore, in the Heisenberg picture (time-dependent operators and time-independent states) the Heisenberg equations read

$$\frac{d}{dt} \hat{q}_i = \frac{i}{\hbar} [\hat{p}_i, \hat{H}],$$

$$\frac{d}{dt} \hat{p}_i = \frac{i}{\hbar} [\hat{q}_i, \hat{H}].$$

Where $\hat{H}$ is the quantum operator Hamiltonian.