

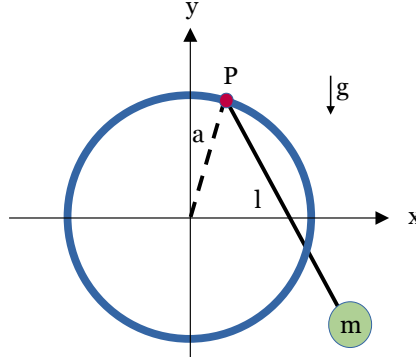


Figure 1: Pendulum attached to a rotating wheel.

Question 1 A very driven pendulum [25 points]

A stiff pendulum of length ℓ and mass m is attached to a pivot point P on a wheel with radius a . The wheel is driven by a motor to rotate with constant angular velocity ω in the clockwise direction. Gravity works in the negative y -direction. There is no friction in the system and all motion is assumed to be in the plane. The system is illustrated in Fig. 1.

The wheel is started so that at $t = 0$ the pivot point (x_P, y_P) is at the top of the wheel, $(x_P, y_P) = (0, a)$.

- a) Find the number of degrees of freedom and define generalised coordinate(s) for the system. [3 points]

Solution: Since we have two-dimensional motion with one object, the pendulum mass (the wheel has its position automatically given by its constant angular velocity), and we have one time-dependent constraint from the distance of the mass to the pivot point, which is time-dependent through the position of the wheel, we have a total of $d = 2N - M = 2 \cdot 1 - 1 = 1$ degrees of freedom. We have several viable choices for generalised coordinate, but here we pick the angle θ that the pendulum makes with the vertical direction (y -axis), with $\theta = 0$ when the pendulum hangs straight down (negative y -direction).

It is also possible to include the position of the wheel through the pivot point as degrees of freedom, this will then be constrained away by two constraints: the angular velocity and the radius of the wheel.

Scoring: 2 points for the d.o.f., where 1 point is for getting the constraints right. 1 point for picking a sensible coordinate.

- b) Show that the Lagrangian for the system can be written as

$$L = \frac{1}{2}m[\ell^2\dot{\theta}^2 + 2a\ell\omega\dot{\theta}\cos(\theta + \omega t) + a^2\omega^2] - mg(a\cos\omega t - \ell\cos\theta). \quad (1)$$

[6 points]

Solution: The distance from the pivot point P to the pendulum mass can be written as

$$x_m = \ell \sin \theta, \quad y_m = -\ell \cos \theta. \quad (2)$$

Adding the position of the pivot point

$$x_P = a \sin \omega t, \quad y_P = a \cos \omega t, \quad (3)$$

the total Cartesian coordinates and their velocities are then

$$\begin{aligned} x &= x_P + x_m = a \sin \omega t + \ell \sin \theta, & \dot{x} &= a\omega \cos \omega t + \ell \dot{\theta} \cos \theta, \\ y &= y_P + y_m = a \cos \omega t - \ell \cos \theta, & \dot{y} &= -a\omega \sin \omega t + \ell \dot{\theta} \sin \theta. \end{aligned}$$

The kinetic energy is then

$$\begin{aligned} K &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m(a^2\omega^2 \cos^2 \omega t + 2a\omega \cos \omega t \ell \dot{\theta} \cos \theta + \ell^2 \dot{\theta}^2 \cos^2 \theta \\ &\quad + a^2\omega^2 \sin^2 \omega t - 2a\omega \sin \omega t \ell \dot{\theta} \sin \theta + \ell^2 \dot{\theta}^2 \sin^2 \theta) \\ &= \frac{1}{2}m[a^2\omega^2 + 2a\omega \ell \dot{\theta}(\cos \omega t \cos \theta - \sin \omega t \sin \theta) + \ell^2 \dot{\theta}^2] \\ &= \frac{1}{2}m[\ell^2 \dot{\theta}^2 + 2a\omega \ell \dot{\theta} \cos(\theta + \omega t) + a^2\omega^2,] \end{aligned}$$

where we have used the trigonometric identity

$$\cos(x + y) = \cos x \cos y - \sin x \sin y. \quad (4)$$

The gravitational potential energy for the mass is

$$V = mgy = mg(a \cos \omega t - \ell \cos \theta). \quad (5)$$

which gives the Lagrangian

$$L = K - V = \frac{1}{2}m[\ell^2 \dot{\theta}^2 + 2a\omega \ell \dot{\theta} \cos(\theta + \omega t) + a^2\omega^2] - mg(a \cos \omega t - \ell \cos \theta). \quad (6)$$

