

Question 1 Mexican hat [21 points]

We begin by considering a particle moving in the one-dimensional Mexican hat potential¹

$$V(x) = -\frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4,$$

shown in Fig. 1, where k and λ are positive parameters. The corresponding Lagrangian is²

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 - \frac{1}{4}\lambda x^4, \quad (1)$$

where x is the generalised coordinate of the system and $\dot{x} = \frac{dx}{dt}$.

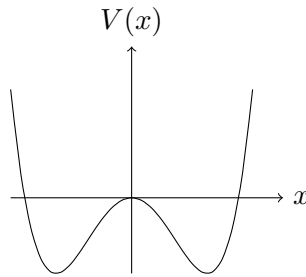


Figure 1: The Mexican hat potential. The name is given from the not so obvious similarity to the famous Mexican sombrero.

a) Find and classify the equilibrium points of the system. [4 points]

Solution: The equilibria are given by the extremum points of the potential in terms of the generalised coordinates, where $\frac{\partial V}{\partial x} = 0$. Since

$$\frac{\partial V}{\partial x} = -kx + \lambda x^3 = \lambda x(x^2 - \frac{k}{\lambda}),$$

either at $x = 0$ or $x = \pm\sqrt{\frac{k}{\lambda}}$. Looking at the second derivative we have

$$\frac{\partial^2 V}{\partial x^2} = -k + 3\lambda x^2,$$

which gives $\frac{\partial^2 V}{\partial x^2} = -k < 0$ and $\frac{\partial^2 V}{\partial x^2} = -k + 3k = 2k > 0$, respectively. Thus $x = 0$ with $V(0) = 0$ is a maximum, and we have to degenerate minima at $x_0 = \pm\sqrt{\frac{k}{\lambda}}$ with

$$V(x_0) = -\frac{1}{2}k\frac{k}{\lambda} + \frac{1}{4}\lambda\frac{k^2}{\lambda^2} = -\frac{k^2}{2\lambda} + \frac{k^2}{4\lambda} = -\frac{k^2}{4\lambda}.$$

¹Since it is one dimensional it is a bit hard to wear, but it works fine mathematically.

²This is a mechanical hat, not a field theory hat, so it is not the problem you studied in Problem Set 12, but it has some distinct similarities.

Scoring: 1 point for the correct criterium, 1 point for finding and classifying each point.

b) Find the equation of motion for the system. [4 points]

Solution: The ingredients to Lagrange's equation are:

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = kx - \lambda x^3, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m\ddot{x},$$

so Lagrange's equation gives

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} - kx + \lambda x^3 = 0.$$

Scoring: 1 point for each ingredient in the Lagrange equation, 1 point for putting it together with the correct sign.

c) Find the Hamiltonian. [4 points]

Solution: The Hamiltonian for one generalised coordinate x is given by

$$H(x, p) = p\dot{x} - L(x, \dot{x}) = p\dot{x} - \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4,$$

where p is the generalised momentum $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$. We solve the generalised momentum for the velocity $\dot{x}$ and insert to get the Hamiltonian in terms of x and p .

$$H(x, p) = p \frac{p}{m} - \frac{1}{2}m \left(\frac{p}{m} \right)^2 - \frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4 = \frac{p^2}{2m} - \frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4.$$

Scoring: 1 point for the correct Hamiltonian expression for one variable, 1 point for inserting correctly, 1 point for finding the generalised momentum, 1 point for removing the velocity in the Hamiltonian.

d) Find one continuous symmetry of this Lagrangian and what is conserved under that symmetry. [3 points]

Solution: There is no explicit time-dependence in the Lagrangian, so it is symmetric under time-translation $t \rightarrow t' = t + a$ where a is a free (continuous) parameter. We have seen in the lectures that this means that the Hamiltonian is conserved since

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0. \quad (2)$$

In our case, since $K = \frac{1}{2}m\dot{x}^2 = \frac{p^2}{2m}$, the Hamiltonian is equal to the total energy $H = K + V$.

