

Question 1 Statutory problem [22 points]

The Italian exchange student Michelangelo has absconded with the statue from the mechanics exam in 2019 in the back of his Fiat pick-up truck, see Fig. 1. He has cleverly engineered the back of the truck so that the statue is subject to a position dependent potential U to avoid the statue falling off. With our department head Professor Viefers in hot pursuit the upright statue starts sliding. Professor Viefers stops to ponder whether the problem is best considered in the Fiat reference frame F or in the reference frame S of a, presumably innocent, student bystander at rest with respect to the road. Please help her decide before Michelangelo gets away!



Figure 1: Statue from the FYS-MEK1110 V19 exam.

The path of the car in S is $\mathbf{R}(t)$ with velocity $\mathbf{V}(t)$, which is a function of time alone. The coordinates of the statue are $\mathbf{r}_0$ in S and $\mathbf{r}$ in F . The Fiat is non-relativistic.

- a) In the student reference frame S the Lagrangian of the statue is

$$L(\mathbf{r}_0, \dot{\mathbf{r}}_0, t) = \frac{1}{2}m\dot{\mathbf{r}}_0^2 - U(\mathbf{r}_0 - \mathbf{R}). \quad (1)$$

Show that in F this Lagrangian can be written as

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{\mathbf{r}}^2 + m\mathbf{V} \cdot \dot{\mathbf{r}} + \frac{1}{2}m\mathbf{V}^2 - U(\mathbf{r}). \quad (2)$$

[3 points]

Solution: The relationship between the positions of the statue in F and S (non-relativistic reference frames) is $\mathbf{r}_0 = \mathbf{R} + \mathbf{r}$. This gives a relationship between the velocities $\dot{\mathbf{r}}_0 = \mathbf{V} + \dot{\mathbf{r}}$. Inserted into the original Lagrangian (1) we get

$$L = \frac{1}{2}m(\mathbf{V} + \dot{\mathbf{r}})^2 - U(\mathbf{r}) = \frac{1}{2}m\mathbf{V}^2 + m\mathbf{V} \cdot \dot{\mathbf{r}} + \frac{1}{2}m\dot{\mathbf{r}}^2 - U(\mathbf{r}). \quad (3)$$

Scoring: 1 point for finding the position relationship between the frames, 1 point for finding the velocity relationship between the frames, 1 point of inserting correctly.

- b) Argue why terms of the form $W(t)$, functions purely of time, can be removed from a Lagrangian without changing the equations of motion.
[3 points]

Solution: If the complete Lagrangian is $L = L' + W(t)$ we can use the fundamental theorem of calculus to write

$$L = L' + W(t) = L' + \frac{d}{dt} \int_a^t W(t') dt'. \quad (4)$$

We know that due to time-invariance total time derivative terms can be removed from the Lagrangian without changing the equations of motion.

Scoring: 1.5 point for realising that the basic idea here is the invariance of the e.o.m. to total time derivative terms. 1.5 points for the re-write using the fundamental theorem of calculus. Alternatively, one can argue directly from the coordinate dependence of Lagrange's equation.

c) Show that the Lagrangian

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - m \dot{\mathbf{V}} \cdot \mathbf{r} - U(\mathbf{r}), \quad (5)$$

has the same equations of motion as the Lagrangian in (2). [4 points]

Solution: By the argument in **b** the term with $\mathbf{V}(t)^2$ does not change the equations of motion. We can also rewrite the second term

$$m \mathbf{V} \cdot \dot{\mathbf{r}} = m \mathbf{V} \cdot \frac{d}{dt} \mathbf{r} = \frac{d}{dt} (m \mathbf{V} \cdot \mathbf{r}) - \frac{d}{dt} (m \mathbf{V}) \cdot \mathbf{r} = \frac{d}{dt} (m \mathbf{V} \cdot \mathbf{r}) - m \dot{\mathbf{V}} \cdot \mathbf{r}, \quad (6)$$

and again we can ignore the total time derivative term.

Scoring: 1 point for using **b** to eliminate the purely time-dependent term present in (2), 1 point for finding the re-write in terms of time-derivatives, 1 point for carrying out this re-write correctly, and 1 point for eliminating the total time-derivative. It is possible to show the equivalence also by finding the equations of motion for both Lagrangians, but this needs a careful consideration of all three coordinates.

d) Find the equations of motion of the statue in F from Eq. (5). Give a brief interpretation of the physical meaning of the terms. [6 points]

Solution: Finding the components of Lagrange's equation

$$\frac{\partial L}{\partial x_i} = -m \dot{V}_i - \frac{\partial U}{\partial x_i}, \quad \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = m \ddot{x}_i. \quad (7)$$

This gives

$$m \ddot{x}_i + m \dot{V}_i + \frac{\partial U}{\partial x_i} = 0, \quad i = 1, 2, 3 \quad (8)$$

The first term in the equations is the acceleration of the statue, the third if the force from the potential U , while the second term is an extra effective force due to the acceleration of the frame F .