Scoring: 1 point for the Cartesian coordinates in terms of generalised coordinates, 0.5 points for setting this up so that it matches the initial condition for the wheel, 1 point for their velocities, 1.5 point for calculating the kinetic energy correctly where 0.5 point is for the trigonometric rewrite, 1 point for the potential energy, 1 point for putting it together in the Lagrangian.

c) Explain why the Lagrangian

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{a\omega^2}{\ell} \cos(\theta - \omega t) + \frac{g}{\ell} \cos \theta, \quad (7)$$

is equivalent to the one found in **b)**, meaning it has the same equations of motion. Do this without finding the equations of motion. [3.5 points]

Solution: The term $a^2\omega^2$ can be removed since adding a constant to a Lagrangian does not change the e.o.m. Such terms disappear by the derivatives in Lagrange's equation. The common factor m can also be removed since multiplying the Lagrangian with a constant does not change the e.o.m. since a constant can be cancelled from all terms in Lagrange's equation.

We have shown that total time-derivative terms can also be removed from the Lagrangian without changing the e.o.m. We can write the term proportional to $\cos \omega t$ as $\frac{d}{dt}(\frac{1}{\omega} \sin \omega t)$, so this is a total time derivative and can be removed. In addition, we can write

$$\frac{d}{dt} \sin(\theta - \omega t) = (\dot{\theta} - \omega) \cos(\theta - \omega t), \quad (8)$$

so that

$$\dot{\theta} \cos(\theta - \omega t) = \omega \cos(\theta - \omega t) + \frac{d}{dt} \sin(\theta - \omega t). \quad (9)$$

which leaves

$$L = \frac{1}{2}[\ell^2 \dot{\theta}^2 + 2a\ell\omega^2 \cos(\theta - \omega t)] + g\ell \cos \theta. \quad (10)$$

Finally we can again divide off a multiplicative constant ℓ^2 giving

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{a\omega^2}{\ell} \cos(\theta - \omega t) + \frac{g}{\ell} \cos \theta. \quad (11)$$

Scoring: 0.5 point for removing the constant term, 1 point for all removing constant factors, 1 point for arguing for the removal of total time-derivatives, 1 point for finding the $\dot{\theta} \cos(\theta - \omega t)$ term.

d) Find the equations of motion. [3.5 points]

Solution: The ingredients for the Lagrange equation are

$$\frac{\partial L}{\partial \theta} = -\frac{a\omega^2}{\ell} \sin(\theta - \omega t) - \frac{g}{\ell} \sin \theta, \quad \frac{\partial L}{\partial \dot{\theta}} = \dot{\theta}, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \ddot{\theta}, \quad (12)$$

which gives the Lagrange equation

$$\ddot{\theta} + \frac{a\omega^2}{\ell} \sin(\theta - \omega t) + \frac{g}{\ell} \sin \theta = 0. \quad (13)$$

Scoring: 1 point for the general form of Lagrange's equation, 0.5 for each ingredient in the e.o.m., except $\frac{\partial L}{\partial \theta}$ which gives 1 point, 0.5 point for getting the relative sign of the terms in Lagrange's equation right.

- e) Show that in the limit of fast rotation of the wheel, $\omega \gg \sqrt{\frac{g}{a}}$, we have the solution $\theta(t) = \omega t$. [2 points]

Solution: In this case the second term in the e.o.m. dominates over the third so our effective e.o.m. is

$$\ddot{\theta} + \frac{a\omega^2}{\ell} \sin(\theta - \omega t) = 0. \quad (14)$$

Inserting $\theta(t) = \omega t$ we see that it is a solution since $\ddot{\theta} = 0$ and $\sin(\theta - \omega t) = \sin 0 = 0$.

