

Question 1 Roundworld [25 points]

Imagine that you are standing on a round world with radius R throwing an object of mass m , with gravity acting towards the centre of the world at coordinates $(x, y, z) = (0, 0, 0)$, with the potential

$$V(x, y) = -\frac{GmM}{\sqrt{x^2 + y^2 + z^2}}, \quad (1)$$

where G is Newton's gravitational constant and M is the mass of the world. You now want to find the path of the object.

To simplify, we will assume that all the motion is in a plane,^{1,2} and we will of course ignore air-resistance. We assume the world is not rotating.

a) Sketch the system. Include yourself as a stick-figure. [2 points]

Scoring: 1 point for a good drawing including axes, 1 point for including the coordinates $(r, \theta, R$ and $h)$. Subtract 0.5 point for missing stick-figure.

b) Find generalised coordinates for the system. [2 points]

Solution: Since we have no constraints and two-dimensional motion we have $d = 2N - M = 2 \cdot 1 - 0 = 2$ degrees of freedom. The most sensible thing is to pick polar coordinates (r, θ) for the system, with $z = 0$, where we define θ to be the angle of the mass with respect to the vertical, and r to be the distance from the centre of the world.

Scoring: 1 point for the d.o.f. 1 point for picking sensible coordinates.

c) Find the Lagrangian. [5 points]

Solution: The Cartesian coordinates and their velocities can be written in terms of the polar coordinates as

$$\begin{aligned} x &= r \sin \theta, & \dot{x} &= \dot{r} \sin \theta + r \cos \theta \dot{\theta}, \\ y &= r \cos \theta, & \dot{y} &= \dot{r} \cos \theta - r \sin \theta \dot{\theta}. \end{aligned}$$

The kinetic energy is then

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2), \quad (2)$$

and the potential

$$V = -\frac{mGM}{r}. \quad (3)$$

This gives the Lagrangian

$$L = K - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{mGM}{r}. \quad (4)$$

¹We mean on a two-dimensional surface, not in an aeroplane.

²In fact, you can prove that this is true (that the motion is in a two-dimensional plane, not that it is not in an aeroplane), but you won't get any extra points for that.

Scoring: 1 point for the Cartesian coordinates in terms of generalised coordinates, 1 point for their velocities, 1 point for calculating the kinetic energy correctly, 1 point for the potential energy, 1 point for putting it together in the Lagrangian.

- d) Identify two conserved quantities in the system and show that they are indeed conserved. [4 points]

Solution: Since θ is a cyclic coordinate its generalised momentum $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ is conserved, *i.e.* $mr^2\dot{\theta} = \ell$, where ℓ is some constant. This quantity is the angular momentum around the z -axis since:

$$\begin{aligned} L_z &= xp_y - yp_x = m(x\dot{y} - y\dot{x}) \\ &= m[r \sin \theta (\dot{r} \cos \theta - r \sin \theta \dot{\theta}) - r \cos \theta (\dot{r} \sin \theta + r \cos \theta \dot{\theta})] \\ &= -mr^2\dot{\theta}. \end{aligned}$$

Additionally, since the Lagrangian has no explicit time-dependence, the Hamiltonian is conserved

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0, \quad (5)$$

and since the potential and the constraints are time and velocity independent the Hamiltonian is the total kinetic energy. Thus total kinetic energy and angular momentum is conserved.

Scoring: 1 point per conserved quantity, 1 point per for argument that they are conserved.

- e) Find the equations of motion. [5 points]

Solution: The ingredients for the two Lagrange equations are

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= 0, & \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta}, \\ \frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - \frac{mGM}{r^2}, & \frac{\partial L}{\partial \dot{r}} &= m\dot{r}, \end{aligned}$$

giving, for the cyclic coordinate θ , the equation

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0, \quad \text{or} \quad mr^2\dot{\theta} = \ell, \quad (6)$$

and, for r , the equation

$$\ddot{r} - r\dot{\theta}^2 + \frac{GM}{r^2} = 0. \quad (7)$$

Scoring: 1 point for the general form of Lagrange's equation, 2 points for the e.o.m. for θ , 2 points for the equation for r .

f) Show that the equations of motion can be reduced to

$$\ddot{r} - \frac{\ell^2}{m^2 r^3} + \frac{GM}{r^2} = 0, \quad (8)$$

where r is the distance of the mass from the centre of the world and ℓ is some constant. [2 points]

