

Question 1 Rollin' [20 points]

A ball of mass m is rolling on the inside of a hollow cylinder at rest as illustrated in Fig. 1. The motion is confined to be in the xy -plane under the influence of gravity. We assume that the ball is always moving fast enough to remain in contact with the cylinder. In the first part of this question we will ignore the moment of inertia of the rolling ball.

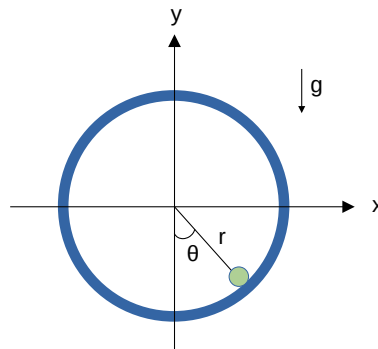


Figure 1: Ball rolling inside cylinder.

- a) Find the degrees of freedom for the motion of the ball and choose reasonable generalised coordinates. [3 points]

Solution: There is only one object moving in two dimensions. There is one constraint from the surface of the cylinder giving $d = 2 \cdot 1 - 1 = 1$ degree of freedom. We choose the angle of the position of the ball θ , measured from the bottom of the cylinder as the generalised coordinate.

Scoring: 2 points for the degrees of freedom, 1 point for choosing a sensible coordinate.

- b) Find the Lagrangian for the system. [4 points]

Solution: The relationship between the generalised coordinate θ and the ordinary Cartesian coordinates are

$$\begin{aligned} x &= r \sin \theta, & y &= -r \cos \theta, \\ \dot{x} &= r \cos \theta \dot{\theta}, & \dot{y} &= r \sin \theta \dot{\theta}. \end{aligned}$$

This gives the kinetic energy

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mr^2\dot{\theta}^2. \quad (1)$$

The potential energy is given from the gravitational pull. Setting $y = 0$ as the zero value $V = 0$ we have

$$V(\theta) = mgy = -mgr \cos \theta. \quad (2)$$

This gives the Lagrangian

$$L = K - V = \frac{1}{2}mr^2\dot{\theta}^2 + mgr \cos \theta. \quad (3)$$

Scoring: 1 point for setting up the correct coordinate relationships, 1 point for the correct kinetic energy, 1 point for the correct potential energy, 1 point for the complete Lagrangian.

c) Find the equation of motion for the system. [4 points]

Solution: We have

$$\frac{\partial L}{\partial \theta} = -mgr \sin \theta, \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = mr^2\ddot{\theta}, \quad (4)$$

giving the following Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} + mgr \sin \theta = 0. \quad (5)$$

Scoring: 1 point for each component in Lagrange's equation, 1 point for putting it together correctly.

d) Find the smallest initial velocity v_0 the ball can have at the bottom of the cylinder so that it can make it around without losing contact with the cylinder. [3 points]

Solution: On the way to the top the ball loses kinetic energy equal to the difference between the potential energies at the top and bottom. This difference is $\Delta V = V(\pi) - V(0) = -mgr \cos \pi + mgr \cos 0 = 2mgr$. To remain in contact with the cylinder at the top the circular acceleration needed to stay in orbit must be greater than that supplied by gravity. This means that $a_{\text{top}} = v_{\text{top}}^2/r > g$, so it must have a kinetic energy of $K_{\text{top}} = \frac{1}{2}mv_{\text{top}}^2 > \frac{1}{2}mgr$. Taken together with the loss in kinetic energy this means that the initial kinetic energy must be $K_{\text{init}} > \frac{5}{2}mgr$, and the velocity must be

$$v_{\text{init}} > \sqrt{5gr}. \quad (6)$$

Scoring: 1 point for realising the acceleration constraint, 1 point for the energy loss, 1 point for a correct calculation.