Scoring: 1 point for finding new e.o.m., 1 point for inserting and showing the solution.

- f) If we look at small perturbations around this fast solution, defining $\phi(t) \equiv \theta(t) - \omega t$, where $\phi \ll 1$, show that the solution for ϕ is the harmonic oscillator like solution $\phi(t) = \phi_0(t) \sin \omega t$, and find $\phi_0(t)$. Comment on what happens when ℓ and a are similar in size. [4 points]

Solution: With the transformation the e.o.m. becomes

$$\ddot{\phi} + \frac{a\omega^2}{\ell} \sin \phi + \frac{g}{\ell} \sin(\phi + \omega t) = 0. \quad (15)$$

For small ϕ we have $\sin \phi \simeq \phi$ and

$$\sin(\phi + \omega t) = \sin \phi \cos \omega t + \sin \omega t \cos \phi \simeq \phi \cos \omega t + \sin \omega t, \quad (16)$$

so that

$$\ddot{\phi} + \left(\frac{a}{\ell} \omega^2 + \frac{g}{\ell} \cos \omega t \right) \phi + \frac{g}{\ell} \sin \omega t = 0. \quad (17)$$

Inserting the proposed solution into the e.o.m. gives

$$-\phi_0 \omega^2 \sin \omega t + \left(\frac{a}{\ell} \omega^2 + \frac{g}{\ell} \cos \omega t \right) \phi_0 \sin \omega t + \frac{g}{\ell} \sin \omega t = 0. \quad (18)$$

This is true if and only if

$$\left(\frac{a}{\ell} \omega^2 + \frac{g}{\ell} \cos \omega t - \omega^2 \right) \phi_0 = -\frac{g}{\ell}, \quad (19)$$

which means

$$\phi_0(t) = \frac{g/\ell}{\omega^2 - (\frac{a}{\ell} \omega^2 + \frac{g}{\ell} \cos \omega t)}. \quad (20)$$

This means that the amplitude for ϕ grows without bounds as ℓ and a get close and the assumptions for the solution break down. Physically this is a resonance.

Scoring: 1 point for rewriting equation into equation for ϕ , 1 point for imposing small ϕ limit, 1 point for checking solution and finding ϕ_0 , 1 point for correct comment on limit.

g) Given the transformation of a Lagrangian $L(q, \dot{q}, t)$,

$$L \rightarrow L' = aL + b + \frac{d}{dt}f(q, t), \quad (21)$$

how does the Hamiltonian $H(q, p, t)$ transform? [3 points]

Solution: The definition of the Hamiltonian is $H \equiv p\dot{q} - L$, which means that we need to take into account how the generalised momentum p transforms. Since we can write

$$\frac{d}{dt}f(q, t) = \frac{\partial f}{\partial q} \frac{dq}{dt} + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial t}, \quad (22)$$

the generalised momentum after the transformation is

$$p' = \frac{\partial L'}{\partial \dot{q}} = ap + \frac{\partial f}{\partial q}. \quad (23)$$

The transformed Hamiltonian is then

$$H' = p'\dot{q} - L' = ap\dot{q} + \frac{\partial f}{\partial q}\dot{q} - aL - b - \frac{d}{dt}f(q, t) = aH - b + \frac{\partial f}{\partial t}. \quad (24)$$

Scoring: 0.5 for the correct definition of the Hamiltonian, 0.5 point for realising that p transforms, 0.5 points for writing the expansion of $\frac{df}{dt}$, 0.5 point for finding p' , 1 point for inserting into the Hamiltonian and re-arranging terms correctly to leave only H , a , b and f .

Question 2 Muon field trip I [28 points]

A muon is an unstable particle, which can be viewed as a more massive cousin of the electron. In this problem we study a muon which is created in the laboratory at $t = 0$ with $\mathbf{v} = 0$, and is then accelerated by a constant – in time and space – electric field $\mathbf{E}$, while there is no magnetic field (as measured in the laboratory reference frame). The mass of the muon is m and its electric charge e . The lifetime of the muon in its rest frame is τ_0 .