Solution: From the conservation of angular momentum we can substitute $\dot{\theta} = \frac{\ell}{mr^2}$ into the equation of motion for r giving

$$\ddot{r} - \frac{\ell^2}{m^2 r^3} + \frac{GM}{r^2} = 0. \quad (9)$$

Scoring: 1 point of identifying the substitution of the cyclic coordinate, 1 point for carrying it out.

g) Find the equations of motion in terms of the height h above the surface of the world, $r = h + R$, in the limit $h \ll R$, and compare these equations to the corresponding situation on our flat Earth. [5 points]

Solution: Substituting $r = h + R$ we get

$$\ddot{h} - \frac{\ell^2}{R^3 m^2 (1 + \frac{h}{R})^3} + \frac{GM}{R^2 (1 + \frac{h}{R})^2} = 0. \quad (10)$$

Series expanding, using from Rottmann,

$$(1 + x)^{-n} = 1 - nx + \frac{1}{2}n(n+1)x^2 + \dots,$$

with $x = h/R$, we get

$$\ddot{h} - \left(\frac{2GM}{R^2} + \frac{3\ell^2}{R^3 m^2} \right) \frac{h}{R} + \frac{GM}{R^2} - \frac{\ell^2}{R^3 m^2} = 0. \quad (11)$$

The equation for the throw height y on flat Earth is $\ddot{y} = -g$, where the gravitational constant would be $g = GM/R^2$, if the radius R and mass M of the world is the same. Ignoring small terms of the order h/R the equations are the same and must thus have the same solutions under the change of constants $g \rightarrow \frac{GM}{R^2} - \frac{\ell^2}{R^3 m^2}$.

Scoring: 1 point for a correct insertion, 1 point for using the series expansion, 1 point for figuring out the correct form of the series expansion, 2 points for a good comparison to the flat Earth case.

Question 2 The Glashow resonance [25 points]

Anti-neutrinos with very high energies created in the most violent processes in the Universe travel through our galaxy and sometimes pass through the Earth. Here they may encounter electrons hanging around not doing very much, and may be turned into massive W -bosons in the process $\bar{\nu} + e^- \rightarrow W^-$. This is called the **Glashow process**.

Assume that the electron is at rest with mass m_e and let the mass and relativistic energy of the incoming anti-neutrino be m_n and E_n , respectively.

- a) Write down the four-momenta of the incoming anti-neutrino p_n^μ and the electron p_e^μ in terms of the variables given above. [4 points]

Solution: Using the relativistic energy-momentum relationship, $E^2 = p^2c^2 + m^2c^4$, and choosing the motion of the neutrino to be in the positive x -direction, their four-momenta are

$$\begin{aligned} p_n^\mu &= (E_n/c, \mathbf{p}_n) = (E_n/c, \sqrt{E_n^2/c^2 - m_n^2c^2}, 0, 0), \\ p_e^\mu &= (E_e/c, 0, 0, 0) = (m_e c, 0, 0, 0). \end{aligned}$$

Scoring: 1 point for writing the general form of the four-momentum, 1 point for the correct electron four-momentum, 2 points for the correct neutrino four-momentum (here 1 point for energy-momentum relationship).

- b) Find the resulting four-momentum of the W -boson p_W^μ . [2 points]

Solution: From conservation of four-momentum we have $p_W^\mu = p_n^\mu + p_e^\mu$ so that

$$p_W^\mu = (E_n/c + m_e c, \sqrt{E_n^2/c^2 - m_n^2c^2}, 0, 0). \quad (12)$$

Scoring: 1 point for the conservation of four-momentum, 1 point for correct expression.