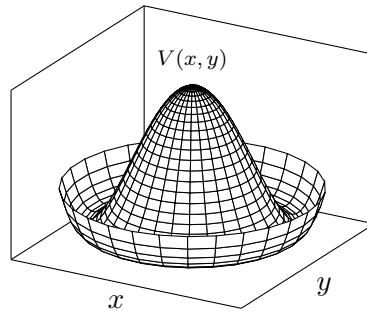


Figure 2: The two-dimensional Mexican hat potential.

Scoring: 1 point for identifying the translation symmetry, 1 point for stating the correct conclusion, 1 point for a correct argument that the Hamiltonian is the total energy.

Suppose now that the particle is moving in the two-dimensional Mexican hat potential shown in Fig. 2, where the Lagrangian can be written

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^2 + y^2) - \frac{1}{4}\lambda(x^2 + y^2)^2. \quad (3)$$

- e) Show that this Lagrangian is symmetric under a rotation in the xy -plane. [3 points]

Solution: With rotation $(x, y) \rightarrow (x', y')$ given by

$$\begin{aligned} x' &= \cos \theta x + \sin \theta y, \\ y' &= -\sin \theta x + \cos \theta y, \end{aligned}$$

we have

$$\begin{aligned} x'^2 + y'^2 &= (\cos \theta x + \sin \theta y)^2 + (-\sin \theta x + \cos \theta y)^2 \\ &= \cos^2 \theta x^2 + 2 \cos \theta \sin \theta xy + \sin^2 \theta y^2 \\ &\quad + \sin^2 \theta x^2 - 2 \sin \theta \cos \theta xy + \cos^2 \theta y^2 \\ &= x^2 + y^2, \end{aligned}$$

and since

$$\begin{aligned} \dot{x}' &= \frac{d}{dt}(\cos \theta x + \sin \theta y) = \cos \theta \dot{x} + \sin \theta \dot{y}, \\ \dot{y}' &= \frac{d}{dt}(-\sin \theta x + \cos \theta y) = -\sin \theta \dot{x} + \cos \theta \dot{y}, \end{aligned}$$

we must similarly have

$$\dot{x}'^2 + \dot{y}'^2 = \dot{x}^2 + \dot{y}^2.$$

Thus $L[x', y', \dot{x}', \dot{y}'] = L[x, y, \dot{x}, \dot{y}]$. This can also be done via a switch to polar coordinates using that θ is cyclic.

Scoring: 1 point for a good way of writing down the effect of rotation, 1 point for demonstrating the invariance of the potential and one for the kinetic energy.

- f) Find and identify the conserved Noether charge associated with the rotation. *Hint:* Write the rotation in terms of an infinitesimal angle $\delta\theta$. [3 points]

Solution: With an infinitesimal angle the rotation is ($\cos \delta\theta = 1$, $\sin \delta\theta = \delta\theta$):

$$\begin{aligned}x' &= x + \delta\theta y, \\y' &= -\delta\theta x + y.\end{aligned}$$

This yields a Noether charge

$$\delta Q = \frac{\partial L}{\partial \dot{x}} \delta x + \frac{\partial L}{\partial \dot{y}} \delta y = m\dot{x}\delta\theta y + m\dot{y}(-\delta\theta x) = m(\dot{x}y - \dot{y}x)\delta\theta,$$

which means that, removing the arbitrary (infinitesimal) transformation parameter $\delta\theta$, that $m(\dot{x}y - \dot{y}x)$ (angular momentum around the z -axis) is the conserved Noether charge.

Scoring: 1 point for writing down the Noether charge correctly and finding partial derivatives, 1 point for finding the infinitesimal transformations, 1 point for identifying this as angular momentum.

Question 2 Some action [19 points]

Consider the action

$$S[x(\tau)] = -mc^2 \int_{\tau_0}^{\tau_1} d\tau, \quad (4)$$

where τ is the proper time for a particle moving in one space dimension x .

- a) Explain the meaning of proper time. [2 points]

Solution: The proper time is the time observed in a reference frame moving along with the particle, or a clock *on* the particle if you will.