We will now take into account that a rolling ball has a moment of inertia $I = \frac{2}{5}ma^2$ around its centre, where a is the radius of the ball ($a \ll r$). The ball rolls without slipping (the distance covered on the inside of the cylinder wall is equal to the distance on the ball surface).

e) Find the Lagrangian in this case. [4 points]

Solution: The moment of inertia adds a kinetic energy to the Lagrangian, $K_{\text{rot}} = \frac{1}{2}I\dot{\phi}^2$, where $\dot{\phi}$ is the angular velocity of the rolling motion. Travelling an angle θ on the inside of the cylinder will cover a length θr , since the ball is not slipping, it will need to roll an angle $\phi = \frac{r}{a}\theta$. This means that $\dot{\phi} = \frac{r}{a}\dot{\theta}$, so the additional kinetic energy will be

$$K_{\text{rot}} = \frac{1}{2}I\dot{\phi}^2 = \frac{1}{2} \frac{2}{5}ma^2 \frac{r^2}{a^2} \dot{\theta}^2 = \frac{1}{5}mr^2\dot{\theta}^2. \quad (7)$$

The Lagrangian is then

$$L = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{5}mr^2\dot{\theta}^2 + mgr \cos \theta = \frac{7}{10}mr^2\dot{\theta}^2 + mgr \cos \theta. \quad (8)$$

Scoring: 1.5 point for realising there is an additional kinetic energy component and what it is in terms of I , 1 point for using the non-slipping condition correctly, 1 point for finding the extra kinetic energy, 0.5 point for the Lagrangian.

f) What is now the minimum velocity to make it around? [2 points]

Solution: Since the kinetic energy in terms of velocity is now $K = \frac{7}{10}mv^2$, where we have used that $v = r\dot{\theta}$ for circular motion, the minimum kinetic energy at the top is now $K_{\text{top}} = \frac{7}{10}mv_{\text{top}}^2 > \frac{7}{10}mgr$. The energy lost to potential energy is the same, so the minimum initial kinetic energy is $K_{\text{init}} > \frac{27}{10}mgr$ and the minimal initial velocity is

$$v_{\text{init}} = \sqrt{\frac{10K_{\text{int}}}{7m}} > \sqrt{\frac{27gr}{7}}. \quad (9)$$

which is slightly smaller than the velocity needed if we do not take the moment of inertia into account.

Scoring: 1 point for identifying the new relationship between the velocity and the kinetic energy (through the angular velocity), 1 point for inserting this correctly into the requirement on the energy and solving for v_{init} .

Question 2 Light-by-light [26 points]

We can consider light as being made up of massless particles γ , called photons. One possible (but rare) process with photons is then the scattering

(collision) of two photons giving two new photons, so called **light-by-light scattering**

$$\gamma + \gamma \rightarrow \gamma + \gamma. \quad (10)$$

- a) Assuming that the relativistic momenta of the two incoming photons are $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, write down the expression for their four-momenta p_1^μ and p_2^μ . [2 points]

Solution: The photons are massless so that their relativistic energies are $E = |\mathbf{p}|c$. This means that their four-momenta are

$$\begin{aligned} p_1^\mu &= (|\mathbf{p}_1|, \mathbf{p}_1), \\ p_2^\mu &= (|\mathbf{p}_2|, \mathbf{p}_2). \end{aligned}$$

Scoring: 1 point for correct photon energy, 1 point for writing getting the form of the four-momentum correct.

- b) Define the centre-of-mass reference frame for the two incoming photons and specialise the four-momenta to that frame. Can you find such a frame for a single photon? [3 points]

Solution: The centre-of-mass reference frame is the reference frame where the sum of the relativistic momenta are zero. For the photons this means the frame where $\mathbf{p}_1 + \mathbf{p}_2 = 0$, meaning $|\mathbf{p}_1| = |\mathbf{p}_2|$. The four-momenta are then

$$\begin{aligned} p_1^\mu &= (|\mathbf{p}_1|, \mathbf{p}_1), \\ p_2^\mu &= (|\mathbf{p}_1|, -\mathbf{p}_1). \end{aligned}$$

A centre-of-mass reference frame does not exist for a single photon. This would mean that its momentum is zero in that frame, and then its energy would also be zero, as well as the velocity.

Scoring: 1 point for the centre-of-mass definition, 1 point for using it correctly, 1 point for a correct argument for the single photon.

- c) Show that in the centre-of-mass reference frame the magnitude of the relativistic momenta of each of the outgoing photons are the same as the incoming photons. [3 points]

Solution: By the conservation of relativistic momentum the sum of the relativistic momenta of the outgoing photons $\mathbf{p}_3$ and $\mathbf{p}_4$ must also be zero, $\mathbf{p}_3 + \mathbf{p}_4 = 0$, thus they are the same in magnitude $|\mathbf{p}_3| = |\mathbf{p}_4|$. By the conservation of relativistic energy we must have $E_1 + E_2 = E_3 + E_4$, but this is $|\mathbf{p}_1|c + |\mathbf{p}_2|c = |\mathbf{p}_3|c + |\mathbf{p}_4|c$, which gives $|\mathbf{p}_1| = |\mathbf{p}_2| = |\mathbf{p}_3| = |\mathbf{p}_4|$.