- c) Show that the relativistic energy of the neutrino in this process can be expressed by the masses of the three particles as

$$E_n = \frac{m_W^2 c^2 - m_n^2 c^2 - m_e^2 c^2}{2m_e}. \quad (13)$$

[5 points]

Solution: Contracting the four-momentum of the W -boson gives $p_W^2 = m_W^2 c^2$. From the above expression of the four-momentum this means that

$$\begin{aligned} m_W^2 c^2 &= (E_n/c + m_e c)^2 - E_n^2/c^2 + m_n^2 c^2 \\ &= E_n^2/c^2 + 2E_n m_e + m_e^2 c^2 - E_n^2/c^2 + m_n^2 c^2 \\ &= 2E_n m_e + m_e^2 c^2 + m_n^2 c^2. \end{aligned} \quad (14)$$

Solving for the energy yields

$$E_n = \frac{m_W^2 c^2 - m_n^2 c^2 - m_e^2 c^2}{2m_e}. \quad (15)$$

Scoring: 1 point for using the conservation of four-momentum, 1 point for using the invariant mass of W , 2 points for contracting the W four-momentum correctly (here 1 point for getting the correct sign on the momentum term), 1 point for completing the calculation.

- d) The mass of the W -boson has been measured to be $m_W = 80.4 \text{ GeV}/c^2$ and the electron is $m_e = 0.511 \text{ MeV}/c^2$. We currently only have an upper limit on the neutrino mass of $m_n < 1 \text{ eV}/c^2$. Find the anti-neutrino energy (in electronvolts eV).³ [3 points]

Solution: The neutrino mass can be safely ignored here since both it and the electron mass gives a vanishing contribution to the the numerator in (13). The numbers then give

$$E_n \simeq \frac{m_W^2 c^4}{2m_e c^2} = \frac{(80.4 \cdot 10^9 \text{ eV})^2}{2 \cdot 0.511 \cdot 10^6 \text{ eV}} = 6.33 \cdot 10^{15} \text{ eV} = 6.33 \text{ PeV}. \quad (16)$$

Scoring: 1 point for realising that the neutrino mass can be ignored, 2 points for calculating correct numbers.

- e) Show that the boost needed to go from the lab-frame considered above (where the electron was at rest) to the centre-of-mass reference frame is given by

$$\beta = \frac{\sqrt{E_n^2 - m_n^2 c^4}}{E_n + m_e c^2}. \quad (17)$$

[6 points]

Solution: In the centre-of-mass reference frame the sum of the (relativistic) momenta are zero. Thus in this frame the produced W -boson is at rest and its four-momentum $p_W'^\mu$ in this frame must be $p_W'^\mu = (m_W c, 0)$. We now need to find the Lorentz transformation between p_W^μ and $p_W'^\mu$. In terms of components of the four-momentum, with relativistic energy E and momentum p (in some direction), this is $p' = \gamma(p - \beta E/c)$, which in our case gives

$$\begin{aligned} 0 &= \gamma(p - \beta E/c) \\ \beta &= \frac{pc}{E} \\ \beta &= \frac{\sqrt{E_n^2/c^2 - m_n^2 c^2} c}{E_n + m_e c^2} = \frac{\sqrt{E_n^2 - m_n^2 c^4}}{E_n + m_e c^2}, \end{aligned} \quad (18)$$

where we have inserted from $p_W'^\mu$ in (12).

³1 GeV is 10^9 eV, 1 MeV is 10^6 eV.

Scoring: 1 point for CoM-frame definition, 1 point for the W 's momentum in CoM-frame, 2 points for writing up a correct Lorentz transformation for the momentum, 1 point for correct insertion of the known quantities, 1 point for solving correctly. Note that there are more complicated paths that lead to correct answers, *e.g.* looking at energy or boosts in the opposite direction.

- f) The W -boson has a very short lifetime of $\tau = 3.16 \cdot 10^{-25}$ s in its rest-frame before it decays. What is the lifetime of the W -boson in the lab-frame? [5 points]

Solution: We know that the relationship between the times in the two different frames is given by a Lorentz transformation $c\tau' = \gamma(c\tau - \beta x)$. In the rest-frame of the W we can place the coordinate system so that the W is at $x = 0$, this gives the standard time-dilation formula $\tau' = \gamma\tau$.

While the direction of the boost is the opposite to the one calculated in e), the magnitude β is the same, so the γ -factor becomes

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - \frac{E_n^2 - m_n^2 c^4}{(E_n + m_e c^2)^2}}} \\ &= \frac{E_n + m_e c^2}{\sqrt{(E_n + m_e c^2)^2 - E_n^2 - m_n^2 c^4}} \\ &= \frac{E_n + m_e c^2}{\sqrt{2E_n m_e c^2 + m_e^2 c^4 - m_n^2 c^4}} \\ &\simeq \sqrt{\frac{E_n}{2m_e c^2}} = \sqrt{\frac{6.33 \cdot 10^{15} \text{ eV}}{2 \cdot 0.511 \cdot 10^6 \text{ eV}}} \simeq 78700,\end{aligned}\quad (19)$$

and a lifetime of $\tau' = 2.49 \cdot 10^{-20}$ s.