Scoring: 2 points for an adequate explanation.

- b) Show that the Lagrangian for the particle in some reference frame with coordinates x and time t is

$$L(x, \dot{x}) = -mc^2 \left(1 - \frac{\dot{x}^2}{c^2}\right)^{1/2}.$$

[3 points]

Solution: We have that $d\tau = \frac{1}{\gamma}dt$, where $\gamma = (1 - \frac{\dot{x}^2}{c^2})^{-1/2}$ is the standard relativistic factor. By changing variables from τ to t in the action we get

$$S[x(t)] = -mc^2 \int_{\tau_0}^{\tau_1} d\tau = -mc^2 \int_{t_0}^{t_1} (1 - \frac{\dot{x}^2}{c^2})^{1/2} dt. \quad (5)$$

This action has the Lagrangian $L = -mc^2(1 - \frac{\dot{x}^2}{c^2})^{1/2}$.

Scoring: 1 point for realising the substitution, 1 point for writing down γ correctly, 1 point for interpreting the Lagrangian correctly.

- c) Find the equation of motion for the particle and show that it reduces to the equation of motion of a free non-relativistic particle. [5 points]

Solution: With the above Lagrangian the equations of motion are

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \dot{x}} = -mc^2 \frac{1}{2} (1 - \frac{\dot{x}^2}{c^2})^{-1/2} (-2 \frac{\dot{x}}{c^2}) = \gamma mc^2 \frac{\dot{x}}{c^2} = \gamma m \dot{x}$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} &= m \frac{d}{dt} \left(\frac{\dot{x}}{(1 - \frac{\dot{x}^2}{c^2})^{1/2}} \right) \\ &= m \frac{\ddot{x}}{(1 - \frac{\dot{x}^2}{c^2})^{1/2}} + m \frac{\dot{x}(-\frac{1}{2})(-2\frac{\dot{x}}{c^2}\ddot{x})}{(1 - \frac{\dot{x}^2}{c^2})^{3/2}} \\ &= \frac{m\ddot{x}}{(1 - \frac{\dot{x}^2}{c^2})^{1/2}} \left(1 + \frac{\frac{\dot{x}^2}{c^2}}{(1 - \frac{\dot{x}^2}{c^2})} \right) \\ &= \frac{m\ddot{x}}{(1 - \frac{\dot{x}^2}{c^2})^{3/2}} = m\gamma^3 \ddot{x}, \end{aligned}$$

so the e.o.m. is $m\gamma^3 \ddot{x} = 0$. Since $\gamma \geq 1$, all solutions here must have $\ddot{x} = 0$, which is the e.o.m. for a free non-relativistic particle.

Scoring: 1 point for each of the two first ingredients to Lagrange's equation, 2 points for the total time differentiation, 1 point for a correct comparison to the free non-relativistic case.

- d) Show directly from the Lorentz transformation

$$t' = \gamma(t - \frac{vx}{c^2}), \quad x' = \gamma(x - vt), \quad (6)$$

that the action in (4) is invariant under Lorentz transformations. Are the equations of motion also invariant? *Hint:* The invariant line element is $ds = \sqrt{c^2 dt^2 - dx^2}$ in one space dimension. [5 points]

Solution: There are multiple ways of doing this, but perhaps the easiest is to show directly that $d\tau$ is invariant starting from its relationship with the invariant line element $ds = cd\tau$, since in the reference frame where τ is the time the particle is at rest. Since the Lorentz transformation is a linear transformation we have:

$$dt' = \gamma(dt - \frac{v}{c^2}dx), \quad dx' = \gamma(dx - vdt), \quad (7)$$

and then

$$\begin{aligned} d\tau' &= cds' = \sqrt{c^2 dt'^2 - dx'^2} \\ &= \sqrt{c^2 \gamma^2 (dt - v/c^2 dx)^2 - \gamma^2 (dx - vdt)^2} \\ &= \gamma \sqrt{c^2 dt^2 + v^2/c^2 dx^2 - dx^2 - v^2 dt^2} \\ &= \gamma \sqrt{(1 - v^2/c^2)(c^2 dt^2 - dx^2)} = \sqrt{c^2 dt^2 - dx^2} = d\tau. \end{aligned}$$

Since $d\tau$ is invariant and the rest of the action is just constants, the action must be Lorentz invariant. Since the action is invariant the equations of motion are automatically invariant since they come from the minimisation of the action.