Scoring: 1 point for using conservation of momentum to show the final state momenta are equal, 1 point for using conservation of energy, 1 point for drawing the right conclusions.

- d) With the four-momenta of the outgoing photons named p_3^μ and p_4^μ , show that

$$p_1 p_2 = (p_1 + p_2) p_3, \quad (11)$$

where for example $p_1 p_2 \equiv p_1^\mu p_{2\mu}$ represents the contraction of the four-momenta p_1^μ and p_2^μ . [4 points]

Solution: The conservation of four-momentum means that $p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$. This can be solved for p_4^μ giving $p_4^\mu = p_1^\mu + p_2^\mu - p_3^\mu$. Contracting a four-momentum p^μ with itself yields $p^2 = m^2 c^2$, but for a massless particle this is $p^2 = 0$. Thus

$$\begin{aligned} p_4^2 &= (p_1^\mu + p_2^\mu - p_3^\mu)(p_{1\mu} + p_{2\mu} - p_{3\mu}) \\ &= p_1^2 + p_1 p_2 - p_1 p_3 + p_2 p_1 + p_2^2 - p_2 p_3 - p_3 p_1 - p_3 p_2 + p_3^2 \\ &= 2p_1 p_2 - 2p_1 p_3 - 2p_2 p_3 = 0, \end{aligned}$$

or $p_1 p_2 = (p_1 + p_2) p_3$. Alternatively, the relationship can be shown to hold for a specific reference frame, for example the centre-of-mass frame, and then argued that this must be true for all frames since this is an invariant.

Scoring: 1 point for the conservation of four-momentum, 1 point for solving for p_4 , 1 point for using the photon invariant mass correctly, 1 point for correct four-vector contraction.

- e) Show that independent of reference frame,

$$p_1 p_2 = |\mathbf{p}_1| |\mathbf{p}_2| (1 - \cos \theta), \quad (12)$$

where θ is the angle between the incoming photons. [3 points]

Solution: Since $p_1 p_2$ is a contraction between four-vectors it is automatically independent of reference frame. The contraction is

$$p_1 p_2 \equiv p_1^\mu p_{2\mu} = |\mathbf{p}_1| |\mathbf{p}_2| - \mathbf{p}_1 \cdot \mathbf{p}_2 = |\mathbf{p}_1| |\mathbf{p}_2| (1 - \cos \theta). \quad (13)$$

Scoring: 1 point for arguing for the invariance, 1 point for writing down the contraction correctly (with correct sign), 1 point for the final answer with the scalar product angle.

- f) For a head-on collision, *i.e.* where $\theta = \pi$, the two incoming and two outgoing photons will always lie in the same plane. Explain why. [2 points]

Solution: For a head-on collision the incoming photons are on a line. Then the direction of one of the outgoing photons forms a plane with the line of the incoming photons. The second outgoing photon must then also lie in that plane. If not, it has a component of momentum so that when the momenta of the outgoing photons are added they are not in the direction of the sum of the incoming momenta, which is forbidden by the conservation of relativistic momentum.

Scoring: 2 points for any correct argument from relativistic momentum conservation.

g) Show that in a reference frame where the collision is head-on but $|\mathbf{p}_1| \neq |\mathbf{p}_2|$, the magnitude of the relativistic momentum of one of the outgoing photons is

$$|\mathbf{p}_3| = \frac{2|\mathbf{p}_1||\mathbf{p}_2|}{|\mathbf{p}_1|(1 - \cos \phi) + |\mathbf{p}_2|(1 + \cos \phi)}, \quad (14)$$

where ϕ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$. [5 points]

Solution: From (12) we have that for $\theta = \pi$

$$p_1 p_2 = |\mathbf{p}_1||\mathbf{p}_2|(1 - \cos \theta) = 2|\mathbf{p}_1||\mathbf{p}_2|. \quad (15)$$