Scoring: 1 point for each of the “ingredients” in Lagrange’s equation, 1 point for putting this together in Lagrange’s equations for the three coordinates. 0.5 point for identifying the acceleration term, 0.5 point for identifying the force from the potential, 1 point for identifying the effective force due to the accelerated frame.

- e) Given that the potential is $U(\mathbf{r}) = \frac{1}{2}m\omega^2(x^2 + y^2)$,¹ where $\mathbf{r} = (x, y, z)$. Describe the movement of the statue in F if it starts at rest at $\mathbf{r} = 0$ at the time $t = 0$, and the car experiences a constant acceleration a in the y -direction for some time, before the acceleration becomes zero again. [6 points]

Solution: Since there is no acceleration and no potential in the z -direction the equation here is trivial, $m\ddot{z} = 0$, giving a constant velocity. This velocity is zero since it started at rest, so there is no vertical movement. In the x -direction the equation of motion is $\ddot{x} + \omega^2 x = 0$, which is the equation for a harmonic oscillator. However, the initial conditions say that it starts at rest, so there is no motion in the x -direction.

For the y -direction we get the inhomogeneous equation

$$\ddot{y} + \omega^2 y + a = 0. \quad (9)$$

This has the solution $y = A \cos(\omega t - \phi_0) - \frac{a}{\omega^2}$, meaning that the harmonic oscillation is around $y_0 = -\frac{a}{\omega^2}$. After the initial acceleration, the equation of motion changes to the homogeneous $\ddot{y} + \omega^2 y = 0$. The oscillation will then continue around $y_0 = 0$ with the momentum from the initial acceleration.

Scoring: 0.5 point for no motion in the z -direction, 1 point for no motion in the x -direction, 1.5 point for writing down the e.o.m. in y -direction, 1.5 point for finding the general solution of that equation, 1.5 point for a correct description of that motion, including the equilibrium point during and after the acceleration.

Question 2 Heffalon production [26 points]

You are being asked to design the new Very Large Hadron Collider (VLHC) at CERN which will collide beams of highly relativistic protons. In order to search for very massive new particles such as the heffalon,² the collider needs to have as large as possible *centre-of-mass energy*, meaning the total energy of the colliding protons in the centre-of-mass reference frame.

¹What did you expect? These are physicists!

²See, e.g., A.J. Barr, C.G. Lester, *A search for direct heffalon production using the ATLAS and CMS experiments at the Large Hadron Collider*, arXiv:1303.7367 [hep-ph].

In the following, let us for concreteness name the relativistic momenta of the colliding particles $\mathbf{p}_1$ and $\mathbf{p}_2$, their relativistic energies E_1 and E_2 , and their mass m_p .

- a) Define the centre-of-mass reference frame for the VLHC. [2 points]

Solution: The centre-of-mass reference frame is the reference frame where the sum of the relativistic momenta are equal. For the VLHC this means the frame where $\mathbf{p}_1 + \mathbf{p}_2 = 0$.

Scoring: 1 point for the general definition, 1 point for an expression in terms of VLHC coordinates.

- b) Explain why the maximum mass of new particles that can be produced at such a collider is given by $M = E/c^2$, where E is the centre-of-mass energy. [3 points]

Solution: Since relativistic energy is conserved the energy available for new particles in *any* reference frame is limited by the energy of the incoming protons in that frame. If particles with total mass M are produced these have a total rest energy of Mc^2 . The largest possible Mc^2 is then when no energy is used for kinetic energy, meaning the momenta of all the particles are zero. However, then the sum of the momenta is also zero, and we are in the centre-of-mass reference frame, and the limiting energy is the centre-of-mass energy, thus $M = E/c^2$.

Scoring: 1 point for using conservation of relativistic energy, 1 point for arguing for particles at rest, 1 point for putting together the whole coherent argument. Partial arguments, such as assuming there is a single particle in the final state get up to 2 points.