Scoring: With this method, 1 point for the idea that we can check transformation of ds , 1 point for correct insertion (subtract 0.5 point if no comment on linearity), 1 point for doing the algebra, 1 point for a statement why this is sufficient, 1 point for a correct statement on the invariance of the e.o.m. Only 1 point for simply pointing out that τ is invariant.

e) Now, consider a more general expression for an action

$$S[x(\lambda)] = -M \int_{\lambda_0}^{\lambda_1} \frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda} d\lambda, \quad (8)$$

where the motion is described in terms of the position four-vector $x^\mu(\lambda)$ is parameterised by some parameter λ , and where M is some constant. Under what conditions will this be equivalent to the action in Eq. (4)? [4 points]

Solution: If we change the variable from λ to τ to compare we get

$$S[x(\lambda)] = -M \int_{\tau_0}^{\tau_1} \frac{d\tau}{d\lambda} \frac{dx^\mu}{d\tau} \frac{d\tau}{d\lambda} \frac{dx_\mu}{d\tau} \frac{d\lambda}{d\tau} d\tau = -Mc^2 \int_{\tau_0}^{\tau_1} \frac{d\tau}{d\lambda} d\tau, \quad (9)$$

where we have used the four-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ to find

$$\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} = U^\mu U_\mu = c^2. \quad (10)$$

We must then have $\frac{d\tau}{d\lambda} = \frac{m}{M}$, so there must be a linear relationship between τ and λ .

Scoring: 1 point for the idea of simply changing variables, 1 point for doing the variable change, 1 point for realising the four-velocity cancels, 1 point for drawing the correct conclusions on M and the relationship between the parameters.

Question 3 How Lorentz found the transformations [19 points]

We will look at a particle with charge q moving at constant velocity v in the z -direction through the origin. We want to find the potentials which are observed in the xy -plane as a result of the charge.

We remind you that the **Liénard-Wiechert potentials** for a point charge with path $\mathbf{r}(t)$ can be written

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0(R - \boldsymbol{\beta} \cdot \mathbf{R})_{\text{ret}}}, \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 qc}{4\pi} \left(\frac{\boldsymbol{\beta}}{R - \boldsymbol{\beta} \cdot \mathbf{R}} \right)_{\text{ret}},$$

where $\mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t)$ is the vector between the observer and the particle, $\boldsymbol{\beta} = \frac{\mathbf{v}(t)}{c}$, and “ret” indicates that the time of evaluation is the retarded time

$$t_- = t - \frac{1}{c}R(t_-) = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}(t_-)|.$$

- a) Explain what the retarded time is, and show that for an observer in the xy -plane it can be expressed as

$$t_- = t - \frac{1}{c}\sqrt{x^2 + y^2 + v^2 t_-^2}. \quad (11)$$

[5 points]

Solution: The retarded time t_- is the time when a change in the e.m. source is visible to the observer at the later time t , and the difference in the times is the time it takes for a light-signal to travel from the source to the observer.

Since the observer is in the xy -plane where $z = 0$ we have $\mathbf{r} = (x, y, 0)$ and the path of the particle can be written $\mathbf{r}(t) = vt\hat{\mathbf{e}}_z = (0, 0, vt)$. We then have

$$|\mathbf{r} - \mathbf{r}(t)| = \sqrt{(\mathbf{r} - \mathbf{r}(t))^2} = \sqrt{r^2 + r(t)^2 - \mathbf{r} \cdot \mathbf{r}(t)} = \sqrt{x^2 + y^2 + v^2 t^2}.$$