Scoring: 1 point for some reasonable argument that the time-dilation formula applies, 1 point for the correct expression for the γ -factor, 1 point to realise we can use the same β as above, 1 point for manipulating the γ -factor until you have an expression that does not evaluate badly numerically, 1 point for the correct numerical value.

Question 3 Circular antenna [28 points]

We will now look at the radiation from a circular antenna in the (x, y) -plane with centre at the origin and radius a . It carries the current

$$I = I_0 \sin \omega t, \quad (20)$$

where the positive direction of current is taken to be in the counter clockwise direction in the (x, y) -plane. The antenna is charge neutral at all times.

We consider the antenna to be thin and the current to be constant across the cross section of the antenna.

We remind you that the expressions for radiation fields to lowest order in $1/r$ can be written

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r c} \left(\ddot{\mathbf{p}} + \frac{1}{c} \ddot{\mathbf{m}} \times \mathbf{n} + \frac{1}{2c} \ddot{\mathbf{D}}_{\mathbf{n}} \right)_{\text{ret}} \times \mathbf{n}, \quad (21)$$

$$\mathbf{E}(\mathbf{r}, t) = -c\mathbf{n} \times \mathbf{B}, \quad (22)$$

where $\mathbf{n} = \frac{\mathbf{r}}{r}$ is the unit vector in the direction of the observer.

- a) Briefly explain the meaning of the word “ret” in the formulae for the radiation fields. [2 points]

Solution: The word “ret” refers to the evaluation of the expression at the retarded time t_r , defined as a time $t_r = t - r/c$, where t and r is the time and distance of the observer. This time is the time when a change in the source is visible to the observer at t , r/c is also the time it takes for a light-signal to travel from the source to the observer.

Scoring: 1 point for the expression for the retarded time, 1 point for a good explanation of what it is. Deduct 0.5 point for giving t_- instead of t_r .

- b) Show that the magnetic dipole moment is

$$\mathbf{m} = \pi a^2 I_0 \sin \omega t \mathbf{e}_z. \quad (23)$$

[5 points]

Solution: Since the current in an infinitesimal volume element for the circular antenna is $\mathbf{j} d^3\mathbf{r} = I d\mathbf{r}$, where $d\mathbf{r} = a d\phi \mathbf{e}_\phi$ is an infinitesimal line element along the antenna that is perpendicular to $\mathbf{r} = a\mathbf{e}_r$, the magnetic dipole moment is

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times \mathbf{j} d^3\mathbf{r} = \frac{1}{2} \oint \mathbf{r} \times I d\mathbf{r} = \frac{1}{2} I_0 \sin \omega t \int_0^{2\pi} a \cdot a d\phi \mathbf{e}_z, \quad (24)$$

where ϕ is the angle of the line element in the (x, y) -plane. The integral gives

$$\mathbf{m} = \pi a^2 I_0 \sin \omega t \mathbf{e}_z. \quad (25)$$

Scoring: 1 point for correct expression, 1 point for getting the integral over to closed contour form, 1 point for getting the contour integral over to angular form, 1 point for finding the direction of $\mathbf{m}$, 1 point for completing the calculation.

- c) Show that Poynting's vector for the resulting radiation at large distances from the source is

$$\mathbf{S}(\mathbf{r}, t) = \frac{\mu_0 a^4 \omega^4}{16 r^2 c^3} I_0^2 \sin^2 \omega t_r \sin^2 \theta \mathbf{n}, \quad (26)$$

and identify the angle θ . [8 points]

Solution: Poynting's vector can be written in terms of the magnetic field as

$$\begin{aligned} \mathbf{S}(\mathbf{r}, t) &= \frac{1}{\mu_0} \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) = \frac{c}{\mu_0} (\mathbf{B} \times \mathbf{n}) \times \mathbf{B} \\ &= \frac{c}{\mu_0} [(\mathbf{B}^2) \mathbf{n} - (\mathbf{n} \cdot \mathbf{B}) \mathbf{B}] = \frac{c}{\mu_0} \mathbf{B}^2 \mathbf{n}, \end{aligned}$$

using the double cross product identity,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a}, \quad (27)$$

and that $\mathbf{n} \cdot \mathbf{B} = 0$ since $\mathbf{n}$ must be perpendicular to $\mathbf{B}$ because the radiation field $\mathbf{B}$ is the result of the cross product of a vector and $\mathbf{n}$.