- c) One design option for the VLHC is a *fixed target collider* where one proton is at rest in a target in the lab and the other proton is shot into it at high velocity. Write down the four-momenta for the two protons for this collider in the lab reference frame. [3 points]

Solution: The four-momenta are

$$p_1^\mu = (E_1/c, \mathbf{p}_1), \quad (10)$$

$$p_2^\mu = (m_p c, 0), \quad (11)$$

where $E_1 = \sqrt{\mathbf{p}_1^2 c^2 + m_p^2 c^4}$.

Scoring: 1 point per four-momentum expression, 1 point for the momentum-energy relationship that is needed.

- d) Find the centre-of-mass energy for the fixed target collider in terms of the incoming proton's energy E_1 in the lab frame. [4 points]

Solution: The total four-momentum in the lab frame is $p^\mu = p_1^\mu + p_2^\mu = (E_1/c + m_p c, \mathbf{p}_1)$. Squaring this gives the invariant

$$\begin{aligned} p^2 &= p^\mu p_\mu = \left(\frac{E_1}{c} + m_p c \right)^2 - \mathbf{p}_1^2 = \frac{E_1^2}{c^2} + 2E_1 m_p + m_p^2 c^2 - \mathbf{p}_1^2 \\ &= 2m_p^2 c^2 + 2E_1 m_p \end{aligned} \quad (12)$$

In the centre-of-mass reference frame this total four-momentum is $p^\mu = (E/c, 0)$, where E is the centre-of-mass energy, giving $p^2 = E^2/c^2$. Solving for the total energy gives

$$E = \sqrt{2E_1 m_p c^2 + 2m_p^2 c^4} \quad (13)$$

Scoring: 2 points for recognising that the centre-of-mass energy can be found from the invariant of the total four-momentum, 2 points for finding that invariant in terms of the lab frame quantities.

- e) The other design option considered is a *symmetric collider* where the protons have the same magnitude of momentum in the lab frame, going in opposite directions. Find the centre-of-mass energy for this collider. [3 points]

Solution: Here the lab frame and the centre-of-mass frame is the same, and the total four-momentum is $p^\mu = (E_1/c + E_2/c, 0)$. Since $E_1 = E_2$ the centre-of-mass energy is $E = 2E_1$.

Scoring: 1 point for recognising that the lab frame and the centre-of-mass frame is the same, 1 point for noting that the energies of the two protons are the same, 1 point for writing down the total centre-of-mass energy.

- f) If the cost of making a collider to accelerate the protons to a given E_1 or E_2 in a single beam in the lab frame is linear in the energy, which design should you choose in order to reach the highest heffalon masses? [3 points]

Solution: The fixed target collider can have twice the proton energy $E_1^{\text{fixed}} = 2E_1^{\text{sym}} = 2E_2^{\text{sym}}$ for the same cost since the symmetric collider must accelerate two beams. This gives a centre-of-mass energy of

$$E^{\text{fixed}} = \sqrt{2E_1^{\text{fixed}} m_p c^2 + 2m_p^2 c^4} \quad (14)$$

$$E^{\text{sym}} = 2E_1^{\text{sym}} = E_1^{\text{fixed}}, \quad (15)$$

for the two colliders. The symmetric collider will eventually for higher (fixed) beam energy give a higher centre-of-mass energy as its dependence on the (fixed) beam energy is linear compared to square root. The two are equal when

$$(E_1^{\text{fixed}})^2 = 2E_1^{\text{fixed}} m_p c^2 + 2m_p^2 c^4, \quad (16)$$

which has (positive) solution $E_1^{\text{fixed}} = (1 + \sqrt{3})m_p c^2$. With $E_1^{\text{fixed}} > (1 + \sqrt{3})m_p c^2$ the centre-of-mass energy of the symmetric collider is then larger. Since we saw in **b** that the centre-of-mass energy is directly connected to the mass of the particles that can be produced, the symmetric collider gives a better reach as long as the lab frame energy of the protons are slightly above twice the proton rest energy, more precisely larger than $(1 + \sqrt{3})m_p c^2$.

Scoring: 1 point for doing the right comparison in terms of equivalent cost, 1 point for arguing that the symmetric collider will eventually give higher energy, 1 point for an exact expression for the threshold energy.

g) In collider physics the centre-of-mass energy is often written in terms of the variable s , with $s = (p_1 + p_2)^2 = (p_1^\mu + p_2^\mu)(p_{1\mu} + p_{2\mu})$ where p_1^μ and p_2^μ are the four-momenta of the colliding particles. Find the relationship between s and the centre-of-mass energy E . How does s change between reference frames? [3 points]

Solution: The variable s (called a Mandelstam variable) does not change since it is an invariant formed by the contraction of a four-vector (the total four-momentum) with itself. In the centre-of-mass frame $p_1^\mu + p_2^\mu = (E/c, 0)$ so that $s = (E/c)^2$.