If the common axis of the colliding photons is chosen to be the x -axis and the common plane of all the photons is the xy -plane, then

$$p_1^\mu = |\mathbf{p}_1|(1, 1, 0, 0), \quad (16)$$

$$p_2^\mu = |\mathbf{p}_2|(1, -1, 0, 0), \quad (17)$$

$$p_3^\mu = |\mathbf{p}_3|(1, \cos \phi, \sin \phi, 0), \quad (18)$$

where ϕ is the angle between $\mathbf{p}_3$ and x -axis (and $\mathbf{p}_1$). Then

$$p_1^\mu + p_2^\mu = (|\mathbf{p}_1| + |\mathbf{p}_2|, |\mathbf{p}_1| - |\mathbf{p}_2|, 0, 0), \quad (19)$$

and

$$(p_1 + p_2)p_3 = |\mathbf{p}_3|[(|\mathbf{p}_1| + |\mathbf{p}_2| - (|\mathbf{p}_1| - |\mathbf{p}_2|)\cos \phi]. \quad (20)$$

From (11) we then have

$$\begin{aligned} |\mathbf{p}_3| &= \frac{2|\mathbf{p}_1||\mathbf{p}_2|}{|\mathbf{p}_1| + |\mathbf{p}_2| - (|\mathbf{p}_1| - |\mathbf{p}_2|)\cos \phi} \\ &= \frac{2|\mathbf{p}_1||\mathbf{p}_2|}{|\mathbf{p}_1|(1 - \cos \phi) + |\mathbf{p}_2|(1 + \cos \phi)}. \end{aligned} \quad (21)$$

Scoring: 1.5 point for setting up the four-momenta correctly, 0.5 point for $p_1 p_2$, 2 points for $(p_1 + p_2)p_3$, 1 point for solving for $|\mathbf{p}_3|$.

- h) Find the boost β needed to go from this frame to the centre-of-mass reference frame. [4 points]

Solution: We will again assume the incoming photons are on the x -axis. We want to boost in the (positive) x -direction so that in the new reference frame $|\mathbf{p}'_1| = |\mathbf{p}'_2|$. The special Lorentz transformation in the x -direction for the components of the four-momentum $p_\mu = (E/c, p, 0, 0)$ is from the formulae collection

$$p' = \gamma(p - \beta \frac{E}{c}). \quad (22)$$

In terms of the momenta and energies of the two incoming photons this is

$$|\mathbf{p}'_1| = \gamma(|\mathbf{p}_1| - \beta |\mathbf{p}_1|) = \gamma(1 - \beta)|\mathbf{p}_1| = \frac{1 - \beta}{\sqrt{1 - \beta^2}} |\mathbf{p}_1| = \sqrt{\frac{1 - \beta}{1 + \beta}} |\mathbf{p}_1|. \quad (23)$$

and

$$|\mathbf{p}'_2| = \gamma(|\mathbf{p}_2| + \beta |\mathbf{p}_2|) = \gamma(1 + \beta)|\mathbf{p}_2| = \frac{1 + \beta}{\sqrt{1 - \beta^2}} |\mathbf{p}_2| = \sqrt{\frac{1 + \beta}{1 - \beta}} |\mathbf{p}_2|. \quad (24)$$

If we now require $|\mathbf{p}'_1| = |\mathbf{p}'_2|$ we get

$$\begin{aligned} \sqrt{\frac{1 - \beta}{1 + \beta}} |\mathbf{p}_1| &= \sqrt{\frac{1 + \beta}{1 - \beta}} |\mathbf{p}_2| \\ \frac{|\mathbf{p}_1|}{|\mathbf{p}_2|} &= \frac{1 + \beta}{1 - \beta} \\ \left(1 + \frac{|\mathbf{p}_1|}{|\mathbf{p}_2|}\right) \beta &= \frac{|\mathbf{p}_1|}{|\mathbf{p}_2|} - 1 \\ \beta &= \frac{|\mathbf{p}_1| - |\mathbf{p}_2|}{|\mathbf{p}_1| + |\mathbf{p}_2|}. \end{aligned} \quad (25)$$

Scoring: 1 point for the Lorentz transformation in terms of energies and momenta, 1 point for finding the momentum of the first photon, 0.5 point for finding the other (change of sign), 0.5 point for setting up the right equation, 1 point for solving for the correct β .