Inserted into the expression for retarded time we get

$$t_- = t - |\mathbf{r} - \mathbf{r}(t_-)|/c = t - \frac{1}{c}\sqrt{x^2 + y^2 + v^2 t_-^2}.$$

Scoring: 2 points for a good explanation of retarded time, 1 point for figuring out the coordinates of the particle and the observer, 1 point for calculating the distance, 1 point for putting it together.

b) Solve for the retarded time in Eq. (11) to show that

$$t_- = \gamma^2 t - \frac{\gamma^2}{c} \sqrt{v^2 t^2 + (x^2 + y^2)/\gamma^2}. \quad (12)$$

[4 points]

Solution: We square Eq. (11) to find

$$c^2(t_- - t)^2 = x^2 + y^2 + v^2 t_-^2 \Rightarrow (c^2 - v^2)t_-^2 - 2c^2 t t_- + c^2 t^2 - x^2 - y^2 = 0 \quad (13)$$

which has two roots (solutions)

$$\begin{aligned} t_- &= \frac{2c^2 t \pm \sqrt{4c^4 t^2 - 4(c^2 - v^2)(c^2 t^2 - x^2 - y^2)}}{2(c^2 - v^2)} \\ &= \frac{2c^2 t \pm \sqrt{4v^2 c^2 t^2 + 4(c^2 - v^2)(x^2 + y^2)}}{2(c^2 - v^2)} \\ &= \frac{t \pm \frac{1}{c} \sqrt{v^2 t^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}}{1 - \frac{v^2}{c^2}} \\ &= \gamma^2 t \pm \frac{\gamma^2}{c} \sqrt{v^2 t^2 + (x^2 + y^2)/\gamma^2}, \end{aligned}$$

where we have used $\gamma^{-2} = 1 - v^2/c^2$. Only the solution with the minus sign corresponds to the retarded time (a time smaller than the observed time t).

Scoring: 1 point for finding the quadratic equation, 2 point for solving it, 1 point for picking out the correct solution.

c) Show that the scalar and vector potentials in the xy -plane can be written as

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(x^2 + y^2)/\gamma^2 + v^2 t^2}}, \quad (14)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q\mu_0}{4\pi} \frac{v\mathbf{e}_z}{\sqrt{(x^2 + y^2)/\gamma^2 + v^2 t^2}}. \quad (15)$$

[6 points]

Solution: We start from the Liénard-Wiechert expression for the potential. The particle velocity is constant, so $\mathbf{v}(t_-) = v\mathbf{e}_z$ and then $\boldsymbol{\beta}(t_-) = \frac{v}{c}\mathbf{e}_z$. Further, $\mathbf{R}(t_-) = \mathbf{r} - \mathbf{r}(t_-) = \mathbf{r} - vt_- \mathbf{e}_z$, so that

$$\boldsymbol{\beta} \cdot \mathbf{R} = \frac{\mathbf{v}(t_-)}{c} \cdot \mathbf{R}(t_-) = \frac{v\mathbf{e}_z}{c} \cdot \mathbf{r} - \frac{v\mathbf{e}_z}{c} \cdot vt_- \mathbf{e}_z = -\frac{v^2}{c} t_-.$$

Solving the retarded time for $\mathbf{R}$ we also have $R(t_-) = c(t - t_-)$. Inserted into the expression we get

$$\begin{aligned}\phi(\mathbf{r}, t) &= \frac{q}{4\pi\epsilon_0(R - \boldsymbol{\beta} \cdot \mathbf{R})_{\text{ret}}} = \frac{q}{4\pi\epsilon_0(c(t - t_-) + \frac{v^2}{c}t_-)} \\ &= \frac{q}{4\pi\epsilon_0 c(t - t_-/\gamma^2)} \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{v^2 t^2 + (x^2 + y^2)/\gamma^2}},\end{aligned}$$

where we have inserted the derived expression (12) for the retarded time. For the vector potential we similarly get