Scoring: 1 point for pointing out that s is an invariant, 0.5 point for explaining why it is an invariant, 1.5 point for finding s (this can be done with reference to **d**).

h) Two other variables t and u are used when the collision produces two (and only two) outgoing particles (say Heffalons) with four-momenta p_3^μ and p_4^μ , defined by $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. Show that

$$s + t + u = 2m_p^2 c^2 + 2m_h^2 c^2, \quad (17)$$

where m_h is the heffalon mass. [5 points]

Solution: We can write

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\ &= p_1^2 + 2p_1 p_2 + p_2^2 + p_1^2 - 2p_1 p_3 + p_3^2 + p_1^2 - 2p_1 p_4 + p_4^2 \\ &= 2m_p^2 c^2 + 2m_h^2 c^2 + 2p_1^2 + 2p_1 p_2 - 2p_1 p_3 - 2p_1 p_4 \\ &= 2m_p^2 c^2 + 2m_h^2 c^2 + 2p_1(p_1 + p_2 - p_3 - p_4) \\ &= 2m_p^2 c^2 + 2m_h^2 c^2, \end{aligned} \quad (18)$$

where we have used that $p^2 = m^2 c^2$, and since four-momentum is conserved $p_1 + p_2 - p_3 - p_4 = 0$.

Scoring: 1.5 point for managing to write out the complete expression for $s + t + u$, 1 point for using the properties of p^2 , 1.5 point for using the conservation of four-momentum, 1 point for completing the algebra of the calculation.

Question 3 Dipole dervish [28 points]

Two charges $q_1 = -q$ and $q_2 = q$ are placed at $\mathbf{r}_1 = (-d/2, 0, 0)$ and $\mathbf{r}_2 = (d/2, 0, 0)$, respectively, and initially held fixed there.

- a) Use the multipole approximation for the scalar potential $\phi(\mathbf{r})$ at large distances from the charges, $r \gg d$, and find the contributions up to and including terms proportional to $1/r^3$. [6 points]

Solution: The contributions up to $1/r^3$ are the monopole, dipole and quadrupole contributions. The electric monopole contribution $\phi_0(\mathbf{r})$ is given in terms of the total charge $Q = q_1 + q_2 = -q + q = 0$, so this vanishes. Using the charge density $\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}')$ for a single charge q at $\mathbf{r}'$ the electric dipole moment $\mathbf{p}$ is

$$\begin{aligned}\mathbf{p} &= \int \rho(\mathbf{r}) \mathbf{r} d^3\mathbf{r} = \int [-q\delta(\mathbf{r} - \mathbf{r}_1) + q\delta(\mathbf{r} - \mathbf{r}_2)] \mathbf{r} d^3\mathbf{r} \\ &= q(\mathbf{r}_2 - \mathbf{r}_1) = q(d, 0, 0).\end{aligned}\quad (19)$$

This gives a dipole contribution

$$\phi_1(\mathbf{r}) = \frac{\mathbf{r} \cdot \mathbf{p}}{4\pi\epsilon_0 r^3} = \frac{q\mathbf{r} \cdot (\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\epsilon_0 r^3} = \frac{qdx}{4\pi\epsilon_0 r^3}.\quad (20)$$

The quadrupole is proportional to the integral ($\hat{\mathbf{n}}$ is the unit vector in the direction of observation $\mathbf{r}$):

$$\begin{aligned}&\int \rho(\mathbf{r})(3(\hat{\mathbf{n}} \cdot \mathbf{r})^2 - r^2) d^3\mathbf{r} \\ &= \int [-q\delta(\mathbf{r} - \mathbf{r}_1) + q\delta(\mathbf{r} - \mathbf{r}_2)](3(\hat{\mathbf{n}} \cdot \mathbf{r})^2 - r^2) d^3\mathbf{r} \\ &= -q(3(\hat{\mathbf{n}} \cdot \mathbf{r}_1)^2 - r_1^2) + q(3(\hat{\mathbf{n}} \cdot \mathbf{r}_2)^2 - r_2^2) \\ &= -q\left[3n_x^2\left(\frac{d}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right] + q\left[3n_x^2\left(\frac{d}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right] = 0.\end{aligned}\quad (21)$$

Thus this contribution also vanishes.

The scalar potential is approximated by the dipole contribution.