For now, we will disregard the effect of the radiation coming from the acceleration of the muon.

a) Use the relativistic form of Newton's second law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad (25)$$

to find the relativistic momentum of the muon $\mathbf{p}(t)$ in the laboratory reference frame. [2 points]

Solution: The Lorentz force acting on the muon is $\mathbf{F} = e\mathbf{E}$. Since $\mathbf{E}$ is constant this can be integrated with respect to time t on both sides of Newton's law giving $\mathbf{p}(t) = e\mathbf{E}t$ when we use that $\mathbf{p}(0) = 0$ from the initial conditions.

Grading: 1 point for finding force, 1 point for integrating correctly using initial conditions.

- b) Find the relativistic energy $\mathcal{E}(t)$ and show that the γ -factor for the muon in the laboratory frame is¹

$$\gamma(t) = \sqrt{1 + k^2 t^2}, \quad \text{where} \quad k = \frac{eE}{mc}.$$

[3 points]

Solution: By the energy-momentum relation we have

$$\mathcal{E}(t) = \sqrt{p^2 c^2 + m^2 c^4} = mc^2 \sqrt{1 + \left(\frac{eE}{mc}\right)^2 t^2} = mc^2 \sqrt{1 + k^2 t^2}.$$

Since the relativistic energy is given as $\mathcal{E} = \gamma mc^2$, the γ -factor is

$$\gamma(t) = \sqrt{1 + k^2 t^2}.$$

Grading: 1 point for using the energy-momentum relationship correctly, 1 point for finding the energy, 1 point for identifying the gamma factor.

- c) Show that the lifetime t_0 of the muon measured in the laboratory is

$$t_0 = \frac{1}{k} \sinh(k\tau_0). \quad (26)$$

[4 points]

Solution: Time measured in the muon's (instantaneous inertial) rest frame is the proper time τ . The relationship to the time t in the laboratory frame is $d\tau = \frac{1}{\gamma} dt$. Thus we have for the lifetime of the muon that

$$\tau_0 = \int_0^{\tau_0} d\tau = \int_0^{t_0} \frac{dt}{\gamma} = \int_0^{t_0} \frac{dt}{\sqrt{1 + k^2 t^2}} = \frac{1}{k} \sinh^{-1}(kt_0).$$

Inverting, we find $t_0 = \frac{1}{k} \sinh(k\tau_0)$.

Grading: 2 points for setting up the correct integral, where 1 point just for the differential relationship between t and τ and 0.5 point for correct integration limits, 1.5 point for doing the integral correctly, and 0.5 point for inverting the expression.

¹We use $\mathcal{E}$ for the energy here so as not to confuse it with the electric field.

- d) Show that the relativistic energy in the laboratory frame, as a function of the proper time τ , is

$$\mathcal{E}(\tau) = mc^2 \cosh(k\tau). \quad (27)$$

[2 points]

Solution: The relationship between laboratory time and proper time found in c) is generic. Thus we have

$$\mathcal{E}(t) = \gamma(t)mc^2 = mc^2 \sqrt{1 + k^2 t^2} = mc^2 \sqrt{1 + \sinh^2(k\tau)} = mc^2 \cosh(k\tau).$$

Grading: 1 point for understanding we know the relationship between the times, 1 point for inserting and rewriting correctly.

- e) Show that the four-acceleration of the muon A^μ can be expressed in the laboratory frame as

$$A^\mu = c(k^2 t, \gamma k \hat{\mathbf{E}}). \quad (28)$$

[5 points]

Solution: We can find the four-acceleration from the four-velocity

$$U^\mu = (\gamma c, \gamma \mathbf{v}) = (\gamma c, \mathbf{p}/m),$$

by direct computation

$$A^\mu \equiv \frac{dU^\mu}{d\tau} = \left(\frac{d\gamma}{d\tau} c, \frac{1}{m} \frac{d\mathbf{p}}{d\tau} \right),$$

where we need to find

$$\frac{d\gamma}{d\tau} = \frac{dt}{d\tau} \frac{d\gamma}{dt} = \gamma \frac{d\gamma}{dt} = \gamma \frac{k^2 t}{\sqrt{1 + k^2 t^2}} = k^2 t,$$

and

$$\frac{d\mathbf{p}}{d\tau} = \frac{dt}{d\tau} \frac{d\mathbf{p}}{dt} = \gamma \frac{d\mathbf{p}}{dt} = \gamma e \mathbf{E}.$$

This gives

$$A^\mu = \left(ck^2 t, \gamma \frac{e}{m} \mathbf{E} \right) = c(k^2 t, \gamma k \hat{\mathbf{E}}),$$

where $\hat{\mathbf{E}}$ is the direction of the electric field.