$$\begin{aligned}\mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q c}{4\pi} \left(\frac{\boldsymbol{\beta}}{R - \boldsymbol{\beta} \cdot \mathbf{R}} \right)_{\text{ret}} = \frac{\mu_0 q c}{4\pi} \left(\frac{\frac{v}{c} \hat{\mathbf{e}}_z}{c(t - t_-) + \frac{v^2}{c}t_-} \right) \\ &= \frac{\mu_0 q}{4\pi} \frac{v}{c(t - t_-/\gamma^2)} \mathbf{e}_z \\ &= \frac{\mu_0 q}{4\pi} \frac{v}{\sqrt{v^2 t^2 + (x^2 + y^2)/\gamma^2}} \mathbf{e}_z.\end{aligned}$$

Scoring: 1 point for finding $\boldsymbol{\beta}$, 1 point for finding $\boldsymbol{\beta} \cdot \mathbf{R}$, 1 point for finding size of R , 1 point of inserting correctly in both expressions with the correct time, 1 point per potential for using the retarded time found earlier and manipulating correctly.

- d) Compare the result for the scalar potential in Eq. (15) to the scalar potential and electric field for a static charge at the origin

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + z^2}}, \quad \mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{r}. \quad (16)$$

Why are these expressions different, and can you explain the differences? [4 points]

Solution: The two potentials are in different reference frames moving with velocity v w.r.t. each other in the z -direction. We therefore expect there to be effects of length contraction and time-dilation. Here, the distance between source and observer in the xy -plane is contracted by a factor of γ in the potential. This is consistent with the result from special relativity that the electric field transforms in the direction perpendicular to the motion

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}).$$

Comment: This is the kind of calculation in electromagnetism where Lorentz first encountered the transformations that now bear his name.

The result in (15) is only derived using electromagnetism and concerns the potentials and fields, but contains the transformations that Einstein later used directly on the coordinates.

Scoring: 1 point for realising these are different frames, 1 point for concluding that we are seeing length contraction, 1 point for noticing it is perpendicular to direction of motion, 1 point for realising that this is what is to be expected given how electric fields transform.

Question 4 Sine-Gordon equation [17 points]

The **sine-Gordon Lagrangian** is given by³

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - m^2(1 - \cos \phi). \quad (17)$$

- a) Show that the resulting equation of motion, the **sine-Gordon equation**, is

$$\square \phi + m^2 \sin \phi = 0. \quad (18)$$

[4 points]

Solution: We find the ingredients of Lagrange's equation

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \sin \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \quad \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi = \square \phi, \quad (19)$$

so Lagrange's equation gives

$$\frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \square \phi + m^2 \sin \phi = 0. \quad (20)$$

Scoring: 1 point for each of the three ingredients in the Lagrange equation, 1 point for putting it together correctly. No commentary on partial/total derivative transition necessary, but expected to start with total and go to partial.

- b) Find the generalised momentum density and the Hamiltonian density.
[4 points]

Solution: The generalised momentum density $\pi(x)$ is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi(x) = \frac{1}{c} \frac{\partial \phi(x)}{\partial t}. \quad (21)$$

³It's a joke. The name, not the exam question. It's almost the Klein-Gordon equation, except with a sine.

The Hamiltonian density is defined by

$$\mathcal{H}(\phi, \nabla\phi, \pi, x) \equiv \pi(x)\partial_0\phi(x) - \mathcal{L}(\phi, \partial_\mu\phi, x). \quad (22)$$

This gives

$$\begin{aligned} \mathcal{H}(\phi, \nabla\phi, \pi, x) &= \partial^0\phi\partial_0\phi - \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + m^2(1 - \cos\phi) \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + m^2(1 - \cos\phi), \end{aligned}$$

where we have removed $\partial_0\phi$ in favour of the generalised momentum π .

Scoring: 1 point for writing down the generalised momentum, 1 point for writing down the Hamiltonian with correct signs on the terms, 2 point for removing the time derivative of the field with correct signs.