Scoring: 1 point for identifying the possible contributions, 1 point for the monopole contribution, 2 points for the dipole contribution where 1.5 point is given for finding the dipole moment, 2 points for showing that the quadrupole contribution is zero.

- b) Find the resulting large distance approximation for the electric and magnetic fields. [3 points]

Solution: For static sources the electric field is given as

$$\begin{aligned}\vec{E}(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \simeq -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \frac{qdx}{4\pi\epsilon_0 r^3} \\ &= \frac{qd}{4\pi\epsilon_0 r^3} \left(\frac{3x^2}{r^2} - 1, \frac{3xy}{r^2}, \frac{3xz}{r^2}\right).\end{aligned}\quad (22)$$

There is no current density, thus no vector potential and no magnetic field.

Scoring: 1 point for the right expression for the electric field in the static case, 1 points for the correct calculation of the field expression. 1 point for arguing why there is no magnetic field.

The charges are now rotated counterclockwise in the xy -plane in a circle of radius $d/2$ around the z -axis with angular frequency ω . We remind you that the scalar potential for time-dependent sources in the long distance approximation is given as

$$\phi(\mathbf{r}, t) = \frac{Q}{4\pi\epsilon_0 r} + \frac{\mathbf{r} \cdot \dot{\mathbf{p}}_{\text{ret}}}{4\pi\epsilon_0 r^2 c} + \frac{\mathbf{r} \cdot \mathbf{p}_{\text{ret}}}{4\pi\epsilon_0 r^3} + \dots, \quad (23)$$

where Q is the total charge and $\mathbf{p}$ is the electric dipole moment, and where “ret” indicates that the quantity is evaluated at the retarded time $t_r = t - r/c$. The vector potential is similarly given as

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} \left(\dot{\mathbf{p}} + \frac{1}{c} \dot{\mathbf{m}} \times \hat{\mathbf{n}} + \frac{1}{2c} \ddot{\mathbf{D}}_{\hat{\mathbf{n}}} + \dots \right)_{\text{ret}}, \quad (24)$$

where $\mathbf{m}$ is the magnetic dipole moment, $\mathbf{D}_{\hat{\mathbf{n}}}$ is the electric quadrupole vector, and $\hat{\mathbf{n}}$ is the unit vector in the $\mathbf{r}$ direction.

- c) Find the electric dipole moment of the charges. [3 points]

Solution: We use the charge density $\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}(t))$ for a single charge q with path $\mathbf{r}(t)$ giving

$$\begin{aligned}\mathbf{p} &= \int \mathbf{r} \rho(\mathbf{r}, t) d^3\mathbf{r} = \int \mathbf{r} q \delta(\mathbf{r} - \mathbf{r}_2) - q \mathbf{r} \delta(\mathbf{r} - \mathbf{r}_1) d^3\mathbf{r} = q(\mathbf{r}_2 - \mathbf{r}_1) \\ &= 2q\mathbf{r}_2 = qd(\cos \omega t, \sin \omega t, 0),\end{aligned}\quad (25)$$

using that $\mathbf{r}_2 + \mathbf{r}_1 = 0$ so that $\mathbf{r}_1 = -\mathbf{r}_2$.

Solution: 0.5 point for the correct definition, 1 point for the δ -function description of the charge density, 1.5 point for doing the integral.

- d) Find the magnetic dipole moment of the charges. [3 points]

Solution: We should expect this to be zero since each charge sets up the same magnetic dipole by its circular motion, however, the opposite charges mean they point in opposite directions. To show this we can use the current density $\mathbf{j}(\mathbf{r}) = q\dot{\mathbf{r}}(t)\delta(\mathbf{r} - \mathbf{r}(t))$ for a single charge q with path $\mathbf{r}(t)$ and velocity $\dot{\mathbf{r}}(t)$, so that the magnetic dipole moment is given by

$$\begin{aligned}\mathbf{m} &= \frac{1}{2} \int \mathbf{r} \times \mathbf{j} d^3\mathbf{r} = \frac{1}{2} \int \mathbf{r} \times [-q\dot{\mathbf{r}}_1\delta(\mathbf{r} - \mathbf{r}_1) + q\dot{\mathbf{r}}_2\delta(\mathbf{r} - \mathbf{r}_2)] d^3\mathbf{r} \\ &= -\frac{q}{2}\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + \frac{q}{2}\mathbf{r}_2 \times \dot{\mathbf{r}}_2 = -\frac{q}{2}\mathbf{r}_2 \times \dot{\mathbf{r}}_2 + \frac{q}{2}\mathbf{r}_2 \times \dot{\mathbf{r}}_2 = 0.\end{aligned}\quad (26)$$

Scoring: 0.5 point for the correct definition, 1 point for the δ -function description of the current density, 1.5 point for completing the calculation (0.5 point is given for using the δ -function property correctly).