Grading: 1 point for starting from the derivative of the four-velocity, 1 point for writing this in terms of the known momentum, 1 point for each of the components with a minimum 1 point for making the correct variable change, 1 point for writing the final answer only in terms of k and t .

- f) Explain what the instantaneous inertial rest frame of the muon is. Show that the proper acceleration of the muon $\mathbf{a}_0$ is a constant vector in this reference frame given by

$$\mathbf{a}_0 = kc\hat{\mathbf{E}}, \quad (29)$$

where $\hat{\mathbf{E}}$ is the unit vector in the direction of the electric field, by using the four-acceleration,² and explain why the motion is linear. [5 points]

Solution: The instantaneous inertial reference frame (IIRF) is the constantly changing (under acceleration) inertial reference frame that moves around with the muon and where the muon is at rest (has zero velocity).

We have the following relation for the IIRF: $A^\mu A_\mu = -a_0^2$, which means that the proper acceleration is given by

$$\begin{aligned} a_0^2 &= -A^\mu A_\mu = -(ck^2t)^2 + (c\gamma k)^2 \\ &= -c^2k^4t^2 + (1 + k^2t^2)c^2k^2 = c^2k^2. \end{aligned} \quad (30)$$

Since the relativistic momentum is always in the direction of $\mathbf{E}$ we have linear motion (and acceleration), and we can write $\mathbf{a}_0 = ck\hat{\mathbf{E}}$.

Grading: 2 point for a good explanation of IIRF where 1 point is for the statement that the velocity is zero there. 1 point for realising how to find the proper acceleration from the four-acceleration, 1 point for doing the algebra with 0.5 point for getting the contraction right, 1 point for a good argument for why the motion is linear and what the direction of acceleration is.

- g) What are the electric and magnetic fields observed by the muon in its rest frame? [3.5 points]

Solution: The fields in the lab-frame are $\mathbf{B} = 0$ and the constant $\mathbf{E}$. The Lorentz transformation to the IIRF (denoted by primes) is

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel}, & \mathbf{E}'_{\perp} &= \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}), \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel}, & \mathbf{B}'_{\perp} &= \gamma(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E}/c^2), \end{aligned}$$

where $\mathbf{v}$ is the velocity of the muon in the lab-frame, and the parallel and perpendicular symbols relate to the direction of the boost. However, since $\mathbf{B} = \mathbf{B}_{\parallel} = \mathbf{B}_{\perp} = 0$, and $\mathbf{E}_{\perp} = 0$ because the velocity is in the direction of the electric field, we have $\mathbf{E}' = \mathbf{E}$, and since $\mathbf{v} \times \mathbf{E} = 0$, again because they are in the same direction, $\mathbf{B}' = 0$. So, in a sense this is a trick question, the fields are the same observed from the muon.

²There is another way of doing this, but that will not score any points. Sorry.

Since the electric field is the same in the muon's reference frame, the other way to find the proper acceleration hinted to above would be to directly find it from the force – but you would need to show that in this frame $\frac{d\mathbf{p}}{dt} = m\mathbf{a}_0$.