Question 3 Death-ray [30 points]

For a charged particle with path $\mathbf{r}(t)$ and acceleration $\mathbf{a}(t)$ the radiation fields at long distances from the particle are given by (see formulae collection):

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi c r} (\mathbf{a} \times \mathbf{n})_{\text{ret}}, \quad \mathbf{E}(\mathbf{r}, t) = c \mathbf{B}(\mathbf{r}, t) \times \mathbf{n}_{\text{ret}}, \quad (26)$$

where $\mathbf{n} = \mathbf{R}/|\mathbf{R}|$ is the unit vector in the direction between the particle and the observer, $\mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t)$. Here, “ret” designates evaluation at the retarded time t_- . These expressions have been derived in the low velocity (non-relativistic) limit.

- a) Define what the retarded time is in this particular situation and explain what it quantifies. [3 points]

Solution: The retarded time is here defined as

$$t_- = t - \frac{|\mathbf{r} - \mathbf{r}(t_-)|}{c}, \quad (27)$$

and it is the time for when a change in the source happens that reaches the observer at $(\mathbf{r}, t)$ travelling at the speed of light (speed of the electromagnetic radiation).

Scoring: 1 point for correct structure for the expression, 0.5 points for using path coordinate $\mathbf{r}(t)$ instead of generic source coordinate, and 0.5 point for using path coordinate at retarded time $\mathbf{r}(t_-)$. 1 point for a good explanation.

- b) Show that Poynting’s vector for this radiation is

$$\mathbf{S}(\mathbf{r}, t) = \frac{\mu_0 q^2}{16\pi^2 c r^2} [(\mathbf{a} \times \mathbf{n})^2 \mathbf{n}]_{\text{ret}}. \quad (28)$$

[4 points]

Solution: Poynting’s vector is

$$\begin{aligned} \mathbf{S}(\mathbf{r}, t) &= \frac{1}{\mu_0} \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) = \frac{c}{\mu_0} (\mathbf{B} \times \mathbf{n}_{\text{ret}}) \times \mathbf{B} \\ &= \frac{c}{\mu_0} [(\mathbf{B}^2) \mathbf{n}_{\text{ret}} - (\mathbf{n}_{\text{ret}} \cdot \mathbf{B}) \mathbf{B}] \\ &= \frac{c}{\mu_0} \frac{\mu_0^2 q^2}{16\pi^2 c^2 r^2} (\mathbf{a} \times \mathbf{n})_{\text{ret}}^2 \mathbf{n}_{\text{ret}} \\ &= \frac{\mu_0 q^2}{16\pi^2 c r^2} [(\mathbf{a} \times \mathbf{n})^2 \mathbf{n}]_{\text{ret}}, \end{aligned} \quad (29)$$

using the double cross product identity $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a}$, and that $\mathbf{n}_{\text{ret}} \cdot \mathbf{B} = 0$ since $\mathbf{n}_{\text{ret}}$ must be perpendicular to $\mathbf{B} \propto (\mathbf{a} \times \mathbf{n})_{\text{ret}}$.

Scoring: 1 point for definition, 1 point for substituting out a field, 1 point for doing the cross product correctly, 1 point for using the perpendicular property (or arguing from plane wave/free field).

c) Show that the total radiated power from the particle is

$$P(t) = \frac{\mu_0 q^2}{6\pi c} a_{\text{ret}}^2. \quad (30)$$

[4 points]

Solution: The total radiated power can be found by integrating Poynting's vector (energy current) over a surface enclosing the source. For simplicity we choose a sphere with surface Σ where the infinitesimal area element is $d\mathbf{A} = \mathbf{n}_{\text{ret}} dA = \mathbf{n}_{\text{ret}} r^2 \sin \theta d\theta d\phi$, using spherical coordinates, which gives

$$\begin{aligned} P(t) &= \oint_{\Sigma} \mathbf{S}(\mathbf{r}, t) \cdot d\mathbf{A} = \int \frac{\mu_0 q^2}{16\pi^2 c r^2} [(\mathbf{a} \times \mathbf{n})^2 \mathbf{n}]_{\text{ret}} \cdot \mathbf{n}_{\text{ret}} dA \\ &= \int \frac{\mu_0 q^2}{16\pi^2 c r^2} (\mathbf{a} \times \mathbf{n})_{\text{ret}}^2 r^2 \sin \theta d\theta d\phi \\ &= \frac{\mu_0 q^2}{16\pi^2 c} \int_0^{2\pi} \int_0^\pi a_{\text{ret}}^2 \sin^3 \theta d\theta d\phi \\ &= \frac{\mu_0 q^2}{8\pi c} a_{\text{ret}}^2 \int_{-1}^1 1 - u^2 du = \frac{\mu_0 q^2}{6\pi c} a_{\text{ret}}^2, \end{aligned} \quad (31)$$

where we have defined θ to be the angle between the acceleration $\mathbf{a}$ and $\mathbf{n}$ at the retarded time, and changed integration variables to $u = \cos \theta$.