- c) When is energy conserved for a field theory? Discuss whether energy is conserved for this model. [5 points]

Solution: Energy (density) conservation comes from translation symmetry (both time and space coordinates). When the Lagrangian has no explicit dependence on the coordinates x^μ as it is here, the the e.o.m. are invariant under translations and we have the required symmetry. By Noether's theorem this means that the components of the energy-momentum tensor $T^{\mu\nu}$ is conserved, *i.e.* $\partial_\mu T^{\mu\nu} = 0$, where $T^{0\nu}$ are the Noether charges. The Noether charge T^{00} is

$$T^{00} = \pi^0\partial^0\phi - \mathcal{L}g^{00} = \partial^0\phi\partial_0\phi - \mathcal{L} = \mathcal{H}, \quad (23)$$

so the energy (Hamiltonian) density does look conserved. However, this is not the complete story. The requirement for the conservation of a Noether charge T^{00} is that the corresponding current

$$T^{i0} = \pi^i\partial^0\phi = \partial^i\phi\partial_0\phi = -\frac{1}{c}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial t},$$

disappears on the boundary. The solution we will see below happens to do this, but it is not obvious since the field does not disappear.

Scoring: 1 point for a correct statement of what we need for energy conservation, 1 point for pointing out that we have translation invariance, 1 point for conservation of energy-momentum tensor, 1 point for concluding that this means that Hamiltonian is conserved as a Noether charge, 1 point if someone notices that the argument is a bit shaky.

- d) Show that in one space dimension, where $\square = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$,

$$\phi_s(x) = 4 \arctan \xi, \quad (24)$$

is a solution to the sine-Gordon equation with $\xi = e^{m\gamma(x-vt)+\delta}$, where v is the (constant) velocity of the wave in the x -direction, γ is the standard relativistic factor that depends on v , and δ a constant.

Hint:

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}, \quad \sin(4 \arctan x) = \frac{4x(1-x^2)}{(1+x^2)^2}. \quad (25)$$

Comment: ϕ_s is called a **soliton** solution. [4 points]

Solution: We have

$$\sin(\phi_s) = \sin(4 \arctan \xi) = \frac{4\xi(1-\xi^2)}{(1+\xi^2)^2}. \quad (26)$$

The derivatives of ξ are

$$\frac{\partial \xi}{\partial x} = m\gamma\xi, \quad \frac{\partial \xi}{\partial t} = -m\gamma v\xi, \quad (27)$$

so by chain rule

$$\frac{\partial \phi_s}{\partial x} = \frac{4m\gamma\xi}{1+\xi^2}, \quad \frac{\partial \phi_s}{\partial t} = \frac{-4m\gamma v\xi}{1+\xi^2}, \quad (28)$$

and

$$\begin{aligned} \frac{\partial^2 \phi_s}{\partial x^2} &= \frac{4m^2\gamma^2\xi \cdot (1+\xi^2) - 4m\gamma\xi \cdot 2m\gamma\xi^2}{(1+\xi^2)^2} = m^2\gamma^2 \frac{4\xi(1-\xi^2)}{(1+\xi^2)^2} \\ \frac{\partial^2 \phi_s}{\partial t^2} &= \frac{4m^2\gamma^2 v^2 \xi \cdot (1+\xi^2) - 4m\gamma v\xi \cdot 2m\gamma v\xi^2}{(1+\xi^2)^2} = m^2\gamma^2 v^2 \frac{4\xi(1-\xi^2)}{(1+\xi^2)^2}. \end{aligned}$$

This gives

$$\begin{aligned} \square \phi_s &= \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi_s = m^2\gamma^2 \left(\frac{v^2}{c^2} - 1 \right) \frac{4\xi(1-\xi^2)}{(1+\xi^2)^2} \\ &= -m^2 \frac{4\xi(1-\xi^2)}{(1+\xi^2)^2}, \end{aligned}$$

using that $\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}}$. So $\square \phi_s + m^2 \sin \phi_s = 0$.

Scoring: Up to four points for a correct calculation, 1 point for finding the sin in terms of ϕ_s , 1 point for finding the first order derivatives, 1 point for second order derivatives, 1 point for putting it together with cancellation using γ .