- e) Find the electric quadrupole vector of the charges. [4 points]

Solution: The electric quadrupole vector is given by

$$\mathbf{D}_{\hat{\mathbf{n}}} = \int \mathbf{r}(\mathbf{r} \cdot \hat{\mathbf{n}})\rho(\mathbf{r}, t)d^3\mathbf{r}.\quad (27)$$

Inserting for the charge density and using the properties of the δ -function we get

$$\begin{aligned}\mathbf{D}_{\hat{\mathbf{n}}} &= q \int \mathbf{r}(\mathbf{r} \cdot \hat{\mathbf{n}})[\delta(\mathbf{r} - \mathbf{r}_2) - \delta(\mathbf{r} - \mathbf{r}_1)]d^3\mathbf{r} \\ &= q[\mathbf{r}_2(\mathbf{r}_2 \cdot \hat{\mathbf{n}}) - \mathbf{r}_1(\mathbf{r}_1 \cdot \hat{\mathbf{n}})] \\ &= q[\mathbf{r}_2(\mathbf{r}_2 \cdot \hat{\mathbf{n}}) - \mathbf{r}_2(\mathbf{r}_2 \cdot \hat{\mathbf{n}})] = 0,\end{aligned}\quad (28)$$

using $\mathbf{r}_1 = -\mathbf{r}_2$.

Scoring: 1 point for the correct definition, 1 point for using the δ -function description correctly, 1 point for using the relationship between the positions, 1 point overall for completing the calculation.

- f) From the approximations in Eqs. (23) and (24), show that up to and including terms of order $1/r^2$ the resulting electric field is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 cr^2} [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \dot{\mathbf{p}}) - \dot{\mathbf{p}}]_{\text{ret}} + \frac{\mu_0}{4\pi r} (\ddot{\mathbf{p}}_{\text{ret}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}.\quad (29)$$

Hint: The following product rule of differentiation may come in handy

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}).\quad (30)$$

This problem is the hardest problem in the exam, and you should only spend significant time on it if you have completed the rest. [5 points]

Solution: The electric field is given from the potentials as

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}. \quad (31)$$

For $\nabla\phi$, using $Q = 0$, the product rule and the property that $\nabla = \hat{\mathbf{n}}\frac{\partial}{\partial r} + \dots$, we have

$$\begin{aligned} \nabla\phi &= \frac{1}{4\pi\epsilon_0 c} \nabla \left(\frac{\mathbf{r} \cdot \dot{\mathbf{p}}_{\text{ret}}}{r^2} \right) + \frac{1}{4\pi\epsilon_0} \nabla \left(\frac{\mathbf{r} \cdot \mathbf{p}_{\text{ret}}}{r^3} \right) \\ &= -\frac{2}{4\pi\epsilon_0 c r^3} \hat{\mathbf{n}}(\mathbf{r} \cdot \dot{\mathbf{p}}_{\text{ret}}) + \frac{1}{4\pi\epsilon_0 r^2 c} \nabla(\mathbf{r} \cdot \dot{\mathbf{p}}_{\text{ret}}) \\ &\quad -\frac{3}{4\pi\epsilon_0 r^4} \hat{\mathbf{n}}(\mathbf{r} \cdot \mathbf{p}_{\text{ret}}) + \frac{1}{4\pi\epsilon_0 r^3} \nabla(\mathbf{r} \cdot \mathbf{p}_{\text{ret}}) \\ &= -\frac{1}{2\pi\epsilon_0 c r^2} \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \dot{\mathbf{p}}_{\text{ret}}) + \frac{1}{4\pi\epsilon_0 c r^2} \nabla(\mathbf{r} \cdot \dot{\mathbf{p}}_{\text{ret}}) + \frac{1}{4\pi\epsilon_0 r^3} \nabla(\mathbf{r} \cdot \mathbf{p}_{\text{ret}}), \end{aligned} \quad (32)$$

where we have removed terms of order $1/r^3$. For the second and third terms we must carefully use the provided product rule. We here consider the second term, but the third term follows in the same manner:

$$\begin{aligned} \nabla(\mathbf{r} \cdot \dot{\mathbf{p}}_{\text{ret}}) &= (\mathbf{r} \cdot \nabla)\dot{\mathbf{p}}_{\text{ret}} + (\dot{\mathbf{p}}_{\text{ret}} \cdot \nabla)\mathbf{r} \\ &\quad + \mathbf{r} \times (\nabla \times \dot{\mathbf{p}}_{\text{ret}}) + \dot{\mathbf{p}}_{\text{ret}} \times (\nabla \times \mathbf{r}) \\ &= (\mathbf{r} \cdot \nabla)\dot{\mathbf{p}}_{\text{ret}} + \dot{\mathbf{p}}_{\text{ret}} + \mathbf{r} \times (\nabla \times \dot{\mathbf{p}}_{\text{ret}}), \end{aligned} \quad (33)$$

where we have simplified terms using $\nabla \times \mathbf{r} = 0$ and $(\dot{\mathbf{p}}_{\text{ret}} \cdot \nabla)\mathbf{r} = \dot{\mathbf{p}}_{\text{ret}}$. While $\mathbf{p}$ does not depend on $\mathbf{r}$ directly since it comes from the source distribution, it depends indirectly through $t_r = t - r/c$. The first term of (33) can be written

$$(\mathbf{r} \cdot \nabla)\dot{\mathbf{p}}_{\text{ret}} = (\mathbf{r} \cdot \nabla t_r)\ddot{\mathbf{p}}_{\text{ret}} = -\frac{1}{c}(\mathbf{r} \cdot \hat{\mathbf{n}})\ddot{\mathbf{p}}_{\text{ret}} = -\frac{r}{c}\ddot{\mathbf{p}}_{\text{ret}} \quad (34)$$

where we have used that $\nabla t_r = \nabla(t - r/c) = -\frac{1}{c}\hat{\mathbf{n}}$. And the last term of (33) can be written

$$\mathbf{r} \times (\nabla \times \dot{\mathbf{p}}_{\text{ret}}) = \mathbf{r} \times (\nabla t_r \times \frac{d}{dt_r}\dot{\mathbf{p}}_{\text{ret}}) = -\frac{1}{c}\mathbf{r} \times (\hat{\mathbf{n}} \times \ddot{\mathbf{p}}_{\text{ret}}). \quad (35)$$

Gathering our terms and inserting this into (32) we get

$$\begin{aligned}
\nabla\phi &= -\frac{1}{2\pi\epsilon_0 cr^2} \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \dot{\mathbf{p}}_{\text{ret}}) + \frac{1}{4\pi\epsilon_0 cr^2} [(\mathbf{r} \cdot \nabla)\dot{\mathbf{p}} + \dot{\mathbf{p}} + \mathbf{r} \times (\nabla \times \dot{\mathbf{p}})]_{\text{ret}} \\
&\quad + \frac{1}{4\pi\epsilon_0 r^3} [(\mathbf{r} \cdot \nabla)\mathbf{p} + \mathbf{p} + \mathbf{r} \times (\nabla \times \mathbf{p})]_{\text{ret}} \\
&= -\frac{1}{2\pi\epsilon_0 cr^2} \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \dot{\mathbf{p}}_{\text{ret}}) + \frac{1}{4\pi\epsilon_0 cr^2} \left[-\frac{r}{c} \ddot{\mathbf{p}} + \dot{\mathbf{p}} - \frac{1}{c} \mathbf{r} \times (\hat{\mathbf{n}} \times \ddot{\mathbf{p}}) \right]_{\text{ret}} \\
&\quad + \frac{1}{4\pi\epsilon_0 r^3} \left[-\frac{r}{c} \dot{\mathbf{p}} + \mathbf{p} - \frac{1}{c} \mathbf{r} \times (\hat{\mathbf{n}} \times \dot{\mathbf{p}}) \right]_{\text{ret}} \\
&= -\frac{1}{4\pi\epsilon_0 cr^2} [2\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \dot{\mathbf{p}}) - \dot{\mathbf{p}}]_{\text{ret}} - \frac{1}{4\pi\epsilon_0 c^2 r} [\ddot{\mathbf{p}} + \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \ddot{\mathbf{p}})]_{\text{ret}} \\
&\quad - \frac{1}{4\pi\epsilon_0 cr^2} [\dot{\mathbf{p}} + \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\mathbf{p}})]_{\text{ret}} \\
&= -\frac{1}{4\pi\epsilon_0 cr^2} [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \dot{\mathbf{p}}) - \dot{\mathbf{p}}]_{\text{ret}} - \frac{\mu_0}{4\pi r} [\ddot{\mathbf{p}} + (\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}]_{\text{ret}}. \quad (36)
\end{aligned}$$

where we again have removed $1/r^3$ terms and used that the double cross product is $\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\mathbf{p}}) = (\hat{\mathbf{n}} \cdot \dot{\mathbf{p}})\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\dot{\mathbf{p}} = (\hat{\mathbf{n}} \cdot \dot{\mathbf{p}})\hat{\mathbf{n}} - \dot{\mathbf{p}}$.