Scoring: 1 point for basic idea of the integral, 1 point for choosing the proper infinitesimal area with a unit vector, 0.5 point for finding the cross product, 0.5 for doing the ϕ -integral, 1 point for carrying out the θ -integral.

d) We now pick as our reference frame S the instantaneous inertial rest frame of the charged particle at the retarded time t_- . Define this frame, and argue why the total radiated power P as given above must be a Lorentz invariant. [3 points]

Solution: The instantaneous inertial rest frame is defined by $v = 0$ so that $v \ll c$ and thus the derivation above applies. The acceleration in the instantaneous inertial reference frame (IIRF) is the proper acceleration $a_{\text{ret}} = a_0$. We know that the contraction of the four-acceleration with itself is $A^\mu A_\mu = -\mathbf{a}_0^2$, thus the power is formed only from Lorentz invariant quantities (a_0 and various constants).

Scoring: 1 point for a correct definition of the IIRF, 1 point for realising it has something to do with the proper acceleration, 1 point for using the invariant.

e) Show that the radiated power in a general reference frame is

$$P(t) = \frac{\mu_0 q^2 \gamma^4}{6\pi c} \left(\mathbf{a}^2 + \gamma^2 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \right), \quad (32)$$

where $\mathbf{v}$ and $\mathbf{a}$ is the velocity and acceleration in that frame. *Hint:* We showed in the lectures an explicit form for the four-acceleration in an inertial reference frame

$$A^\mu = (A^0, \mathbf{A}) = \gamma^2 \left(\gamma^2 \frac{\mathbf{v} \cdot \mathbf{a}}{c}, \mathbf{a} + \gamma^2 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \mathbf{v} \right). \quad (33)$$

[3 points]

Solution: Since the invariant $A^\mu A_\mu = -\mathbf{a}_0^2$ must be the same in all reference frames, and

$$\begin{aligned} A^\mu A_\mu &= \gamma^4 \left(\gamma^4 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} - (\mathbf{a} + \gamma^2 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \mathbf{v}) \cdot (\mathbf{a} + \gamma^2 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \mathbf{v}) \right) \\ &= \gamma^4 \left(\gamma^4 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} - \mathbf{a}^2 - 2\gamma^2 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \mathbf{v} \cdot \mathbf{a} - \gamma^4 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^4} \mathbf{v}^2 \right) \\ &= \gamma^4 \left(\gamma^4 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \left(1 - \frac{\mathbf{v}^2}{c^2} \right) - \mathbf{a}^2 - 2\gamma^2 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \right) \\ &= \gamma^4 \left(-\mathbf{a}^2 - \gamma^2 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \right), \end{aligned} \quad (34)$$

using $\gamma^{-2} = (1 - \mathbf{v}^2/c^2)$, we must have $\mathbf{a}_0^2 = \gamma^4 \left(\mathbf{a}^2 + \gamma^2 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \right)$. Thus

$$P(t) = \frac{\mu_0 q^2}{6\pi c} a_0^2 = \frac{\mu_0 q^2 \gamma^4}{6\pi c} \left(\mathbf{a}^2 + \gamma^2 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \right). \quad (35)$$

Scoring: 1 point for realising how the proper acceleration can be substituted, 1 point for a correct contraction, 1 point for doing the algebra correctly.

At the Large Hadron Collider (LHC) at CERN protons are run in a circular orbit around a ring with radius $r = 4243$ m, with relativistic momentum $p = 6.8$ TeV/c. The LHC runs with two beams with 2808 bunches¹ of $1.15 \cdot 10^{11}$ protons each. Some useful constants for the following problems: the proton has a mass of $m = 938$ MeV/c²,² the proton charge is $q = 1.6 \cdot 10^{-19}$ C, and the vacuum permittivity is $\epsilon_0 = 8.85 \cdot 10^{-12}$ C²/Nm².

f) Find the gamma factor of protons at the LHC. [3 points]

¹For practical reasons the protons are bunched up instead of in a continuous current, however, for the purposes of this problem it will not matter, except that you will need to multiply big numbers.