Since the current loop is neutral the charge density is $\rho = 0$ and thus both the electric dipole moment and the quadrupole term that depend on the charge density in the integrand must be zero. The magnetic radiation field at large distances is then

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r c^2} (\ddot{\mathbf{m}}_{\text{ret}} \times \mathbf{n}) \times \mathbf{n}, \quad (28)$$

and with the time derivative of the dipole moment

$$\ddot{\mathbf{m}}_{\text{ret}} = \frac{d^2}{dt^2} \mathbf{m}_{\text{ret}} = -\omega^2 \mathbf{m}_{\text{ret}} = -\pi a^2 I_0 \omega^2 \sin \omega t_r \mathbf{e}_z, \quad (29)$$

we get

$$\begin{aligned} \mathbf{S}(\mathbf{r}, t) &= \frac{c}{\mu_0} \frac{\mu_0^2}{16\pi^2 r^2 c^4} \ddot{\mathbf{m}}_{\text{ret}}^2 \sin^2 \theta \mathbf{n} = \frac{\mu_0}{16\pi^2 r^2 c^3} \ddot{\mathbf{m}}_{\text{ret}}^2 \sin^2 \theta \mathbf{n} \\ &= \frac{\mu_0}{16\pi^2 r^2 c^3} \pi^2 a^4 I_0^2 \omega^4 \sin^2 \omega t_r \sin^2 \theta \mathbf{n} \\ &= \frac{\mu_0 a^4 \omega^4}{16 r^2 c^3} I_0^2 \sin^2 \omega t_r \sin^2 \theta \mathbf{n}, \end{aligned} \quad (30)$$

where θ is the angle between $\mathbf{e}_z$ and $\mathbf{n}$ involved in the cross-product giving the magnetic field.⁴

⁴Since $\mathbf{n}$ is also the unit radial vector we could use spherical coordinates to show that $\mathbf{e}_z \times \mathbf{n} = \mathbf{e}_\phi$ and $(\mathbf{e}_z \times \mathbf{n}) \times \mathbf{n} = \mathbf{e}_\phi \times \mathbf{n} = \mathbf{e}_\theta$.

Scoring: 1 point for definition of Poynting's vector, 1 point for substituting out a field, 1 point for doing the double cross product correctly, 1 point for using the perpendicular property (or arguing from plane wave/free field), 1 point for finding the B -field expression (zero contributions), 1 point for finding $\ddot{\mathbf{m}}$, 1 point for identifying θ , 1 point for putting it together correctly.

- d) Describe the polarisation of the radiation (type and direction).
[3 points]

Solution: The electric field is

$$\mathbf{E}(\mathbf{r}, t) = -c\mathbf{n} \times \mathbf{B} = \frac{\mu_0}{4\pi r c} [(\ddot{\mathbf{m}}_{\text{ret}} \times \mathbf{n}) \times \mathbf{n}] \times \mathbf{n}. \quad (31)$$

Since $\ddot{\mathbf{m}}_{\text{ret}}$ has a fixed direction (but switches sign with time) the electric field must have a fixed direction. This means that the radiation is linearly polarised. The direction can be found from the cross product identity (27)

$$(\ddot{\mathbf{m}}_{\text{ret}} \times \mathbf{n}) \times \mathbf{n} = (\ddot{\mathbf{m}}_{\text{ret}} \cdot \mathbf{n})\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})\ddot{\mathbf{m}}_{\text{ret}} = (\ddot{\mathbf{m}}_{\text{ret}} \cdot \mathbf{n})\mathbf{n} - \ddot{\mathbf{m}}_{\text{ret}}, \quad (32)$$

giving

$$[(\ddot{\mathbf{m}}_{\text{ret}} \times \mathbf{n}) \times \mathbf{n}] \times \mathbf{n} = [(\ddot{\mathbf{m}}_{\text{ret}} \cdot \mathbf{n})\mathbf{n}] \times \mathbf{n} - \ddot{\mathbf{m}}_{\text{ret}} \times \mathbf{n} = -\ddot{\mathbf{m}}_{\text{ret}} \times \mathbf{n}, \quad (33)$$

so, since $\ddot{\mathbf{m}}_{\text{ret}}$ is proportional to $\mathbf{e}_z$, the polarisation is in the $\mathbf{e}_z \times \mathbf{n}$ direction.