Grading: 1 point for finding the correct transformations, 1 point for inserting the zero magnetic field, 0.5 point for $\mathbf{E}_\perp = 0$, 1 point for realising $\mathbf{v} \times \mathbf{E} = 0$,

h) Show that the acceleration in the laboratory frame is

$$\mathbf{a} = \frac{\mathbf{a}_0}{\gamma^3}. \quad (31)$$

[3.5 points]

Solution: For the scalar component of four-acceleration $A^\mu = (A^0, \mathbf{A})$ we have

$$A^0 = \frac{dU^0}{d\tau} = \frac{d(c\gamma)}{d\tau} = c\gamma \frac{d\gamma}{dt} = c\gamma \frac{d}{dt} \left(1 - \frac{\mathbf{v}^2}{c^2} \right)^{-\frac{1}{2}} = c\gamma \gamma^3 \frac{\mathbf{v}}{c^2} \cdot \frac{d\mathbf{v}}{dt} = \gamma^4 \frac{\mathbf{v} \cdot \mathbf{a}}{c}, \quad (32)$$

where $\mathbf{v}$ and $\mathbf{a}$ are the velocity and acceleration in any inertial reference frame, for example the laboratory frame. We can compare this to our result $A^0 = ck^2t$ above in the laboratory frame, and since $\mathbf{v} \cdot \mathbf{a} = va$ because the motion is linear in the direction of $\hat{\mathbf{E}}$, we get

$$\gamma^4 \frac{\mathbf{v} \cdot \mathbf{a}}{c} = \gamma^3 \frac{\gamma m v a}{mc} = \gamma^3 \frac{pa}{mc} = \gamma^3 \frac{eEta}{mc} = \gamma^3 kta = ck^2t, \quad (33)$$

which means $\mathbf{a} = \frac{ck}{\gamma^3} \hat{\mathbf{E}} = \frac{\mathbf{a}_0}{\gamma^3}$.

Grading: 1.5 point for finding a comparison between our specific calculation and a general expression for A^μ in terms of $\mathbf{a}$, 1 point for finding what the general expression is in our case, and 1 point for solving for $\mathbf{a}$.

Question 3 Muon field trip II [23 points]

We now look at the radiation from the muon in the previous question.

a) Find the electric dipole moment of the muon as observed in the laboratory in terms of its path $\mathbf{r}(t)$ measured from the origin. [3 points]

Solution: For a single particle we can write the charge density as $\rho(\mathbf{r}, t) = e\delta(\mathbf{r} - \mathbf{r}(t))$ where $\mathbf{r}(t)$ is the path of the particle. This gives the dipole moment

$$\mathbf{p} = \int \mathbf{r} \rho(\mathbf{r}, t) d^3r = e \int \mathbf{r} \delta(\mathbf{r} - \mathbf{r}(t)) d^3r = e\mathbf{r}(t), \quad (34)$$

by the properties of the delta-function.

Grading: 1 point for correct definition, 1 point for correct charge density, 1 point for carrying out the integral.

- b) Find the magnetic dipole moment of the muon as observed in the laboratory. [4 points]

Solution: For a single particle we can write the current density as $\mathbf{j}(\mathbf{r}, t) = e\mathbf{v}(t)\rho(\mathbf{r}, t) = e\mathbf{v}(t)\delta(\mathbf{r} - \mathbf{r}(t))$ where $\mathbf{r}(t)$ is the path of the particle and $\mathbf{v}(t)$ its velocity. This gives the dipole moment

$$\begin{aligned}\mathbf{m} &= \frac{1}{2} \int \mathbf{r} \times \mathbf{j}(\mathbf{r}, t) d^3r = \frac{e}{2m} \int (\mathbf{r} \times m\mathbf{v}(t)) \delta(\mathbf{r} - \mathbf{r}(t)) d^3r \\ &= \frac{e}{2m} (\mathbf{r}(t) \times m\mathbf{v}(t)) = 0,\end{aligned}$$

where we have used that the muon has linear acceleration so the velocity and path is pointing in the same direction.

Grading: 1 point for definition, 1 point for current density, 1 point for doing the integral correctly, and 1 point for understanding that it is zero.

- c) Using the Liénard-Wiechert potentials one can show³ that the radiation part of the fields for a relativistic particle far from the observer can be written in terms of the electric dipole moment $\mathbf{p}$ as

$$\mathbf{E}_{\text{rad}}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} \left[\frac{\ddot{\mathbf{p}} \times (\hat{\mathbf{n}} - \frac{\mathbf{v}}{c})}{(1 - \hat{\mathbf{n}} \cdot \frac{\mathbf{v}}{c})^3} \right]_{\text{ret}} \times \hat{\mathbf{n}}, \quad (35)$$

$$\mathbf{B}_{\text{rad}}(\mathbf{r}, t) = \frac{1}{c} \hat{\mathbf{n}} \times \mathbf{E}_{\text{rad}}(\mathbf{r}, t), \quad (36)$$

where $\hat{\mathbf{n}}$ is the unit vector in the direction of the observer and “ret” signifies evaluation at the retarded time $t_r = t - r/c$.