²1 TeV = 10^{12} eV and 1 MeV = 10^6 eV.

Solution: With relativistic energy $E = \gamma mc^2$ and the relativistic-energy momentum relationship $E^2 = p^2 c^2 + m^2 c^4$ we have

$$\gamma = \frac{\sqrt{p^2 c^2 + m^2 c^4}}{mc^2} \simeq \frac{6.8 \text{ TeV}}{938 \text{ MeV}} = 7249. \quad (36)$$

Scoring: 2 points for finding the correct expression for γ , 1 point for getting the numbers right.

g) Find the radiated power for a single proton and the total radiated power of the LHC. [3 points]

Solution: For a single proton we use Eq. (32) with a circular orbit so that $\mathbf{v} \cdot \mathbf{a} = 0$. The magnitude of the acceleration for the circular motion is $a = v^2/r \simeq c^2/r$ since the speed is so close to light-speed, so that

$$\begin{aligned} P(t) &= \frac{\mu_0 q^2 \gamma^4}{6\pi c} \mathbf{a}^2 \simeq \frac{\mu_0 q^2 \gamma^4 c^4}{6\pi c r^2} = \frac{q^2 \gamma^4 c^4}{6\pi \epsilon_0 c^3 r^2} = \frac{q^2 \gamma^4 c}{6\pi \epsilon_0 r^2} \\ &= \frac{(1.6 \cdot 10^{-19} \text{ C})^2 \cdot 7249^4 \cdot 3.0 \cdot 10^8 \text{ m/s}}{6\pi \cdot 8.85 \cdot 10^{-12} \text{ C}^2/\text{Nm}^2 \cdot (4243 \text{ m})^2} \simeq 7.1 \cdot 10^{-12} \text{ W} \end{aligned} \quad (37)$$

where we also used the relationship between the constants $c^2 = 1/\epsilon_0 \mu_0$. Multiplying up by the number of protons gives a power loss of

$$P_{\text{tot}} = 2 \cdot 2808 \cdot 1.15 \cdot 10^{11} \cdot 7.1 \cdot 10^{-12} \text{ W} \simeq 4560 \text{ W}, \quad (38)$$

so the equivalent of a few electric heaters blasting at full power.

Scoring: 1 point for using the correct formula with the circular motion simplification, 0.5 point for finding the acceleration, 0.5 point for doing the constant substitutions, 0.5 point for doing the numbers correctly, 0.5 point for the total power.

h) The power per unit solid angle is given via Poynting's vector as

$$\frac{dP}{d\Omega}(\theta, \phi) = |\mathbf{S}|r^2 \quad (39)$$

where $d\Omega = \sin\theta d\theta d\phi = d(\cos\theta)d\phi$ is the infinitesimal solid angle. Show that for a proton at the LHC in the instantaneous inertial reference frame, with Poynting's vector given in Eq. (28), this is

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a_0^2}{16\pi^2 c} (1 - \cos^2\phi \sin^2\theta), \quad (40)$$

where we have chosen the proper acceleration a_0 to be in the negative x -direction at the retarded time. [3 points]

Solution: From (28) we have

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c r^2} |(\mathbf{a}_0 \times \mathbf{n})^2|_{\text{ret}} r^2 = \frac{\mu_0 q^2}{16\pi^2 c} (\mathbf{a}_0 \times \mathbf{n})_{\text{ret}}^2. \quad (41)$$

Using spherical coordinates for the observer position

$$x = r \cos \phi \sin \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \theta, \quad (42)$$

we have $\mathbf{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ and $\mathbf{a}_0 = (-a_0, 0, 0)$, so that $\mathbf{a}_0 \times \mathbf{n} = (0, a_0 \cos \theta, -a_0 \sin \phi \sin \theta)$, giving

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a_0^2}{16\pi^2 c} (\cos^2 \theta + \sin^2 \phi \sin^2 \theta) = \frac{\mu_0 q^2 a_0^2}{16\pi^2 c} (1 - \cos^2 \phi \sin^2 \theta). \quad (43)$$

Scoring: 1 point for setting up the observer and acceleration vectors, 1 point for evaluating the cross product (squared), 1 point for final result.