With (24) we easily get the vector potential contribution

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial \mathbf{A}}{\partial t_r} \simeq \frac{\mu_0}{4\pi r} \left(\ddot{\mathbf{p}} + \frac{1}{c} \ddot{\mathbf{m}} \times \hat{\mathbf{n}} + \frac{1}{2c} \ddot{\mathbf{D}}_{\hat{\mathbf{n}}} \right)_{\text{ret}} = \frac{\mu_0}{4\pi r} \ddot{\mathbf{p}}_{\text{ret}}, \quad (37)$$

since the magnetic dipole moment and electric quadrupole vector are zero. Inserting our results from both the scalar and vector potential into (31) we finally arrive at

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0 cr^2} [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \dot{\mathbf{p}}) - \dot{\mathbf{p}}]_{\text{ret}} + \frac{\mu_0}{4\pi r} [\ddot{\mathbf{p}} + (\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}]_{\text{ret}} - \frac{\mu_0}{4\pi r} \ddot{\mathbf{p}}_{\text{ret}} \\
&= \frac{1}{4\pi\epsilon_0 cr^2} [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \dot{\mathbf{p}}) - \dot{\mathbf{p}}]_{\text{ret}} + \frac{\mu_0}{4\pi r} (\ddot{\mathbf{p}}_{\text{ret}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}. \quad (38)
\end{aligned}$$

observer r/c .

Scoring: 0.5 point for the definition of the electric field, 0.5 point for removing the monopole term from the scalar potential, 3 points for carrying through the differentiation of the scalar potential, where 1 point is given for using the indirect position dependence through the retarded time, 0.5 point for removing the higher-order terms when needed, 1 point for correctly using the provided product rule, and 0.5 point for the final reorganisation and cancellation of terms from the scalar potential contribution, 1 point for the vector potential contribution.

g) Compare the results in **b** and **f**. Comment in particular on the physical interpretation of the terms in Eq. (29). [4 points]

Solution: The $1/r$ term in (29) that dominates at large distances is a radiation term due to the movement of the charges and will naturally not appear in the static case in **b**. The $1/r^2$ terms come from the existence of a fixed electric dipole at a particular location at some time in the past reaching the observer at some later time and can be compared to the result in **b**. With the electric dipole moment from **c**,

$$\hat{\mathbf{n}} \cdot \dot{\mathbf{p}} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) \cdot (-qd\omega \sin \omega t, qd\omega \cos \omega t, 0) = qd\omega \left(-\frac{x}{r} \sin \omega t + \frac{y}{r} \cos \omega t \right), \quad (39)$$

the $1/r^2$ terms can be written as

$$\begin{aligned} \mathbf{E} &= \frac{qd\omega}{4\pi\epsilon_0 cr^2} \left[3 \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) \left(-\frac{x}{r} \sin \omega t + \frac{y}{r} \cos \omega t \right) - (\cos \omega t, \sin \omega t, 0) \right]_{\text{ret}} \\ &= \frac{qd\omega}{4\pi\epsilon_0 cr^2} \left[\left(\frac{3xy}{r^2} - 1 \right) \cos \omega t - \frac{3x^2}{r^2} \sin \omega t, \right. \\ &\quad \left. - \left(\frac{3xy}{r^2} + 1 \right) \sin \omega t + \frac{3y^2}{r^2} \cos \omega t, -\frac{3xz}{r^2} \sin \omega t + \frac{3yz}{r^2} \cos \omega t \right]_{\text{ret}} \quad (40) \end{aligned}$$

Compared to (22) we see that the term in the parenthesis is identical in the x and y -directions, modulo that the dipole changes its direction, e.g. the same expression is found for $t = 0$. The prefactor is different by a factor $r\omega/c$. This corresponds to the total angle of rotation the dipole makes in the time a light signal takes from the dipole to the observer r/c .

Scoring: 1.5 point for pointing out that the second term in **f** is the radiation term dominating at long distances that does not exist in **b**, 1.5 point for discussing the similarity between **b** and the first term in **f**, 1 point for pointing out the prefactor governing the constructive/destructive interference of this term (bunching of field lines).