Scoring: 1 point for identifying that the radiation is linear, 1 point for finding the direction, 1 point for reducing the direction into the simple $\mathbf{e}_z \times \mathbf{n}$.

- e) What force does a charge on the z -axis observe as an effect of the antenna radiation? [3 points]

Solution: For an observer on the z -axis $\mathbf{n} = \mathbf{e}_z$ so that

$$\mathbf{B} \propto (\ddot{\mathbf{m}}_{\text{ret}} \times \mathbf{e}_z) \times \mathbf{e}_z, \quad (34)$$

and since $\ddot{\mathbf{m}} \propto \mathbf{e}_z$ we have $\mathbf{B} \propto (\ddot{\mathbf{m}}_{\text{ret}} \times \mathbf{e}_z) \times \mathbf{e}_z \propto (\mathbf{e}_z \times \mathbf{e}_z) \times \mathbf{e}_z = 0$ to the accuracy of the $1/r$ expansion. Similarly, the E -field in this direction is zero since $\mathbf{E} = -c\mathbf{n} \times \mathbf{B}$. As a result there is no force on a charge on the z -axis.

Scoring: 1 point for the realisation that the E - and B -fields must be calculated to find the Lorentz force, 1 point for inserting the correct direction for the magnetic dipole, 1 point for realising that the cross products give zero.

- f) What is the force and torque felt by the antenna if an external constant magnetic field $\mathbf{B}_{\text{ext}} = B_0 \mathbf{e}_x$ is applied in the x -direction? [3 points]

Solution: The force is given as

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}_{\text{ext}}) = \nabla(\pi a^2 I_0 \sin \omega t \mathbf{e}_z \cdot B_0 \mathbf{e}_x) = 0, \quad (35)$$

because the vectors are perpendicular, while the torque is

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}_{\text{ext}} = (\pi a^2 I_0 \sin \omega t \mathbf{e}_z \times B_0 \mathbf{e}_x) = \pi a^2 I_0 B_0 \sin \omega t \mathbf{e}_y, \quad (36)$$

thus it will twist the antenna giving it an angular momentum in the y -direction.

Scoring: 1 point for the correct definitions (0.5 points each), 1 point for realising the force is zero due to the vectors, 1 point for finding the correct torque.

- g) Find the power emitted by the antenna. [4 points]

Solution: The power P is found by integrating Poynting's vector (flux of energy) over a sphere Σ surrounding the antenna using spherical coordinates with fixed radius r :

$$\begin{aligned} P(t) &= \int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} = \frac{\mu_0 a^4 \omega^4}{16 r^2 c^3} I_0^2 \sin^2 \omega t_r \int_{\Sigma} \sin^2 \theta \mathbf{n} \cdot \mathbf{n} dA \\ &= \frac{\mu_0 a^4 \omega^4}{16 r^2 c^3} I_0^2 \sin^2 \omega t_r \int_0^{2\pi} \int_0^{\pi} \sin^2 \theta r^2 \sin \theta d\theta d\phi \\ &= \frac{\pi \mu_0 a^4 \omega^4}{8 c^3} I_0^2 \sin^2 \omega t_r \int_0^{\pi} \sin^3 \theta d\theta \\ &= \frac{\pi \mu_0 a^4 \omega^4}{6 c^3} I_0^2 \sin^2 \omega t_r \end{aligned} \quad (37)$$

where we have used that with the substitution $u = \cos \theta$,

$$\int_0^{\pi} \sin^3 \theta d\theta = \int_{-1}^1 \sin^3 \theta \frac{du}{\sin \theta} = \int_{-1}^1 1 - u^2 du = u - \frac{1}{3} u^3 = \frac{4}{3}. \quad (38)$$

Scoring: 1 point for the correct idea of the integration, 1 point for reducing the integration to an area integral, 1 point for writing the integral in terms of spherical coordinates, 1 point for doing the spherical coordinate integrals