Use this to find the radiation fields $\mathbf{E}_{\text{rad}}(\mathbf{r}, t)$ and $\mathbf{B}_{\text{rad}}(\mathbf{r}, t)$ for the muon in the laboratory frame. Comment on the direction of the fields with respect to the observer. *Hint:* Try to remove the velocity to leave only k and $\hat{\mathbf{E}}$ from Question 2 as free parameters apart from the standard constants. We will give points for partial answers. [6 points]

Solution: We need the time-derivatives of the electric dipole moment which is just $\ddot{\mathbf{p}} = e\mathbf{a}(t)$. We then have

$$\mathbf{E}_{\text{rad}}(\mathbf{r}, t) = \frac{\mu_0 e}{4\pi r} \left[\frac{\mathbf{a} \times (\hat{\mathbf{n}} - \frac{\mathbf{v}}{c})}{(1 - \hat{\mathbf{n}} \cdot \frac{\mathbf{v}}{c})^3} \right]_{\text{ret}} \times \hat{\mathbf{n}} = \frac{\mu_0 e}{4\pi r} \left[\frac{\mathbf{a} \times \hat{\mathbf{n}}}{(1 - \hat{\mathbf{n}} \cdot \frac{\mathbf{v}}{c})^3} \right]_{\text{ret}} \times \hat{\mathbf{n}}, \quad (37)$$

³Since we are decent people we won't torture you with the details on this exam.

where we use that $\mathbf{v}$ and $\mathbf{a}$ are parallel. With $\mathbf{a} = \frac{\mathbf{a}_0}{\gamma^3} = \frac{ck\hat{\mathbf{E}}}{\gamma^3}$ we have

$$\begin{aligned}\mathbf{E}_{\text{rad}}(\mathbf{r}, t) &= \frac{\mu_0 e}{4\pi r} ck \left[\frac{\hat{\mathbf{E}} \times \hat{\mathbf{n}}}{\gamma^3 (1 - \hat{\mathbf{n}} \cdot \frac{\mathbf{v}}{c})^3} \right]_{\text{ret}} \times \hat{\mathbf{n}} \\ &= \frac{\mu_0 e}{4\pi r} \frac{ck}{(\gamma - \hat{\mathbf{n}} \cdot \frac{\gamma m \mathbf{v}}{mc})_{\text{ret}}^3} (\hat{\mathbf{E}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}} \\ &= \frac{\mu_0 e}{4\pi r} \frac{ck}{(\gamma - kt\hat{\mathbf{n}} \cdot \hat{\mathbf{E}})_{\text{ret}}^3} (\hat{\mathbf{E}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}.\end{aligned}\quad (38)$$

The magnetic field is simply

$$\mathbf{B}_{\text{rad}}(\mathbf{r}, t) = \frac{1}{c} \hat{\mathbf{n}} \times \mathbf{E}_{\text{rad}}(\mathbf{r}, t) = \frac{\mu_0 e}{4\pi r} \frac{k}{(\gamma - kt\hat{\mathbf{n}} \cdot \hat{\mathbf{E}})_{\text{ret}}^3} \hat{\mathbf{E}} \times \hat{\mathbf{n}}. \quad (39)$$

Both radiation fields are perpendicular to the direction to the observer since $\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{rad}} = \hat{\mathbf{n}} \cdot \mathbf{B}_{\text{rad}} = 0$, because the cross-product with $\hat{\mathbf{n}}$ in the fields make them perpendicular to $\hat{\mathbf{n}}$.