- i) Find the power per unit solid angle for the LHC proton in the lab frame S' . *Hint:* Use the fact that you can choose how to set axis and origin of the reference frames so that at $t = t' = 0$ the proton is at the origin $r = r' = 0$. Choose your coordinate system so that the proton is moving in the z -direction at the time the radiation is emitted and the acceleration points in the negative x -direction. [4 points]

Solution: We assume that at $t = t' = 0$ the proton in the IIRF is moving in the z -direction, and that the acceleration is in the negative x -direction so the motion lies in the xz -plane. The Lorentz transformation from the coordinates in the S frame to the lab frame S' is given by

$$ct' = \gamma(ct + \beta z), \quad x' = x, \quad y' = y, \quad z' = \gamma(z + \beta ct). \quad (44)$$

The radiation radiated at the retarded time $t = t' = 0$ will travel radially from the proton and reach the observer at $r = ct$ and $r' = ct'$ in the two frames. Using this, and rewriting the Lorentz transformations in spherical coordinates gives the relationships

$$r' = \gamma r(1 + \beta \cos \theta), \quad (45)$$

$$r' \cos' \phi' \sin \theta' = r \cos \phi \sin \theta, \quad (46)$$

$$r' \sin' \phi' \sin \theta' = r \sin \phi \sin \theta, \quad (47)$$

$$r' \cos \theta' = \gamma r(\cos \theta + \beta). \quad (48)$$

Dividing (47) by (46) we get $\tan \phi' = \tan \phi$ which means that $\phi' = \phi$ and the angle in the plane perpendicular to the motion is unaffected by the transformation (the ease of this equation was the reason for

choosing the motion to be in the xz -plane). Inserting this back into (47) also gives $r' \sin \theta' = r \sin \theta$. Using (45) then gives

$$\gamma(1 + \beta \cos \theta) \sin \theta' = \sin \theta. \quad (49)$$

Comparing the remaining two equations (45) by (48) the radial coordinates (and γ) can be removed, giving

$$(1 + \beta \cos \theta) \cos \theta' = \cos \theta + \beta. \quad (50)$$

We can now solve these equations for $\sin \theta$ and $\cos \theta$ giving

$$\cos \theta = \frac{\cos \theta' + \beta}{1 - \beta \cos \theta'}, \quad \sin \theta = \frac{1}{\gamma} \frac{\sin \theta'}{1 - \beta \cos \theta'}. \quad (51)$$

We saw above that the total power $P = \int \frac{dP}{d\Omega} d\Omega$ is a Lorentz invariant, this means that the differential power in the two frames is related by

$$\frac{dP}{d\Omega}(\theta, \phi) d\Omega = \frac{dP'}{d\Omega'}(\theta', \phi') d\Omega'. \quad (52)$$

Using $d\Omega = \sin \theta d\theta d\phi = d(\cos \theta) d\phi$ we have

$$\frac{d\Omega}{d\Omega'} = \frac{d(\cos \theta) d\phi}{d(\cos \theta') d\phi'} = \frac{d(\cos \theta)}{d(\cos \theta')} = \frac{1}{\gamma^2 (1 - \beta \cos \theta')^2}, \quad (53)$$

and using (40) with the acceleration in the lab frame $a_0 = \gamma^2 a$,

$$\begin{aligned} \frac{dP'}{d\Omega'} &= \frac{\mu_0 q^2 \gamma^4 a^2}{16\pi^2 c} \left[1 - \cos^2 \phi' \left(\frac{1}{\gamma} \frac{\sin \theta'}{1 - \beta \cos \theta'} \right)^2 \right] \frac{1}{\gamma^2 (1 - \beta \cos \theta')^2} \\ &= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\gamma^2}{(1 - \beta \cos \theta')^2} \left(1 - \frac{1}{\gamma^2} \frac{\cos^2 \phi' \sin^2 \theta'}{(1 - \beta \cos \theta')^2} \right). \end{aligned} \quad (54)$$

Scoring: 1 point for setting up the correct Lorentz transformations, 1 point for finding them in spherical coordinates, 1 point for solving for the angles in S , 1 point for constructing the differential power, including the Jacobian factor for the transformation.