Grading: 1 point for finding the second derivative of the dipole, 1 point for using the linear motion, 1 point for inserting the correct acceleration, 1 point for rewriting the last velocity factor, 0.5 point for keeping the retarded time in the correct place, 0.5 point for giving both fields (it is not necessary to do the fancy re-write of the cross products), 1 point for a good comment on the direction of the fields.

d) Find Poynting's vector for the radiation fields and comment on its angular dependence. [6 points]

Solution: We have

$$\begin{aligned}\mathbf{S}(\mathbf{r}, t) &= \frac{1}{\mu_0} \mathbf{E}_{\text{rad}} \times \mathbf{B}_{\text{rad}} = \frac{1}{\mu_0} \mathbf{E}_{\text{rad}} \times \frac{1}{c} (\hat{\mathbf{n}} \times \mathbf{E}_{\text{rad}}) \\ &= \frac{1}{\mu_0 c} \hat{\mathbf{n}} E_{\text{rad}}^2 - \frac{1}{\mu_0 c} \mathbf{E}_{\text{rad}} (\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{rad}}) \\ &= \frac{\mu_0 e^2}{16\pi^2 r^2 c} \frac{c^2 k^2 \sin^2 \theta}{(\gamma - kt \cos \theta)_{\text{ret}}^6} \hat{\mathbf{n}}\end{aligned}\quad (40)$$

where we have used the vector relationship $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, and that $\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{rad}} = 0$. The angle θ is defined as the angle between the direction of motion of the muon $\hat{\mathbf{E}}$ and the observer $\hat{\mathbf{n}}$.

Poynting's vector is pointing (sorry!) in the direction of the observer $\hat{\mathbf{n}}$. The dependence on θ through the sin-term means that there is no energy flowing in the directly forward $\theta = 0$ (and backward $\theta = \pi$) direction of the muon. However, because of the cos-term there is also more radiation near the forward direction.

Grading: 0.5 point for the definition, 1 point for removing one field, 1 point for rewriting using expression for double cross product, 1 point for removing term due to perpendicular direction, 1 point for inserting field and getting angular dependence in both positions, 0.5 for completing the algebra, 1 point for a correct comment on the directional dependence.

- e) Power P can be shown to be a Lorentz invariant quantity. Starting from the non-relativistic form found in the formulae collection, use this to argue why the relativistic form of Larmor's formula giving the radiated power from single charged particle of charge q is

$$P = \frac{\mu_0 q^2}{6\pi c} \mathbf{a}_0^2, \quad (41)$$

where $\mathbf{a}_0$ is the proper acceleration. Use this formula to show that the total energy the muon radiates in its lifetime t_0 , as measured in the laboratory, is

$$\mathcal{E}_{\text{rad}} = \frac{\mu_0 e^2}{6\pi} k c \sinh(k\tau_0). \quad (42)$$

[4 points]

Solution: The non-relativistic Larmor formula is

$$P = \frac{\mu_0 q^2}{6\pi c} a^2, \quad (43)$$

where a is the acceleration of the particle. This must be true in the rest frame of the particle where the acceleration is the proper acceleration $\mathbf{a}_0$ since the particle is non-relativistic there. But since the proper acceleration squared is Lorentz invariant (it is the contraction of the four-acceleration), then the expression is Lorentz invariant and must be true in all frames.

The radiated power for the muon is then constant and equal to

$$P = \frac{\mu_0 e^2}{6\pi c} \mathbf{a}_0^2 = \frac{\mu_0 e^2}{6\pi} k^2 c.$$

Since this is Lorentz invariant we can look at the total energy in the lab reference frame where we know that the muon decays after time $t_0 = \frac{1}{k} \sinh(k\tau_0)$. Thus the total radiated energy there is

$$\mathcal{E}_{\text{rad}} = P t_0 = \frac{\mu_0 e^2}{6\pi} k^2 c \frac{1}{k} \sinh(k\tau_0) = \frac{\mu_0 e^2}{6\pi} k c \sinh(k\tau_0). \quad (44)$$

Grading: Up to 2 points for a good argument for the the formula, here 1 point is for the non-relativistic formula being applicable to the IIRF, and 1 point for arguing that the expression with the proper acceleration is Lorentz invariant. 0.5 point for finding the power for the muon, 1 point for realising how to complete the calculation with constant power, and 0.5 point for the final answer.